Tomography and its role in Quantum Computation

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TOMOGRAPHY AND ITS ROLE IN QUANTUM COMPUTATION

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Quantum computation depends on quantum entanglement, a correlation between subsystems that cannot occur classically. A variety of theoretical measures exist for quantifying the degree entanglement in such schemes, all of which are functions of the system density matrix. How can the entanglement be measured experimentally ? Using quantum tomography techniques developed for two photon entangled states, the density matrix can be reconstructed from the appropriate experimental data. In this case the state tomography gives the complete characterization of the physical system (for the relevant degree of freedom, such as spin or polarization). It gives information on both the degree of nonclassical correlation, that is entanglement, as well as the amount of decoherence in the system. In this proceedings we discuss the general state tomography procedure required to characterize a few qubit quantum computer, for any architecture.

1 Introduction

One of the distinguishing features of quantum mechanics, not found in classical physics, is the possibility of entanglement between subsystems. It lies at the core of many applications in the emerging field of quantum information science[1], such as quantum teleportation[2] and quantum error correction[3]. Quantum entanglement refers to correlations between the results of measurements made on component subsystems of a larger physical system which cannot be explained in terms of correlations between local classical properties inherent in those same subsystems. Alternatively, an entangled state cannot be prepared by local operations and local measurements on each subsystem. Thus one often says that an entangled composite system is nonseparable. Formally, the state of a composite system, pure or mixed, is *separable* if the state has an ensemble decomposition in terms of product states. A separable state has no quantum entanglement, and a nonseparable state is entangled.

The nonclassical nature of quantum entanglement has been recognized for many years[4,5] but only recently has considerable attention been focused on trying to understand and characterize its properties precisely. We now have a good understanding of entanglement for a pair of qubits[6], however, how does one determine the extent to which a real physical few qubit system is entangled? What measurements are actually required? There are a number of possible techniques but arguably the simplest (if not the most efficient) is to perform appropriate measurements to reconstruct the density matrix and then use the theoretical measures currently known. This reconstruction technique, known as Quantum state tomography, has a long history. Arguably, the first such experimental technique for determining the state of quantum system was devised in 1852 by George Stokes[7]. He found that four parameters allow one to uniquely determine the polarization state of a light beam. Such techniques can be applied to any ensemble of two-level quantum mechanical systems and allows one to determine the density matrix describing this ensemble. More recently, experimental techniques for the measurement of the more subtle quantum properties of light have been the subject of intensive investigation (see ref. 8 and references their in). Tomographic techniques, in which the density matrix of a quantum state have been applied to experiments such as the homodyne measurement of the Wigner function of a single mode of light [9] and of the density matrix of the polarization degrees of freedom of a pair of entangled photons[10,11].

For the characterization of a few qubit quantum computer, quantum state tomography provides invaluable information on the system. The degree of entanglement and the degree of mixture (entropy for instance) can be calculated. If one were to consider quantum process tomography (tomography associated with the evolution of the state) then effects such as decoherence can be investigated. Two caveats must be made: firstly, there must be a large enough number of copies of an identically prepared quantum system to allow to a reasonable approximation the reconstruction of the state (this may be time consuming in many architectures where the system must be re-initialized after each measurement); secondly, more measurements are preformed in the reconstruction that what is likely to be needed to get the degree of entanglement. However we believe these disadvantages are outweighed by the other information one can obtain from the reconstructed states.

We structure this paper as follows, we first introduce a suitable notation to describe arbitrary n qubit states and then describe a set of simple measurements necessary to reconstruct the state of the system. Several examples are used to illustrate the technique.

2 Multi Qubit states

Let us consider an arbitrary n qubit state (shown schematically in Figure (1)). This n

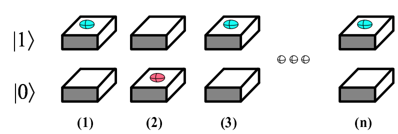

Figure 1. Schematic representation of an n qubit state.

qubit state can be mathematically described by a density matrix of the form,

$$\rho = \frac{1}{2^n} \sum_{i_1,\ldots,i_n=0}^{3} c_{i_1,\ldots,i_n} \lambda_{i_1}^{(1)} \otimes \lambda_{i_2}^{(2)} \otimes \ldots \otimes \lambda_{i_n}^{(n)}, \tag{1}$$

where the λ_i matrices[12] are given by,

$$\lambda_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

and the j superscript in $\lambda_i^{(j)}$ labels the qubit. The $c_{i_1,\ldots,i_n}$ are the coefficients that specify the state. There are 4^n of these that need to be determined however the normalization criterion ($Tr(\rho) = 1$) ensures that $c_{0,\ldots,0} = 1$ leaving $4^n - 1$ parameters to be determined or specified . Noting that,

$$\langle \lambda_{i_1}^{(1)} \lambda_{i_2}^{(2)} \ldots \lambda_{i_n}^{(3)} \rangle = \frac{1}{2^n} \quad c_{i_1,\ldots,i_n}, \tag{3}$$

we now observe the procedure to reconstruct the state. By measuring all the expectation values $\langle \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_n} \rangle$, for $i_1, i_2, \ldots, i_n = 0, 1, 2, 3$ one determines the coefficients $c_{i_1,\ldots,i_n}$ and hence the state. How we actually measure these expectation values$\langle\lambda_{i_1}\lambda_{i_2}...\lambda_{i_n}\rangle$ depends heavily on the physical architecture. However one can always measure the probability of the system being in the ground state $|0\rangle$. From such measurements we can then calculate expectation values like $\langle\lambda_{i_1}\lambda_{i_2}...\lambda_{i_n}\rangle$ (for $i_i = 0$ *or* 3). With appropriate single qubit rotations:

$$\hat{U}_j^k(\phi) = \exp\left[-ik\frac{\pi}{2}\left(|1\rangle_j\langle 0|e^{-i\phi} + |0\rangle_j\langle 1|e^{i\phi}\right)\right], \tag{4}$$

on the individual qubits (where j label the particular qubit, $k\pi$ is the length of the pulse and ϕ the polarization) followed by the ground state measurement, all the expectation values $\langle\lambda_{i_1}\lambda_{i_2}...\lambda_{i_n}\rangle$ (for $i_i = 0,1,2,3$) can be determined and hence the state reconstructed. It is easy also to write the reconstructed state directly in terms of these single qubit rotations and ground measurements.

In any real situations, the information used to reconstruct the state will contain uncertainties due to small experimental errors. This errors make it possible that this reconstruction procedure for the state will not produce a physically acceptable state. While the resulting density matrix will be trace preserving and Hermitian, it may process small negative eigenvalues. Using a maximum likelihood technique[13] physically acceptable density matrices can be obtained. Let us now illustrate this procedure with two examples.

2.1 Single qubit and two qubit reconstruction

A single qubit density matrix can be written as $\rho = \frac{1}{2}\left(I_2 + \sum_{i=1}^3 c_i\lambda_i\right)$, where I_2 is the 2×2 identity matrix, and the coefficients c_i are given by the measured expectation values $c_i = 2\langle\lambda_i\rangle$. In matrix form this is written as,

$$\rho = \frac{1}{2}\begin{pmatrix} 1+\langle\lambda_3\rangle & \langle\lambda_1\rangle - i\langle\lambda_2\rangle \\ \langle\lambda_1\rangle + i\langle\lambda_2\rangle & 1-\langle\lambda_3\rangle \end{pmatrix}. \tag{5}$$

As an example, if one were to measure the correlations $\langle\lambda_1\rangle, \langle\lambda_2\rangle, \langle\lambda_3\rangle$ and find them all zero, the system is the maximally mixed state $\rho = I_2/2$ (diagonal elements only).

Let us consider the simplest states that may contain entanglement, namely two qubit states. Such states can be expressed in the form,

$$\rho = \frac{I_2\otimes I_2}{4} + \sum_{\substack{i_1,i_2=0\\ i_1=i_2\neq 0}}^{3} c_{i_1,i_2} \quad \lambda_{i_1}^{(1)}\otimes\lambda_{i_2}^{(2)}. \tag{6}$$

So by measuring the moments $\langle\lambda_{i_1}^{(1)}\lambda_{i_2}^{(2)}\rangle$, the coefficients c_{i_1,i_2} are determined and hence the density matrix specified. As an example consider the result of the measurement of $\langle\lambda_{i_1}^{(1)}\lambda_{i_2}^{(2)}\rangle$ where the only nonzero zero measurements are given by $\langle\lambda_1^{(1)}\lambda_1^{(2)}\rangle = -\langle\lambda_2^{(1)}\lambda_2^{(2)}\rangle = \langle\lambda_3^{(1)}\lambda_3^{(2)}\rangle = \gamma/4$ and $\langle\lambda_0^{(1)}\lambda_0^{(2)}\rangle = 1/4$. The state is then the Werner state[14] given by,

$$\rho = \begin{pmatrix} \frac{1+\gamma}{4} & 0 & 0 & \frac{\gamma}{2} \\ 0 & \frac{1-\gamma}{4} & 0 & 0 \\ 0 & 0 & \frac{1-\gamma}{4} & 0 \\ \frac{\gamma}{2} & 0 & 0 & \frac{1+\gamma}{4} \end{pmatrix}.$$

With this reconstructed state properties like the degree of entanglement and entropy can

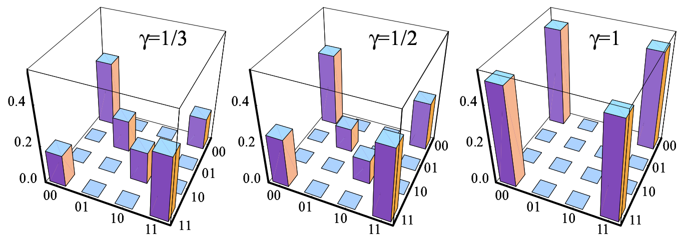

Figure 2. Graphical representation of the two qubit state reconstruction for the Werner state. For $\gamma = 1/3$ the state is separable, while for $\gamma = 1/2, 1$ the state is entangled with $\gamma = 1$ corresponding to the maximally entangled Bell state.

now be calculated. For instance it is straightforward to calculate the entanglement of formation (EOF)[15] and entropy S. At $\gamma = 1/2$ for the example given, we find

$$E_F(\hat{\rho}(\gamma = 1/2)) \approx 0.12 \qquad S(\hat{\rho}(\gamma = 1/2)) \approx 0.77$$

For systems with several more qubits the tomography procedure can be easily implemented however the number of measurements increases as 4^n. Currently their is no analytical formula for the degree of entanglement, however numerical techniques do exist and can be calculated from the density matrix. The degree of mixture is straightforward to calculate.

To summarize, in this proceedings we have shown a simple method by which the state of a few qubit quantum architecture can be reconstructed and hence the degree of entanglement determined. This characterization will be essential for early proof of principle quantum computation experiments.

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