Burgers turbulence and dynamical systems

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1. Introduction

In this paper we discuss recent progress in the understanding of Burgers turbulence. Significant part of this development is connected with applications of the theory of dynamical systems. Certainly dynamical systems were used in hydrodynamics for a very long time. However, the main features of the approach which we describe below are connected with the fact that dynamical systems methods are used not only to construct stationary distributions for solutions of hydrodynamic equations, but also to obtain mathematical results on structure of singularities for typical stationary solutions. The resulting geometrical picture leads to important quantitative predictions for universal scaling exponents related to the pdf (probability distribution function) for the velocity gradients. It also allows to analyse analytic properties of the structure functions. In this short article it is not possible to comment on all the important aspects of mathematical hydrodynamics. We discuss here only problems which are related to turbulence driven by a random force. The proofs of the results which we discuss here will be published in [22]. Results in the one-dimensional case are based on joint papers of one of us with Weinan E, Alex Mazel and Yakov Sinai (see [9], [10]).

Most of the results which we discuss below are obtained for Burgers equation:

$$\partial_t u + (u \cdot \nabla)u = \nu \Delta u + f(y,t) \tag{1}$$

where $u(y,t) = (u_i(y,t), 1 \le i \le d)$ is a velocity field, $y = (y_i, 1 \le i \le d) \in \mathbb{R}^d$, and $f(y,t) = (f_i(t,y), 1 \le i \le d)$ is an external force. However, for general discussion we shall also refer to the Navier-Stokes equation:

$$\partial_t u + (u \cdot \nabla)u = \nu \Delta u - \nabla p + f(y,t), \quad \text{div } u = 0. \tag{2}$$

A general setting involves one of the hydrodynamics equations listed above driven by random force. The forcing term is responsible for the constant pumping of energy into the system. The energy also dissipates through different dissipation mechanisms. The balance between pumping and dissipation should in principle lead to the establishment of a stationary probability distribution for solutions. This is a general philosophy and the first task of a mathematical theory of turbulence consists in establishing of the existence and, preferably, uniqueness of this stationary distribution, or, in dynamical systems terminology, the invariant measure. The difficulty of the problem depends on the mechanism of dissipation involved. This mechanism obviously depends on the type of equation and the dimension d. In general the difficulty increases when one passes from Burgers equation to Navier-Stokes equation, from viscous case $\nu > 0$ to inviscid limit $\nu \to 0$, and from dimension $d = 1$ (for Burgers equation) or $d = 2$ (for Navier-Stokes) equation to higher dimensions.

It is important to mention, that the construction of the stationary distribution or the invariant measure corresponds to the limit $t \to \infty$. At the same time, the inviscid case, which is mostly interesting for turbulence theory, requires another limiting process $\nu \to 0$. It is tempting, first, to construct an invariant measure μ_ν for $\nu > 0$, that is first to take limit as $t \to \infty$ and then study limit of μ_ν as $\nu \to 0$. However, it is extremely difficult to control the last limiting process. Experience of the random Burgers equation, for which invariant measure can be constructed for both viscous and inviscid cases, suggests that it is more productive to consider from the very beginning $\nu = 0$, construct unique invariant measure μ, and only later prove that $\mu_\nu \to \mu$ as $\nu \to 0$.

As we have mentioned above, a mathematical theory starts with establishing the existence and uniqueness of an invariant measure. However, from the physical point of view the most interesting problem is the analysis of statistical properties of stationary solutions. In other words, how a typical solution looks like? What are the leading singularities? How the power spectrum decays? What are the asymptotic properties of pdf's and the structure functions? Those questions are much harder to answer. They form the core of the problem of turbulence. It is probably fair to say, that the problem of turbulence splits into "soft part" of existence-uniqueness statements and a "hard part" of analysing properties of typical solutions. In the rest of the paper we discuss only Burgers equation, where on can complete the "soft part" in a very general situation, and also answer many of the "hard part" questions (especially in the one-dimensional case). On the contrary, despite of recent progress (see [23],[14],[5]), even situation with two-dimensional Navier-Stokes equation remains largely open. In the most important case of finite number of modes being excited by the random force one can prove uniqueness of the stationary distribution only for large enough viscosity ν. Correspondingly, in the case of Navier-Stokes equation there are no mathematical results on the asymptotic behaviour in the limit $\nu \to 0$.

2. Random setting

To make the general discussion more precise we have to specify assumptions on the random force $f(y,t)$. The most natural, from the turbulence theory point of view, is a setting where $f(y,t)$ is a smooth function in space variable y and quite irregular in time t. This brings us to a popular model, where

$$f(y,t) = -\nabla F(y,t), \;\; F(y,t) = \sum_{k=1}^{N} F_k(y) B_k(t). \tag{3}$$

Here $F_k(y)$ are smooth potentials and $B_k(t) = \dot{W}_k(t)$ are independent white noises, corresponding to independent Wiener processes $W_k(t)$. Summation in (3) can be finite or infinite. In the latter case one has to make sure that potentials $F_k(y)$ decay fast enough so that the series converges and defines a smooth forcing term $f(y,t)$. Under assumption (3) the hydrodynamics equations (1), (2) become stochastic PDE's. However, one can essentially eliminate stochastic part by considering another popular model for a random force, namely random kicking force. Let

$$f(y,t) = -\nabla F(y,t), \;\; F(y,t) = \sum_{j \in \mathbb{Z}} F^j(y) \delta(t - t_j). \tag{4}$$

The force in (4) corresponds to the situation when the system evolves without any force between moments of kicks t_{j-1} and t_j, so that $u(y, t_j-)$ is obtained from $u(y, t_{j-1}+)$ as a result of free evolution of a system. Then, suddenly, we apply a kicking force which change solution discontinuously, i.e.

$$u(y, t_j+) = u(y, t_j-) - \nabla F^j(y). \tag{5}$$

Smooth potentials $\{F^j(y), j \in \mathbb{Z}\}$ form a realization of some stationary potential-valued random process. The simplest situation corresponds to Bernoulli process, when different F^j are chosen independently according to some probability distribution $\chi \in \mathcal{P}(C^\infty(T^d))$ in a space of smooth potentials. Here and below we denote by $\mathcal{P}(M)$ the set of all probability distributions on the measure space M. One can also make different assumptions on kicking times t_j. We shall just indicate two natural cases. The first one corresponds to a periodic sequence $t_j = \bar{t}j, \; j \in \mathbb{Z}$. As we shall see later, this assumption in the case of Burgers equation leads to a problem, which is closely related to random Aubry-Mather theory. In the second situation kicking times are determined by the realization of a stationary random process which is independent from the random process $F^j(y)$. For example, $\{t_j, j \in \mathbb{Z}\}$ can be a realization of Poisson process. In this paper we consider only the simplest case when kicking potentials form Bernoulli sequence, and kicking times are periodic with $\bar{t} = 1$.

In both "white" and "kicked" settings the force is given by a stationary random process. Denote by $(\Omega, \mathcal{B}, P)$ the probability space corresponding to this random process. Sometimes we shall use the notation $f^\omega(y,t), F^\omega(y,t), F^{j,\omega}, \; \omega \in \Omega$, for the forcing term and the potentials, indicating their randomness. The time shift

$f(y,t) \to f(y,t+\tau)$ corresponds to the flow of automorphisms of the probability space Ω :

$$\theta^\tau : \Omega \to \Omega,\ \theta^\tau P = P,\ F^\omega(y,t+\tau) = F^{\theta^\tau \omega}(y,t),\ \tau \in \mathbb{R}, \tag{6}$$

Equality $\theta^\tau P = P$ in (6) expresses the stationarity of the random force. Obviously, in the case of periodic kicking times one should consider only the values $\tau = j, j \in \mathbb{Z}$ in (6). We can define now a standard skew-product structure which leads to the definition of invariant measure. Consider the Cauchy problem for one of the equations (1), (2). Denote by $\mathcal{U}$ some natural functional space, which is invariant under the evolution given by (1), (2), or an invariant set in such space. This means that $u(y,t) \in \mathcal{U}$ for all $t > 0$, provided $u(y,0) \in \mathcal{U}$. Denote by $S^\omega(\tau)$ a nonlinear random transformation on $\mathcal{U}$ corresponding to the solution of the Cauchy problem on the time interval $[0,\tau]$ with the random force $f^\omega(y,t)$. In other words, $u(y,\tau) = S^\omega(\tau)u(y,0)$. Clearly, solution of the equation for longer times implies iteration of $S^\omega(\tau)$ for shifted ω. It is easy to see that this process corresponds to non-random transformation $\hat{S}$ acting on the product space $\mathcal{U} \times \Omega$:

$$\hat{S}(\tau)(u,\omega) = (S^\omega(\tau)u, \theta^\tau \omega). \tag{7}$$

Definition 2.1. *A probability measure $\nu \in \mathcal{P}(\mathcal{U} \times \Omega)$ is called an invariant measure for the random equation if*

1. For any τ, $\hat{S}(\tau)$ preserves ν : $\hat{S}(\tau)\nu = \nu$.

2. Marginal distribution of ν on Ω is given by P, i.e.

$$\nu(du, d\omega) = \nu^\omega(du)P(d\omega),$$

where $\nu^\omega(du)$ are the conditional distributions on $\mathcal{U}$ under the condition of fixed ω.

3. The conditional distributions ν^ω are measurable with respect to the σ - algebra $\mathcal{B}^0_{-\infty}$ generated by the random process on the time interval $(-\infty, 0]$.

The last condition means that ν^ω does not depend on the future distribution of a random force $f^\omega(y,t), t > 0$. It may look a bit artificial but one may say that all physically relevant invariant measures always satisfy this condition.

We shall describe now a slightly different point of view which also leads to the notion of a stationary distribution for solutions of the random equation. It is easy to see that $S^\omega(\tau)$ defines a stationary Markov process on $\mathcal{U}$ with transition probabilities given by:

$$p_\tau(A \mid u_0) = P(\omega : u(y,\tau) = S^\omega(\tau)u_0(y) \in\ A). \tag{8}$$

Definition 2.2. *A probability measure $\mu \in \mathcal{P}(\mathcal{U})$ is called a stationary distribution for the random equation if it is a stationary distribution for the Markov process (8), i.e.*

$$\mu(A) = \int_{\mathcal{U}} p_\tau(A \mid u_o)\mu(du_0),\ \tau > 0.$$

The notions of invariant measure and stationary distribution for a random equation are closely connected. Namely, if ν is an invariant measure then its marginal distribution $\mu(du) = \int_\Omega \nu^\omega(du)P(d\omega)$ is a stationary distribution for the corresponding Markov process. Notice that this property holds only if an invariant measure ν satisfies condition 3 in Definition 2.1.

3. Random Hamilton-Jacobi equation and minimizers

Our main results which we shall formulate in the next section are proven for the Burgers equation on a compact connected Riemannian manifold. For simplicity we consider here only the case of the torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$ which corresponds to periodic potentials F_k in (3), or F^j in (4). Everywhere below we assume that potentials $F_k,\ F^j$ in (3), (4) belong to $C^\infty(\mathbb{R}^d)$ and are $\mathbb{Z}^d-$periodic functions:

$$F_k(y+m) = F_k(y),\ F^j(y+m) = F^j(y), m \in \mathbb{Z}^d, y \in \mathbb{R}^d. \tag{9}$$

We shall denote points of T^d by x and points of the universal cover $\mathbb{R}^d$ by y.

It is well known that due to the creation of shocks the inviscid Burgers equation has no strong, or, in other words, smooth solutions. However, there exists a unique "physical" weak solution for the Cauchy problem, which is called the "viscosity" solution. Given initial data $u(x,0) = u_0(x)$ one can construct the "viscosity" solutions by considering the viscous Burgers equation and taking the limit as $\nu \to 0$. We shall always assume that u_0 is a gradient-like function, i.e. $u_0 = \nabla\phi_0$, where ϕ_0 is a Lipschitz function. It is easy to see that the Burgers equation with a potential force is an evolution equation in a space of gradient-like functions. That means that the set of gradient-like functions is an invariant set. Taking $u(x,t) = \nabla\phi(x,t)$ we obtain the Hamilton-Jacobi equation for ϕ:

$$\partial_t\phi(y,t) + \frac{1}{2}(\nabla\phi(y,t))^2 + F^\omega(y,t) = 0. \tag{10}$$

As in the case of the Burgers equation there are many weak solutions to the Hamilton-Jacobi equation. However, there exists a unique "viscosity" solution corresponding to the Burgers viscosity solution. The exact expression for the viscosity solution is given by the Hopf-Lax-Oleinik variational principle. That is, the "viscosity" solution of the Cauchy problem for (10) in $y \in \mathbb{R}^d, t \in [0,T]$ with the initial condition $\phi(\cdot,0) = \phi_0$ is given by:

$$\phi(y,t) = \inf\left[\phi_0(\gamma(0)) + \int_0^t \left(\frac{1}{2}\dot\gamma^2 - F^\omega(\gamma(\tau),\tau)\right) d\tau\right],\ t \in [0,T], \tag{11}$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0,t] \to \mathbb{R}^d$ such that $\gamma(t) = y$ ([21, 24, 31]). Notice, that if potentials F^ω are $\mathbb{Z}^d$-periodic functions, and $\phi_0(y) = b\cdot y + \psi_0(y)$, where $b \in \mathbb{R}^d$ and $\psi_0(y)$ is $\mathbb{Z}^d$-periodic, then the solution (11) has the same form. Namely,

$$\phi(y,t) = b\cdot y + \psi(y,t), t > 0 \tag{12}$$

where $\psi(y,t)$ is a $\mathbb{Z}^d$-periodic function of y. The linear form $b\cdot y$ can be considered as a first integral of the Hamilton-Jacobi equation. For the Burgers equation b corresponds to the average velocity

$$b=\int_{T^d} u(y,t)dx \ ,$$

which is well known to be a first integral. From now one we shall assume that the value of the first integral b is fixed.

Using formula (11) we get the variational principle for $\psi(x,t)$, which we consider now as a function on the torus T^d. For fixed b denote the *action* of a curve $\gamma:[s,t]\to T^d$ by

$$A^{\omega,b}_{s,t}(\gamma)=\int_s^t\left(\frac{1}{2}(\dot\gamma(\tau)-b)^2-F^{\omega}(\gamma(\tau),\tau)-\frac{b^2}{2}\right)d\tau. \tag{13}$$

Then,

$$\psi(x,t)=\inf_{\gamma\in AC(0;x,t)}(\psi_0(\gamma(0))+A^{\omega,b}_{0,t}(\gamma)), \tag{14}$$

where $AC(s;x,t)$ is the set of all absolutely continuous curves $\gamma:[s,t]\to T^d$ on the torus T^d such that $\gamma(t)=x$. We shall also use the following notations. Denote $AC(x_1,s;x_2,t)$ the set of absolutely continuous curves $\gamma:[s,t]\to T^d$ such that $\gamma(s)=x_1$ and $\gamma(t)=x_2$; $AC(x,t)$ the set of absolutely continuous curves $\gamma:(-\infty,t]\to T^d$ such that $\gamma(t)=x$; AC the set of all absolutely continuous curves $\gamma:(-\infty,+\infty)\to T^d$. Finally, we define the Lax operator $\mathcal{L}^{\omega,b}_{s,t}$:

$$\mathcal{L}^{\omega,b}_{s,t}\psi(x)=\inf_{\gamma\in AC(s;x,t)}\psi(\gamma(s))+A^{\omega,b}_{s,t}(\gamma). \tag{15}$$

Obviously, a function $\phi(y,t),\ y\in T^d,\ t\in[t_1,t_2]$ is a "viscosity" solution of the Hamilton - Jacobi equation if $\phi(y,t)=b\cdot y+\psi(y,t)$, where ψ is $\mathbb{Z}^d$-periodic and for all $t_1\le s<t\le t_2$:

$$\mathcal{L}^{\omega,b}_{s,t}\psi(\cdot,s)=\psi(\cdot,t).$$

The variational principle (11, 14) can be easily extended to the "kicked" case. Let $u(y,n\pm)=\nabla\phi(y,n\pm),\phi(y,n\pm)=b\cdot y+\psi(y,n\pm),\psi(y,n+)=\psi(y,n-)-F^{n,\omega}(y)$. Denote by $\mathcal{A}^{\omega,b}_{m,n}(\mathbf{X})$ the "kicked" action of the sequence $\mathbf{X}=\{x_i\in T^d,m\le i\le n\}$:

$$\mathcal{A}^{\omega,b}_{m,n}(\mathbf{X})=\sum_{i=m}^{n-1}\left(\frac{1}{2}\rho_b^2(x_{i+1},x_i)-F^{i,\omega}(x_i)-\frac{b^2}{2}\right), \tag{16}$$

where $\rho_b(x,x')=\min_{k\in\mathbb{Z}^d}(\|x-x'-b-k\|)$. Then,

$$\psi(x,n)=\inf_{\mathbf{X}\in S(0;x,n)}\left[\psi(x_0,0-)+\mathcal{A}^{\omega,b}_{m,n}(\mathbf{X})\right], \tag{17}$$

where $S(m;x,n)$ is the set of sequences $\mathbf{X}=\{x_i\in T^d,m\le i\le n\}$ such that $x_n=x$.

In analogy with the continuous case we shall also use the following notations: $S(x',m;x,n)$ is the set of all sequences $\{x_i\in T^d,m\le i\le n\}$ such that $x_m=$

$x', x_n = x$; $S(x,n)$ is the set of infinite sequences $\{x_i \in T^d, -\infty < i \le n\}$ such that $x_n = x$; S is the set of all double-infinite sequences $\{x_i \in T^d, -\infty < i - \infty\}$.

We next define an important notion of a minimizer.

Definition 3.1. *1. A curve $\gamma : [s,t] \to T^d$ is called a minimizer if*

$$A^{\omega,b}_{s,t}(\gamma) = \min_{\sigma \in AC(\gamma(s),s;\gamma(t),t)} A^{\omega,b}_{s,t}(\sigma).$$

2. A curve $\gamma : [s,t] \to T^d$ is called a $\psi-$minimizer if

$$\psi(\gamma(s)) + A^{\omega,b}_{s,t}(\gamma) = \min_{\sigma \in AC(s;\gamma(t),t)} \Big(\psi(\sigma(s)) + A^{\omega,b}_{s,t}(\sigma)\Big).$$

3. A curve $\gamma : (-\infty, t_0] \to T^d$ is called a one-sided minimizer if it is a minimizer for all $(s,t), -\infty < s < t \le t_0$.

4. A curve $\gamma : (-\infty, \infty) \to T^d$ is called a global minimizer if it is a minimizer for all $(s,t), -\infty < s < t < \infty$.

Similar definitions apply to the "kicked" case. One has just to replace curves γ by sequences $\mathbf{X}$, the action $A^{\omega,b}_{s,t}$ by the "kicked" action $\mathcal{A}^{\omega,b}_{m,n}$ and pairs (s,t) by the pairs of integer moments of time (m,n).

Clearly, minimizers are special trajectories of the Lagrangian flow $L^\omega_s, s \in \mathbb{R}$ which is generated by solutions of the Euler-Lagrange system of equations:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\nabla F^\omega(x,t). \end{aligned} \tag{18}$$

The Lagrangian flow L^ω_s, is a stochastic flow of diffeomorphisms of $T^d \times \mathbb{R}^d$:

$$L^\omega_s : (x_0, v_0) \mapsto (x_s, v_s), \;\; s \in \mathbb{R}. \tag{19}$$

We shall also define the skew-product extension $\hat{L}_s$ of the Lagrangian flow. Namely, we consider the flow of non-random transformations of $T^d \times \mathbb{R}^d \times \Omega$ given by:

$$\hat{L}_s : (x_0, v_0, \omega) \mapsto (L^\omega_s(x_0, v_0), \theta^s\omega) = (x_s, v_s, \theta^s\omega), \;\; s \in \mathbb{R}. \tag{20}$$

In the "kicked" case the Lagrangian dynamics is given by a random Standard-like map (instead of Euler-Lagrange equation):

$$\begin{aligned} x_{n+1} &= x_n + v_n - \nabla F^{n,\omega}(x_n) \\ v_{n+1} &= v_n - \nabla F^{n,\omega}(x_n). \end{aligned} \tag{21}$$

Then the Lagrangian flow L^ω_n and its skew-product extension is defined exactly in the same way as above

$$L^\omega_n : (x_0, v_0) \mapsto (x_n, v_n), \;\; \hat{L}_n : (x_0, v_0, \omega) \mapsto (x_n, v_n, \theta^n\omega), \;\; n \in \mathbb{Z}. \tag{22}$$

We shall see in the next section that global minimizers correspond to non-trivial invariant measures of $\hat{L}$.

4. Formulation of the main results

Consider the random Hamilton-Jacobi equation (10, 3) in a semi-infinite interval of time $(-\infty, t_0]$. It turns out that for any $b \in \mathbb{R}^d$ there exists a unique (up to a constant) "viscosity" solution. More precisely, the following theorem holds.

Theorem 4.1. *1. For almost all ω and all $b \in \mathbb{R}^d$, $t_0 \in \mathbb{R}$ there exists a unique (up to an aditive constant) function $\phi_b^\omega(y,t)$, $y \in \mathbb{R}^d, t \in (-\infty, t_0]$, such that*

$$\phi_b^\omega(y,t) = b \cdot y + \psi_b^\omega(y,t)$$

where ψ_b^ω is a $\mathbb{Z}^d-$periodic function and for all $-\infty < s < t \leq t_0$

$$\psi_b^\omega(y,t) = \mathcal{L}_{s,t}^\omega \psi_b^\omega(y,s).$$

2. The function ϕ_b^ω is Lipschitz in y. If $x \in T^d$ is a point of differentiability of $\phi_b^\omega(x,t)$ then there exists a unique one-sided minimizer $\gamma_{x,t}^\omega$ at (x,t) and its velocity is given by the gradient of ϕ: $\dot{\gamma}_{x,t}^\omega(t) = \nabla \phi_b^\omega(x,t)$.

3. For almost all ω and arbitrary family of continuous functions $\zeta_s(y) = b \cdot y + \eta_s(y)$, where η_s are $\mathbb{Z}^d-$periodic functions, the following convergence holds:

$$\lim_{s \to -\infty} \mathcal{L}_{s,t}^\omega \eta_s = \psi_b^\omega(x,t) \quad (mod \quad constant).$$

Moreover, if x is a point of differentiability of $\phi_b^\omega(x,t)$ then

$$\lim_{s \to -\infty} \dot{\gamma}_{\eta_s,s;x,t}^\omega(t) = \dot{\gamma}_{x,t}^\omega(t) = \nabla \phi_b^\omega(x,t),$$

where $\gamma_{\eta_s,s;x,t}^\omega$ is a η_s-minimizer.

Similar theorem holds in the "kicked" case. Suppose the probability distribution χ for the kicking potentials $F^{j,\omega}(x)$ satisfy the following assumption.

Assumption 1. *For any $x \in T^d$ there exists a potential F belonging to the support of χ and such that F has a unique global non-degenerate maximum at x.*

Theorem 4.2. *Suppose probability distribution $\chi \in \mathcal{P}(C^\infty(T^d))$ satisfies Assumption 1. Then all the statements of Theorem 4.1 remain true with replacement of t by $n+$ and $n-$, $n \in \mathbb{Z}$.*

The following statement is a simple consequence of Theorems 4.1, 4.2.

Corollary 4.3. *1. Consider the random Burgers equation (1, 3) or (1, 4) in a semi-infinite interval of time $(-\infty, t_0]$. Assume that probability distribution $\chi \in \mathcal{P}(C^\infty(T^d))$ satisfies Assumption 1. Then in both "white" and "kicked" cases, almost surely for all $b \in \mathbb{R}^d$ there exists a unique "viscosity" solution $u_b^\omega(x,t)$ with the average velocity b.*

2. For Lebesgue-almost all x

$$u_b^\omega(x,t) = \nabla \phi_b^\omega(x,t) = \dot{\gamma}_{x,t}^\omega(t),$$

where $\gamma_{x,t}^\omega$ is the unique one-sided minimizer at (x,t).

3. Almost surely for all all $b \in \mathbb{R}^d$, $t \in (\infty, t_0]$ and arbitrary family of gradient-like functions $w_s(x)$, such that $\int_{T^d} w_s(x)dx = b$ the following convergence holds for Lebesgue-almost all $x \in T^d$:

$$\lim_{s \to -\infty} w_s(x,t) = u_b^\omega(x,t),$$

where $w_s(x,t)$ is the solution of the Cauchy problem for the forced Burgers equation with the initial data at time s given by $w_s(x)$.

Corollary 4.3 implies uniqueness of the stationary distribution for the random Burgers equation. Denote by $\mathcal{U}_b = \{u(x) : \ u(x) = b + \nabla\psi(x), \psi \in Lip(T^d)\}$. Obviously, $\mathcal{U}_b$ is an invariant set for the Burgers dynamics. We shall regard $\mathcal{U}_b$ as a subset of $L^p(T^d, dx), p > 0$. This embedding defines a measurable structure on $\mathcal{U}_b$. It is easy to see that a mapping $\omega \mapsto u_b^\omega(x,0)$ (or, $u_b^\omega(x,0+)$ in the "kicked" case) gives a measurable transformation $U : \Omega \to \mathcal{U}_b$. Denote by $\delta_b^\omega(du)$ the atomic measure on $\mathcal{U}_b$ with the atom at $u_b^\omega(x,0)$ (or at the $u_b^\omega(x,0+)$ in the "kicked case"), and define a probability measure ν_b on $\mathcal{P}(\mathcal{U}_b \times \Omega)$ by $\nu_b(du, d\omega) = \delta_b^\omega(du)P(d\omega)$. We shall also consider the marginal distribution of ν_b on $\mathcal{U}_b$. Denote this marginal distribution by $\mu_b : \mu_b(du) = \int_\Omega \delta_b^\omega(du)P(d\omega)$. As in the previous theorem we assume that probability distribution $\chi \in \mathcal{P}(C^\infty(T^d))$ satisfies Assumption 1.

Theorem 4.4. *Consider the evolution given by the random Burgers equation (1, 3), or (1, 4). Then for all $b \in \mathbb{R}^d$ the following statements hold.*

1. The probability measure $\nu_b \in \mathcal{P}(\mathcal{U}_b \times \Omega)$ is the unique invariant measure for the skew-product dynamics (7) on $\mathcal{P}(\mathcal{U}_b \times \Omega)$.

2. The probability measure $\mu_b \in \mathcal{P}(\mathcal{U}_b)$ is the unique stationary distribution for the Markov process (8) on $\mathcal{U}_b$.

We next discuss the properties of minimizers. It follows from Theorem 4.1 that for a fixed time t and for Lebesgue-almost all points x there exists a unique one-sided minimizer. In fact, an easy compactness argument gives the existence of at least one one-sided minimizers for all $x \in T^d$. Points of non-uniqueness are called shock waves, or simply shocks. Every one-sided minimizer γ_{x,t_0} is a trajectory of the Lagrangian flow corresponding to the Euler-Lagrange system (18), or the random Standard map (21). It can be continued as a trajectory of Lagrangian flow for all t. However, its continuation is likely not to be a one-sided minimizer for large enough $t > t_0$. Global minimizers correspond to exactly those one-sided minimizers that can be continued as minimizers for all t. It turns out that in a very general situation there exists a unique global minimizer. We shall first formulate a theorem in the "white" force case.

Theorem 4.5. *Suppose that the potentials $F_k(x), 1 \le k \le N$, satisfy the following property: for some $1 \le l \le N$ a mapping*

$$(F_1(x), ...F_l(x), F_{l+1}(y), ..., F_N(y)) : T^d \times T^d \to \mathbb{R}^N$$

is an embedding. Then with probability 1 there exists only one global minimizer.

In the "kicked" case we shall assume that the kicking potentials $F^{j,\omega}(x)$ are given by linear combination of the smooth non-random potentials $F_i, 1 \le i \le M$ with random coefficients.

Assumption 2.

$$F^{j,\omega}(x) = \sum_{i=1}^{M} \xi_i^j(\omega) F_i(x),\ j \in \mathbb{Z},$$

where the random vectors $\xi^j(\omega) = \{\xi_i^j(\omega), 1 \le i \le M\}$ are independent identically distributed vectors in $\mathbb{R}^M$ with an absolutely continuous distribution.

Theorem 4.6. *Assume that Assumptions 1,2 hold, and in addition the potentials $F_i(x), 1 \le i \le M$, are such that the mapping*

$$(F_1(x), ..., F_M(x)) : T^d \to \mathbb{R}^M$$

is one-to-one. Then with probability 1 there exist only one global minimizer.

Denote by x_g^ω the position of the global minimizer at time $t = 0$. For fixed $t > 0$ consider all one-sided minimizers $\gamma_{x,t}^\omega,\ x \in T^d$. Denote by $A^\omega(t) = \{x_0 = \gamma_{x,t}^\omega(0)\}$ the set of points at which one-sided minimizers cross T^d at $t = 0$. Notice, that $x_g^\omega \in A^\omega(t)$ for all $t \ge 0$. The following statement follows immediately from the uniqueness of the global minimizer.

Corollary 3. *If there exists a unique global minimizer then $diam A^\omega(t) \to 0$ as $t \to \infty$.*

Corollary 3 implies that $\gamma_{x,t}^\omega(0)$ approaches x_g^ω as $t \to \infty$. However, in the multi-dimensional case we cannot control the rate of convergence. At the same time, in the one-dimensional case convergence to the global minimizer is exponentially fast. It follows from the hyperbolicity of the global minimizer. Denote by v_g^ω the velocity of the global minimizer at time $t = 0$. Define a probability measure $\kappa \in \mathcal{P}(T^d \times \mathbb{R}^d \times \Omega)$ by

$$\kappa(dx, dv, d\omega) = \delta_{x_g^\omega, v_g^\omega}(dx, dv) P(d\omega),$$

where $\delta_{x_g^\omega, v_g^\omega}(dx, dv)$ is an atomic measure in $\mathcal{P}(T^d \times \mathbb{R}^d)$ with the atom at (x_g^ω, v_g^ω). The following theorem holds if $d = 1$.

Theorem 4.7. *(see* [9]*) 1. The probability measure κ is a hyperbolic ergodic invariant measure for the skew-product extension of the Lagrangian flow. In particular, it implies that with probability 1 the global minimizer is a hyperbolic trajectory of the Lagrangian flow with non-random positive Lyapunov exponent $\lambda > 0$.*

2. With probability 1 there exists a one-dimensional C^∞-smooth unstable manifold passing through (x_g^ω, v_g^ω), which consists of all (x, v) whose Lagrangian trajectories are asymptotic to the global minimizer as $t \to -\infty$.

3. With probability 1 for all b the function $u_b^\omega(x, 0)$ is piecewise C^∞-smooth with finite number of shocks (jump discontinuities). The graph of each smooth piece of $u_b^\omega(x, 0)$ belongs to the unstable manifold of the global minimizer.

4. All the shocks except one correspond to double-folds of the unstable manifold. The only special shock, which is called the main shock, has a topological nature. It is the only shock which can be traced ever in the past. All other shocks have finite history and have appeared from preshocks at some negative moments of time t.

Remark 4.8. *Part of the statement 1 of theorem 4.7 remains valid in the multi-dimensional case. Namely, it is still true that the probability measure κ is an ergodic invariant measure for the skew-product extension of the Lagrangian flow. However, hyperbolicity of the measure κ is still an open problem.*

The proof of all the theorems (except the last one) will be published in [22]. The results of the last theorem have been proved in [9] for the "white" force case. The proof requires an additional assumption (see nondegeneracy condition (A1) in [9], p.900), which is of the same nature as the condition of Theorem 4.5 above.

5. Concluding remarks.

1. Most of the results which we have formulated in the previous section are related to the multi-dimensional Burgers equation driven by a random force. Similar results in the one-dimensional case have been proven in [9]. However, the methods in [9] were purely one-dimensional. The new approach which we apply here is more general and straightforward. Advantage of the method is connected with a systematic use of Lagrangian formalism and the Hamilton-Jacobi equation. This allows us to prove results on the uniqueness of stationary distribution in any dimension. Our method also gives a proof of the uniqueness of global minimizer, which is a fundamental property of random Lagrangian systems. In fact, one can consider the work presented here as a study of minimizing orbits of random Lagrangian systems. This problem is quite interesting on its own without any connection with the hydrodynamics and the Burgers equation. The theory of minimizing orbits for Lagrangian systems with autonomous or time-periodic potentials has a long history. It goes back to the works of M. Morse and G. Hedlund (see [30], [20])). In the last 15 years this subject became a very active research area in connection with Aubry-Mather theory (see [1, 28, 29, 26, 27, 15, 8]). R. Mañé has conjectured that for a generic time-periodic potential there exists a unique global minimizer which is a hyperbolic periodic trajectory of the Lagrangian system. This conjecture remains open. However, a related result on the uniqueness of a minimizing measure have been proven in [26]. It was also shown in [7] that the invariant measure is hyperbolic provided it is supported by a single periodic orbit. In the random case we can prove not only the uniqueness of a minimizing measure, but also show that this measure is supported by the unique global minimizer. He hope that the uniqueness of the global minimizer will make it possible to prove that it is a hyperbolic trajectory with probability 1. There is an interesting connection between our hyperbolicity conjecture in the random case and Mane's conjecture. Indeed, random

Lagrangian systems are generic with probability 1 in the measure-theoretic sense, while Mane's conjecture is related to topologically generic Lagrangian systems.

2. The case of "kicking" potentials corresponds to a random generalisation of Aubry - Mather theory (see [1, 28]). The Lagrangian in this case can be considered as a random Frenkel-Kontorova type model, which describes an infinite chain of particles placed in periodic potentials and connected by elastic springs. In the random situation the potentials are chosen independently for each particle, contrary to the classical case when one periodic potential serves all of them. In the non-random case the structure of global minimizers is very complicated. It crucially depends on the arithmetic properties of the rotation vector, which in turn depends non-trivially on the vector b in the linear form $b \cdot y$. In the case of fixed Diophantine rotation number ($d = 1$) it is believed that one has transition from elliptic minimizers for small potentials to hyperbolic global minimizers when potentials are large enough. This transition is related to destruction of invariant KAM curves which correspond to elliptic minimizers. All this complicated picture should be compared with the uniqueness of the global minimizer for all b in the random case (see Theorems 4.5, 4.6). In the random one-dimensional case one can also prove (see Theorem 4.7) the hyperbolicity of the global minimizer, while in the classical Aubry-Mather theory the hyperbolicity of minimizers after destruction of KAM curves remains an outstanding open problem.

3. The uniqueness of the "viscosity" solutions of the random Hamilton-Jacobi and Burgers equations for $-\infty < t \leq t_0$ (Theorems 4.1, 4.2, Corollary 4.3) demonstrates the extreme stability of those equations. The uniqueness result can be formulated as the "one force - one solution" principle. We just mention that in the case of the Navier-Stokes equation situation is much more complicated. The "one force - one solution" principle is valid only for large enough values of the viscosity.

4. Finally, we shall discuss the quantitative predictions which we have mentioned at the beginning of the article. One-dimensional random Burgers equation with large scale forcing have been discussed very intensively in the physical literature in the last 5 years (see [11, 32, 6, 19, 2, 18, 3, 4, 17]). The methods used vary from numerical simulations to quantum field theoretic closure techniques. It was shown that the pdf g(z) for the velocity gradient $z = \frac{du^\omega}{dx}$ decays super exponentially for large positive values of z, and algebraically for negative values, i.e. $g(z) \sim \frac{C}{|z|^\alpha}$ as $z \to -\infty$. The critical exponent α is universal. It means that α is independent of the precise formula for a driven force. In ([32], [2]) it was predicted that α is equal to $\frac{5}{2}$ or 3. However, as it was shown in [10]) the geometrical picture formulated above (see Theorem 4.7) suggests that the main contribution to $g(z)$ for large negative values of z comes from small neighbourhoods of the preshock events. Then an easy calculation implies that $\alpha = \frac{7}{2}$ (see also [12, 13]. This prediction, although not rigorous, seems to be more and more accepted by the physical community. The prediction for α is based on the hyperbolic geometrical picture which implies that the number of shocks is finite (rather than infinite with shocks

being everywhere dense on $S^1 = \mathbb{R}/\mathbb{Z}$). This suggests that the preshocks events form a set of isolated points on the (x,t) space, and, hence, leads to the predicted value $\alpha = \frac{7}{2}$. We hope that in the multidimensional case one will be able to get nontrivial quantitative results, provided that the hyperbolicity is established.

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