Directed Percolation and Tandem Queues

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growth models

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Directed percolation and tandem queues

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Abstract

We outline some of the connections which exist between directed site percolation and tandem queues, and show how these can be (and have been, although generally not in the language of queueing theory) used to explicit shape theorems for certain first and last passage percolation problems.

1 Directed site percolation in $\mathbb{Z}^2$

For $(i,j) \le (k,l)$ in $\mathbb{Z}^2$, denote by $\Pi_1((i,j),(k,l))$ the set of increasing paths of adjacent points $(i,j)=(i_1,j_1)<(i_2,j_2)<\cdots(i_m,j_m)=(k,l)$. Denote by $\Pi_2((i,j),(k,l))$ the set of paths $(i,j_1),(i+1,j_2),\ldots,(k,j_{k-i+1})$ where $l \le j_1 \le j_2 \le \ldots j_{k-i+1} \le l$. Let $\{s(i,j):\ (i,j)\in\mathbb{Z}^2\}$ be a collection of iid random variables, and for each $\gamma \in \Pi_1((i,j),(k,l)) \cup \Pi_2((i,j),(k,l))$ set $S(\gamma)=\sum_{(i,j)\in\gamma} s(i,j)$. The first and last passage percolation variables we consider are, for $i=1,2$:

$$F_i((i,j),(k,l)) = \min_{\gamma\in\Pi_i((i,j),(k,l))} S(\gamma)$$

and

$$L_i((i,j),(k,l)) = \max_{\gamma\in\Pi_i((i,j),(k,l))} S(\gamma).$$

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By Kingman's subadditive ergodic theorem, the following limits exist almost surely for $x > 0$ (possibly infinite):

$$f_i(x) = \lim_{n\to\infty} \frac{1}{n} F_i((0,0),([xn],n))$$

and

$$l_i(x) = \lim_{n\to\infty} \frac{1}{n} L_i((0,0),([xn],n)).$$

Sufficient conditions for these limits to be finite are given in [2, 5, 7]. Explicit formulae have only been found in some special cases; these are:

If the $s(i,j)$ are exponentially distributed with parameter 1, then [14]

$$l_1(x) = (\sqrt{x}+1)^2.$$

If the $s(i,j)$ are positive geometrically distributed with parameter q, then [10, 4]

$$l_1(x) = \frac{x + 2\sqrt{qx} + 1}{1-q}.$$

If the $s(i,j)$ are Bernoulli with parameter p, then [15]

$$f_2(x) = \left(\max\left\{\sqrt{qx} - \sqrt{(1-q)}, 0\right\}\right)^2$$

and

$$l_2(x) = \begin{cases} \left(\sqrt{qx} + \sqrt{(1-q)}\right)^2 - 1 & \text{for } x > q/(1-q) \\ x & \text{otherwise.} \end{cases}$$

These are often referred to as 'hydrodynamic limits', or 'shape theorems', and are often stated in terms of 'interface' or 'growth' models.

Under suitable moment conditions where necessary, these functions have the following properties (see, for example, [2] for the case of l_2).

l_1 and l_2 are concave;

f_1 and f_2 are convex;

$$l_1(0^+) = \lim_{x\to\infty} l_1(x)/x = \lim_{x\to\infty} l_2(x)/x = Es(0,0);$$

$$f_1(0^+) = \lim_{x\to\infty} f_1(x)/x = \lim_{x\to\infty} f_2(x)/x = Es(0,0);$$

$$l_2(0^+) = f_2(0^+) = 0.$$

One of the (many) big open problems in this area is to determine the last-passage shape function l_1 (or, equivalently, f_1) in the case where the $s(i,j)$ are Bernoulli. This would in turn determine the critical probability for directed site percolation in $\mathbb{Z}^2$, which is presently unknown (it can be verified using concentration inequalities that percolation occurs if, and only if, $l_1(1) = 2$).

Fluctuation theorems for L_1 in the cases of geometric and exponential random variables have been obtained by Johansson [11]. Large deviations results for the same problem are also presented in [11] (lower tails) and by Seppalainen [17] (upper tails); large deviations results are also given in [15] for F_2 in the Bernoulli case.

2 Variational formula for average waiting time in a tandem queueing system

To begin with, consider a series of n single-server queues, each with unlimited waiting space and first-in-first-out service discipline. Initially the system is empty and k customers are placed in the first queue. The service time of the customer i at the queue j is given by $s(i,j)$, and these variables are assumed to be iid and non-negative. Denote by $D(k,n)$ the departure time of customer k from queue n. Then the following formula, which first appeared in [13], holds

$$D(k,n) = L_1((1,1),(k,n)). \tag{1}$$

Glynn and Whitt [9] use this identity to infer asymptotic results for $D(k,n)$ from known asymptotics for last passage percolation, which had earlier been obtained by Rost [14], in the case where the $s(i,j)$ are iid exponentials.

It was recently observed by Seppalainen [16] that an 'equilibrium' version of the identity (1) can be used to go the other way and explicitly compute the hydrodynamic limit for the last passage percolation problem using known results for interacting particle systems. In a queueing context, the fundamental identity which underpins the approach was first presented in [18]. (A related formula was presented in [1] in the context of Hammersley's process and the longest increasing subsequence problem.) We assume the system has been running for an infinitely long time, and customers are labeled by the integers. For $k \in \mathbb{Z}$, let $a(k)$ denote the interarrival time between customers k and $k+1$, at the first queue; we shall assume these are iid random variables. As before, $s(i,j)$ denotes the service time of customer i at queue j, and again we assume these are iid random variables with $\tau = Ea(0) > Es(0,0) = \sigma$ for stability. Denote by $v(i,j)$ the sojourn time of customer i at queue j. Szczotka and Kelly [18] obtain the following identity:

$$\sum_{j=1}^{n} v(0,j) = \sup_{K \geq 1} \left\{ L_1((-K,1),(0,n)) - \sum_{k=1}^{K} a(-k) \right\}. \tag{2}$$

This formula can be used to compute the hydrodynamic limit l_1 in special cases. Suppose that the interdeparture times from the first queue are iid with the same law as the interarrival times, and the random variables $v(0,1),\ldots,v(0,n)$ are also iid. Suppose that for each $\tau > \sigma$ we can choose an interarrival time distribution with $Ea(0) = \tau$ such that this is the case, and $g(\tau) := Ev(0,1) < \infty$. Then, under suitable moment conditions on $s(0,0)$ (see, for example, [2, Theorem 5.3], where it is shown that '$3+\epsilon$ moments suffice', or [16] and references therein for related results), we can divide both sides of (2) by n and let $n \to \infty$ to obtain:

$$g(\tau) = \sup_{x>0} \{l_1(x) - \tau x\}, \tag{3}$$

for $\tau > \sigma$. This variational formula, which is essentially a Legendre transform, can be inverted to get:

$$l_1(x) = \inf_{\tau>\sigma} \{\tau x + g(\tau)\}. \tag{4}$$

We now present two examples where this ingeneous method can be applied.

Example 1. The first is the case where the $s(i,j)$ are exponentially distributed with mean σ and the $a(i)$ are exponentially distributed with mean τ, for any $\tau > \sigma$. The fact that all of the above is satisfied in this case is a consequence of Burke's theorem (see, for example, Kelly [12]). In this case, $g(\tau) = \sigma\tau/(\tau - \sigma)$ and we can apply (4) to obtain Rost's result [14] that

$$l_1(x) = \sigma(\sqrt{x} + 1)^2.$$

Example 2. The second example is a discrete analogue of the first. Take the $s(i,j)$ to be *positive* geometrically distributed with parameter q, that is,

$$P(s = k) = q^{k-1}(1 - q),$$

for $k \geq 1$. Take the $a(j)$ to be geometrically distributed with parameter $p > q$. Again, all of the above conditions are satisfied with

$$Ev(0,1) = \frac{p}{p - q}$$

(see, for example, [19, §4.3]) and we obtain the known result [10, 4] that

$$l_1(x) = \frac{x + 2\sqrt{qx} + 1}{1 - q}.$$

3 Variational formula for the average queue-length in a tandem queueing system

In this section we use an analogue of the identity (2) for the *total occupancy* in a series of tandem queues, operating in discrete time, to compute ϕ_2 in special cases. The first of these is the case where the $s(i,j)$ are Bernoulli, which had previously been solved by Seppalainen [15] using methods which are related but less direct. The other cases we consider are apparently new, namely the cases of geometric and exponential distributions; here we exploit a recent observation due to Bedekar and Azizoglu [3].

We consider queues which operate as follows. At time k, $a(k)$ customers arrive at the queue and at most $s(k)$ customers depart; if $q(k)$ denotes the number of customers in the queue at time k, then q satisfies the Lindley recursion:

$$q(k) = \max\{q(k-1) + a(k) - s(k), 0\}. \tag{5}$$

The number of departures from the queue at time k is given by

$$d(k) = a(k) + q(k-1) - q(k). \tag{6}$$

Note that this queueing system is well defined even if the a's and s's are not integers. As in the previous section we consider a series of n such queues, with departures from one queue immediately entering the next queue in the series. Let $a(k,j)$ denote the number of arrivals at queue j at time k, $s(k,j)$ denote the service available, $q(k,j)$ denote the number of customers in the queue and $d(k,j)$ denote the number of departures; then

$$q(k,j) = \max\{q(k-1,j) + a(k,j) - s(k,j), 0\}, \tag{7}$$

and

$$a(k,j) = d(k,j-1) = a(k,j-1) + q(k-1,j-1) - q(k,j-1). \tag{8}$$

If the $s(i,j)$ are iid, and the $a(k,1)$ are iid with $Ea(0,1) < Es(0,1)$, then we can set

$$q(0,j) = \sup_{k\geq 0} \sum_{i=-k+1}^{0} [a(i,j) - s(i,j)] \tag{9}$$

and this represents a stable equilibrium. (By convention we take the empty sum to be zero.) We can now combine (7), (8) and (9) recursively to obtain:

$$\sum_{j=1}^{n} q(0,j) = \sup_{k\geq 0} \left\{ \sum_{i=-k+1}^{0} a(i,1) - F_2((-k+1,1),(0,n)) \right\}. \tag{10}$$

As far as I am aware, this formula first appeared in [8]. As before, suppose that for each $\nu < \mu = Es(0,1)$ we can choose the distribution of $a(0,1)$ such that $Ea(0,1) = \nu$ and the $q(0,j)$ are iid with $h(\nu) = Eq(0,j) < \infty$. Then (under suitable moment conditions which can be identified by mimicking the arguments set out in [2], for example) we can scale the identity (10) and pass to the limit:

$$h(\nu) = \sup_{x \geq 0}\{\nu x - f_2(x)\}. \tag{11}$$

Since this holds for each $\nu < \mu$ we can take the Legendre transform to get

$$f_2(x) = \sup_{0<\nu<\mu}\{\nu x - h(\nu)\}. \tag{12}$$

Let's ignore technical issues and apply the formula (12) to some examples.

Example 1. Consider the case where the $s(i,j)$ are Bernoulli with parameter q and the $a(i,1)$ are Bernoulli with parameter $p < q$. Then, for each j, the sequence $\{a(i,j), j \in \mathbb{Z}\}$ is iid Bernoulli with parameter p and the $q(0,j)$ are iid geometric random variables with mean $h(p) = p(1-q)/(q-p)$ (see, for example, [19]). Applying the formula (12) we obtain Seppalainen's result that

$$f_2(x) = \left(\max\left\{\sqrt{qx} - \sqrt{(1-q)}, 0\right\}\right)^2.$$

Note that we can immediately deduce from this (by switching 1's and 0's) that for $s(i,j)$ Bernoulli with parameter $r = 1-q$,

$$l_2(x) = \begin{cases} \left(\sqrt{rx} + \sqrt{(1-r)}\right)^2 - 1 & \text{for } x > r/(1-r) \\ x & \text{otherwise.} \end{cases}$$

Example 2. Now suppose that the $s(i,j)$ are iid geometric random variables with parameter b, that is $P(s = k) = b^k(1-b)$ for $k \geq 0$. It was recently shown by Bedekar and Azizoglu [3] that if the $a(k,1)$ are iid geometric with parameter $a < b$ then (in equilibrium) the departures are also iid geometric with parameter a, and the $q(0,j)$ are iid with distribution

$$P(q=l) = \begin{cases} \frac{b-a}{b(1-a)} & l=0 \\ \frac{a(1-b)}{b(1-a)}(1-a/b)(a/b)^{l-1} & l \geq 1 \end{cases}$$

Thus, setting $\nu = a/(1-a)$ and $\mu = b/(1-b)$, we have

$$h(\nu) = Eq(0,1) = \frac{\nu(1+\nu)}{\mu-\nu},$$

and we can apply the formula (12) to obtain

$$f_2(x) = \mu\left(\sqrt{1+x} - \sqrt{1+1/\mu}\right)^2$$

for $x \geq 1/\mu$ and zero otherwise.

Example 3. We treat the exponential case as a continuum version of the geometric case. It follows easily from the result of Bedekar and Azizoglu [3] that for iid exponential $s(i,j)$ with mean β and iid exponential $a(k,1)$ with mean $\alpha < \beta$, the departures are iid exponentials with mean α the $q(0,j)$ are iid exponentials with mean $\alpha^2/(\beta-\alpha)$. Applying (12) in this case we obtain

$$f_2(x) = \beta\left(\sqrt{1+x} - 1\right)^2.$$

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