The Non-local Content of Quantum Operations

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We show that quantum operations on multi-particle systems have a non-local content; this mirrors the non-local content of quantum states. We introduce a general framework for discussing the non-local content of quantum operations, and give a number of examples. Quantitative relations between quantum actions and the entanglement and classical communication resources needed to implement these actions are also described. We also show how entanglement can catalyse classical communication from a quantum action.

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We show that quantum operations on multi-particle systems have a non-local content; this mirrors the non-local content of quantum states. We introduce a general framework for discussing the non-local content of quantum operations, and give a number of examples. Quantitative relations between quantum actions and the entanglement and classical communication resources needed to implement these actions are also described. We also show how entanglement can catalyse classical communication from a quantum action.

I. INTRODUCTION

In the past, most of the research on quantum non-locality has been devoted to the issue of non-locality of *quantum states*. However we feel that an equally important issue is that of non-locality of *quantum evolutions*. That is, in parallel with the understanding of non-locality of quantum kinematics one should also develop an understanding of the non-locality of quantum dynamics.

Let us start with a simple example. Consider two qubits situated far from each other, one held by Alice and the other one by Bob. Suppose they would like to implement a two qubit quantum evolution described by the unitary operator U. (We wish to be able to apply U on *any* initial state of the two qubits). With exception of the case when U is a product of two local unitary operators, $U = U_A \otimes U_B$, no other quantum evolution can be accomplished by local means only. Thus almost all quantum evolutions are non-local. The main question we address in this paper is how to describe, qualitatively and quantitatively, the non-locality of quantum evolutions.

In order to be able to describe the amount of non-locality contained by the unitary operator U we suggest the following approach. We consider that Alice and Bob, in addition of being able to perform any local operations, they also have *additional resources*, namely they share entangled states, and they are able to communicate classically. The question then reduces to finding out how much of these resources is needed to implement U.

The above framework has also been put-forward by Chefles, Gilson and Barnett [1].

We emphasise that although we have largely discussed the role of quantum entanglement above, the role of the classical communication is equally important. Understanding the character of a quantum evolution requires knowing both the amount of entanglement and the amount of classical communication needed.

II. GENERAL SUFFICIENCY CONDITIONS

First of all, it is important to note that *any* unitary evolution can be implemented given enough shared entanglement and classical communication. Indeed, consider the case of two qubits, one held by Alice and one by Bob. Any unitary transformation U on these two qubits can be accomplished by having Alice teleport her qubit to Bob, Bob performing U locally and finally Bob teleporting Alice's qubit back to Alice. The resources needed for the two teleportation actions are: (1 e-bit plus two classical bits transmitted from Alice to Bob for the Alice to Bob teleportation) plus (1 e-bit plus two classical bits transmitted from Bob to Alice for the Bob to Alice teleportation). It is obvious now that any unitary operation involving any number of parties and any number of qubits can be accomplished by a similar procedure (teleporting all states to a single location, performing U locally and teleporting back the qubits to their original locations).

The "double teleportation" procedure shown above is *sufficient* to implement any quantum evolution. The question is however whether so much resources are actually *needed*. We will discuss a couple of specific example below.

III. THE SWAP OPERATION ON TWO QUBITS

The SWAP operation defined by:

$$U_{\mathrm{SWAP}}|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle \tag{1}$$

is a particularly intriguing case, since although it takes product states to product states, it is, as we now show, the most non-local operation possible in the sense described above. That is, we will prove that in order to implement a SWAP on two qubits it is not only sufficient but also *necessary* to use 2 e-bits plus 2 bits of classical communication from Alice to Bob plus 2 bits of classical communication from Bob to Alice.

Proof: To prove that the SWAP operation *needs* as non-local resources 2 e-bits, we will show that if we have an apparatus able to implement the SWAP operation we can use it in order to create 2 e-bits. Thus, since entanglement cannot be created *ex nihilo*, the apparatus which implements the SWAP *must* use 2 e-bits as an internal non-local resource.

Let us show how to generate two singlets using the SWAP operation. Firstly Alice and Bob prepare singlets locally

$$\uparrow_A\uparrow_a + \downarrow_A\downarrow_a \quad \text{and} \quad \uparrow_B\uparrow_b + \downarrow_B\downarrow_b, \tag{2}$$

Alice's spins are labelled A and a and Bob's B and b (here and in what follows we will leave out normalisation factors for states). Now perform the SWAP operation on spins A and B:

$$(\uparrow_A\uparrow_a + \downarrow_A\downarrow_a)\,(\uparrow_B\uparrow_b + \downarrow_B\downarrow_b) \mapsto$$
$$(\uparrow_B\uparrow_a + \downarrow_B\downarrow_a)\,(\uparrow_A\uparrow_b + \downarrow_A\downarrow_b). \tag{3}$$

This state contains two singlets held between Alice and Bob.

To find the classical communication resources *needed* to implement the SWAP operation we will adapt an argument first given in [2]. We show that if we have an apparatus able to implement the SWAP operation we can use it in order to communicate 2 bits from Alice to Bob plus 2 bits from Bob to Alice. From this follows that it must be the case that the SWAP apparatus uses 2 bits of classical communication from Alice to Bob plus 2 bits of classical communication from Bob to Alice as an internal resource, otherwise Alice could receive information from Bob transmitted faster than light.

For suppose that the SWAP operation requires less than four bits of classical communication (two bits each way). Alice and Bob can produce an instantaneous SWAP operation which works correctly with probability greater than one sixteenth in the following way. Alice and Bob run the usual SWAP protocol, but instead of waiting for classical communication from each other, they simply guess the bits that they would have received. Since we have assumed that the SWAP operation requires less than 4 bits, the probability that Alice and Bob guess correctly is greater than one sixteenth and hence the SWAP operation also succeeds with probability greater than one sixteenth.

Thus using the protocol described previously can now use this imperfect, but instantaneous SWAP to communicate 4 bits instantaneously. The bits arrive correctly when the SWAP is implemented correctly. Hence the probability that 4 bits arrive correctly is larger than one sixteenth; 4 bits communicated correctly with probability greater than one sixteenth represents a non-zero amount of information. Thus Alice and Bob have managed to convey some information to each other instantaneously. We conclude therefore that the SWAP operation cannot be done with less that 4 bits of classical communication; otherwise it allows communication faster than the speed of light.

Earlier in this section we showed that the SWAP operation can be used to generate two singlets. We now show that the SWAP operation can be also be used to perform four bits of classical communication (two bits each way): the main idea is that of "super-dense coding" [3]. Suppose that initially Alice and Bob share two singlets:

$$\uparrow_A\uparrow_B + \downarrow_A\downarrow_B \quad \text{and} \quad \uparrow_a\uparrow_b + \downarrow_a\downarrow_b\,. \tag{4}$$

Now Alice chooses one of four local unitary operations 1 (identity), σ_x, σ_y, σ_z and performs it on her spin A. This causes the first singlet to be in one of the four Bell states. Bob also, independently chooses one of these four locally unitaries and performs it on his spin b, putting the second singlet into one of the Bell states. Then the SWAP operation is performed on spins A and b. Now both Bob and Alice have one of the Bell states locally; which one they have depends on which operation the other performed. By measurement, they can work out which of the four unitaries the other performed. Thus the SWAP operation has enabled two bits of classical communication to be performed each way.

IV. THE CNOT OPERATION ON TWO QUBITS

Another important quantum operation is CNOT, defined as

$$\uparrow\uparrow\mapsto\uparrow\uparrow \tag{5}$$

$$\uparrow\downarrow\mapsto\uparrow\downarrow \tag{6}$$

$$\downarrow\uparrow\mapsto\downarrow\downarrow \tag{7}$$

$$\downarrow\downarrow\mapsto\downarrow\uparrow\,. \tag{8}$$

As we prove below, the necessary and sufficient resources for CNOT are 1 e-bit plus 1 bit of classical communication from Alice to Bob plus 1 bit of classical communication from Bob to Alice.

Proof: Constructing a CNOT We now show how to construct the CNOT operation using one singlet and two bits of classical communication. We then show how to generate one singlet or perform two bits of classical communication using the CNOT.

Firstly we will show how, using one singlet and one bit of classical communication each way, we can perform a CNOT on the state

$$(\alpha \uparrow_A + \beta \downarrow_A)\,(\gamma \uparrow_B + \delta \downarrow_B) \tag{9}$$

i.e. transform it to

$$\alpha \uparrow_A (\gamma \uparrow_B + \delta \downarrow_B) + \beta \downarrow_A (\gamma \downarrow_B + \delta \uparrow_B). \tag{10}$$

Since the operation behaves linearly, the protocol performs the CNOT on any input state (i.e. even if the qubits are entangled with each other or with other systems).

Step 1 The first step is to append a singlet held between Alice and Bob to the state (9):

$$(\alpha \uparrow_A + \beta \downarrow_A)\,(\uparrow_a\uparrow_b + \downarrow_a\downarrow_b)\,(\gamma \uparrow_B + \delta \downarrow_B), \tag{11}$$

then for Alice to measure the total spin of her spins A and a.

If the total spin is one, then the state becomes

$$(\alpha \uparrow_A\uparrow_a\uparrow_b + \beta \downarrow_A\downarrow_a\downarrow_b)\,(\gamma \uparrow_B + \delta \downarrow_B). \tag{12}$$

Now Alice disentangles the singlet spin by performing the following (local) operation:

$$\uparrow_A\uparrow_a \mapsto \uparrow_A\uparrow_a; \qquad \downarrow_A\downarrow_a \mapsto \downarrow_A\uparrow_a, \tag{13}$$

and the state becomes

$$(\alpha \uparrow_A\uparrow_b + \beta \downarrow_A\downarrow_b)\,(\gamma \uparrow_B + \delta \downarrow_B) \uparrow_a . \tag{14}$$

If the total spin had been zero, then rather than (12) the state becomes

$$(\alpha \uparrow_A\downarrow_a\downarrow_b + \beta \downarrow_A\uparrow_a\uparrow_b)\,(\gamma \uparrow_B + \delta \downarrow_B). \tag{15}$$

In this case Alice can disentangle the a spin by

$$\uparrow_A\downarrow_a \mapsto \uparrow_A\uparrow_a; \qquad \downarrow_A\uparrow_a \mapsto \downarrow_A\uparrow_a, \tag{16}$$

leading to

$$(\alpha \uparrow_A\downarrow_b + \beta \downarrow_A\uparrow_b)\,(\gamma \uparrow_B + \delta \downarrow_B) \uparrow_a . \tag{17}$$

In order to get this state in the correct form, Bob needs to invert his b spin. Thus Alice must communicate one bit to Bob to tell him whether she found total spin one or zero, and thus whether he needs to invert his spin or not.

After these operations, the state is

$$(\alpha \uparrow_A\uparrow_b + \beta \downarrow_A\downarrow_b)\,(\gamma \uparrow_B + \delta \downarrow_B) \uparrow_a . \tag{18}$$

Step 2 Now Bob performs a CNOT on the b and B spins, thus the total state is

$$[\alpha \uparrow_A\uparrow_b (\gamma \uparrow_B + \delta \downarrow_B) + \beta \downarrow_A\downarrow_b (\gamma \downarrow_B + \delta \uparrow_B)] \uparrow_a . \tag{19}$$

Step 3 Bob now measures σ_x on his part of the singlet b. Either the state becomes

$$[\alpha \uparrow_A (\gamma \uparrow_B + \delta \downarrow_B) + \beta \downarrow_A (\gamma \downarrow_B + \delta \uparrow_B)] \otimes \uparrow_a \; (\uparrow_b + \downarrow_b), \tag{20}$$

or

$$[\alpha \uparrow_A (\gamma \uparrow_B + \delta \downarrow_B) - \beta \downarrow_A (\gamma \downarrow_B + \delta \uparrow_B)] \otimes \; \uparrow_a (\uparrow_b - \downarrow_b), \tag{21}$$

In the former case (i.e. the x component of spin was $+$) we have performed the protocol as desired. In the latter, Alice needs to perform a σ_z rotation by π. Thus Bob needs to communicate one bit to Alice to tell her whether or not to perform the rotation.

We have thus shown how to perform a CNOT using one singlet and one bit of classical communication each way.

Creating entanglement by CNOT We show now that a CNOT apparatus can be used to create 1 e-bit between Alice and Bob; thus (since entanglement cannot be increased by local operations) 1 e-bit is a necessary resource for constructing a CNOT.

Creating 1 e-bit by a CNOT is straightforward:

$$(\uparrow_A + \downarrow_A) \uparrow_B \mapsto \uparrow_A\uparrow_B + \downarrow_A\downarrow_B . \tag{22}$$

Classical communication by CNOT

Suppose that Alice and Bob have an apparatus which implements a CNOT and also share 1 e-bit. They can use these resources to communicate *at the same time* 1 classical bit from Alice to Bob and 1 classical bit from Bob to Alice. This proves (see preceding section) that communicating 1 classical bit each way is a necessary resource for constructing a CNOT.

Suppose the initial state is

$$\uparrow_a\uparrow_b + \downarrow_a\downarrow_b . \tag{23}$$

Alice can encode a "0" by not doing anything to the state and a "1" by flipping her qubit. Bob can encode a "0" by not doing anything to the state and a "1" by changing the phase as follows: $\uparrow \to \uparrow$ and $\downarrow \to -\downarrow$.

The four states corresponding to the different bit combinations are thus

$$\uparrow_a\uparrow_b + \downarrow_a\downarrow_b \quad \textit{corresponds to} \;\; 0_A 0_B. \tag{24}$$

$$\downarrow_a\uparrow_b + \uparrow_a\downarrow_b \quad \textit{corresponds to} \;\; 1_A 0_B. \tag{25}$$

$$\uparrow_a\uparrow_b - \downarrow_a\downarrow_b \quad \textit{corresponds to} \;\; 0_A 1_B. \tag{26}$$

$$\downarrow_a\uparrow_b - \uparrow_a\downarrow_b \quad \textit{corresponds to} \;\; 1_A 1_B. \tag{27}$$

After encoding their bits, Alice and Bob apply the CNOT operation. This results in the corresponding four states

$$\uparrow_a\uparrow_b + \downarrow_a\uparrow_b = (\uparrow_a + \downarrow_a) \uparrow_b \quad \textit{corresponds to} \;\; 0_A 0_B \tag{28}$$

$$\downarrow_a\downarrow_b + \uparrow_a\downarrow_b = (\uparrow_a + \downarrow_a) \downarrow_b \quad \textit{corresponds to} \;\; 1_A 0_B \tag{29}$$

$$\uparrow_a\uparrow_b - \downarrow_a\uparrow_b = (\uparrow_a - \downarrow_a)\uparrow_b \quad \textit{corresponds to} \quad 0_A 1_B \tag{30}$$

$$\downarrow_a\downarrow_b - \uparrow_a\downarrow_b = (\downarrow_a - \uparrow_a)\downarrow_b \quad \textit{corresponds to} \quad 1_A 1_B. \tag{31}$$

Bob can now find out Alice's bit by measuring his qubit in the $\{\uparrow_b, \downarrow_b\}$ basis while Alice can find out Bob's bit by measuring her qubit in the $\{\uparrow_a + \downarrow_a, \uparrow_a - \downarrow_a\}$ basis.

V. THE DOUBLE CNOT OPERATION ON TWO QUBITS

One might have thought that the SWAP operation was the *unique* maximally non-local operation, at least in the terms used in this paper. We here demonstrate that there is another maximally non-local operator, which is the "Double CNOT", or "DCNOT" gate, formed by acting a CNOT from particle 1 onto particle 2, and then a second CNOT from particle 2 onto particle 1. It is defined by

$$\uparrow\uparrow\mapsto\uparrow\uparrow \tag{32}$$

$$\uparrow\downarrow\mapsto\downarrow\downarrow \tag{33}$$

$$\downarrow\uparrow\mapsto\uparrow\downarrow \tag{34}$$

$$\downarrow\downarrow\mapsto\downarrow\uparrow\,. \tag{35}$$

To show that DCNOT is maximally non-local, we shall first demonstrate that it can be used to create 2 e-bits. We shall then show that it can be used to communicate 2 bits of information from Alice to Bob, and simultaneously to send 2 bits from Bob to Alice. The argument used for the SWAP operation then proves that to build a DCNOT we need 2 e-bits plus 2 bits of classical communication from Alice to Bob plus 2 bits of classical communication from Bob to Alice. Since any transformation on two qubits can be performed using these resources via teleportation, we will then have shown that the DCNOT is maximally non-local, in terms of resources.

Creating 2 e-bits is easy. Alice and Bob prepare singlets locally, and then perform the DCNOT on spins A and B:

$$(\uparrow_A\uparrow_a + \downarrow_A\downarrow_a)(\uparrow_B\uparrow_b + \downarrow_B\downarrow_b) \mapsto$$

$$\uparrow_A\uparrow_a\uparrow_B\uparrow_b + \downarrow_A\uparrow_a\downarrow_B\downarrow_b + \uparrow_A\downarrow_a\downarrow_B\uparrow_b + \downarrow_A\downarrow_a\uparrow_B\downarrow_b\,. \tag{36}$$

We now have a Schmidt decomposition of rank 4, ie. a 2 party state which is locally equivalent to 2 e-bits between Alice and Bob.

Transmitting 2 bits of information in both directions at the same time is a little more tricky. Alice and Bob need to have 2 e-bits in addition to the DCNOT operation. They first transform their e-bits (locally) into the state

$$\uparrow_A\uparrow_a\uparrow_B\uparrow_b + \downarrow_A\uparrow_a\uparrow_B\downarrow_b + \downarrow_A\downarrow_a\downarrow_B\uparrow_b + \uparrow_A\downarrow_a\downarrow_B\downarrow_b\,. \tag{37}$$

Alice now encodes 1 bit of information in the state by either applying, or not applying $\sigma_z \otimes \sigma_z$ to her 2 spins. She encodes a second bit of information by applying, or not applying σ_x to her first spin, A. Bob similarly encodes two bits of information, using the transformation σ_z on spin B to encode his first bit, and $\sigma_x \otimes \sigma_x$ to encode his second bit.

Having encoded the information, they make it locally accessible by applying the DCNOT to spins A and B. It is not obvious, but simple to check, that Alice and Bob now each have one of the 4 Bell states locally, and that Alice's particular state corresponds to Bob's encoded bits, and vice-versa.

VI. MULTI-PARTITE OPERATIONS

In the previous sections we studied different bi-partite operations. What about multi-partite operations, such as the Toffoli or the Fredkin gates on three qubits? As we showed in section II, they can all be implemented by using the "double teleportation" method. On the other hand, finding the *necessary* resources is far more difficult than in the bi-partite case; indeed it is not possible at present. The reason is that there exist different inequivalent types of multi-partite entanglement [4,5]. For example, it is known that singlets and GHZ states are inequivalent in the sense that they cannot be reversibly transformed into each other, not even in the asymptotic limit. Although GHZs (as all other entangled states) can be built out of singlets, such a procedure is wasteful. Hence, when investigating the minimal entanglement resources needed to implement multi-partite quantum operations, we have to use the different inequivalent types of entanglement. Unfortunately, at present multi-partite entanglement is far from being fully understood.

VII. "CONSERVATION" RELATIONS

In studying the non-locality of quantum states a most important issue is that of "manipulating" entanglement, i.e. of transforming some states into others [6]. Similarly we can ask: Given a unitary evolution, can we use it to implement some other unitary evolution?

In particular, for pure quantum states we have *conservation* relations [6,7]. For example, when Alice and Bob share a large number n of pairs of particles, each pair in the same state Ψ, they could use these pairs to generate some other number, k, of pairs in some other state Φ. In the limit of large n, this transformation can be performed *reversibly*, meaning that the total amount of non-locality contained in the n copies of the state Ψ is the same as the total amount of non-locality contained in the k copies of the state Φ. Is something similar taking place for unitary transformations?

For unitary transformations we have not yet studied the case of the asymptotic limit, i.e. performing the same transformation U on many pairs of particles. However, an interesting pattern emerges even at the level of a single copy.

Consider first the case of SWAP. We know what the minimal resources needed to implement a SWAP are. But suppose now that we are given a device which implements a SWAP. Could we could use it to get back the original resources needed to create the SWAP?

The balance of resources needed to implement a SWAP can be written as

$$\text{2e-bits} + \text{2bits}_{A\to B} + \text{2bits}_{B\to A} => \text{SWAP}. \tag{38}$$

The question is whether

$$\text{SWAP} => \text{2e-bits} + \text{2bits}_{A\to B} + \text{2bits}_{B\to A}? \tag{39}$$

Though we do not have yet a complete proof, it appears that the answer to the above question is "No". That is, combining entanglement and classical communication resources to yield a SWAP is an *irreversible process* - we cannot use the SWAP to get the resources back.

On the other hand, looking back to the proof of the resources needed for SWAP, we see that we can write the following tight "implications":

$$\text{2e-bits} + \text{2bits}_{A\to B} + \text{2bits}_{B\to A} => \text{1SWAP}. \tag{40}$$

$$\text{2e-bits} + \text{1SWAP} => \text{2bits}_{A\to B} + \text{2bits}_{B\to A}. \tag{41}$$

$$\text{1SWAP} => \text{2e-bits}. \tag{42}$$

The first of these three implications is to be read as "given 2e-bits and 2bits$_{A\to B}$ and 2bits$_{A\to B}$ we can produce the SWAP operation; also if we wish to produce the SWAP operation with e-bits, and bits communicated from Alice to Bob and vice-versa, we cannot do so with fewer than 2e-bits and 2bits$_{A\to B}$ and 2bits$_{A\to B}$."

The second and third implications have a slightly different meaning. For example we read the second implication as "given 1 SWAP and 2 e-bits, we can communicate 4 classical bits (two each way); also we cannot communicate more than 4 classical bits (two each way) ". On the other hand, it does not mean that "1 SWAP and 2 e-bits are necessary for communicating 4 classical bits (two each way) " - for example we can implement this classical communication with 2 SWAPs.

Exactly the same implications apply for the DCNOT.

$$\text{2e-bits} + \text{2bits}_{A\to B} + \text{2bits}_{B\to A} => \text{1DCNOT}. \tag{43}$$

$$\text{2e-bits} + \text{1DCNOT} => \text{2bits}_{A\to B} + \text{2bits}_{B\to A}. \tag{44}$$

$$\text{1DCNOT} => \text{2e-bits}. \tag{45}$$

Furthermore, very similar implications can be written for the CNOT:

$$\text{1e-bit} + \text{1bit}_{A\to B} + \text{1bit}_{B\to A} => \text{1CNOT}. \tag{46}$$

$$\text{1e-bit} + \text{1CNOT} => \text{1bit}_{A\to B} + \text{1bit}_{B\to A}. \tag{47}$$

$$\text{1CNOT} => \text{1e-bit}. \tag{48}$$

In fact these implications are very similar to the implications which describe teleportation and super-dense coding which appear, together with many other similar implications on Bennett's famous transparency presented at almost all early quantum information conferences:

$$\text{1e-bit} + \text{2bits}_{A\to B} => \text{1qubit} \tag{49}$$

$$\text{1e-bit} + \text{1qubit} => \text{2bits}_{A\to B} \tag{50}$$

$$\text{1qubit} => \text{1e-bit} \tag{51}$$

The above three implications (49,50,51) are generally thought to describe relations between classical information, quantum information and entanglement. However, we would like to argue that their true meaning is may be more closely related to dynamics, and that a more illuminating form is probably

$$\text{1e-bit} + \text{2bits}_{A\to B} => \text{1teleportation}_{A\to B} \tag{52}$$

$$\text{1e-bit} + \text{1teleportation}_{A\to B} => \text{2bits}_{A\to B} \tag{53}$$

$$\text{1teleportation}_{A\to B} => \text{1e-bit} \tag{54}$$

We conjecture that similar relations hold between any quantum action and the resources needed to implement it, that is

$$Entanglement + ClassicalCommunication => Action \tag{55}$$

$$Entanglement + Action => ClassicalCommunication \tag{56}$$

$$Action => Entanglement \tag{57}$$

It may be that these relations hold, in general, only in the asymptotic limit of many copies of the quantum action.

VIII. DIFFERENT WAYS OF ACHIEVING THE SAME TASK

It is interesting to note that although the transformation from resources to unitary actions is irreversible, sometimes the same end product can be achieved in two different ways. For example, there are two alternative ways to implement

$$2\text{CNOTs} => 1\text{bit}_{A\to B} + 1\text{bit}_{B\to A}. \tag{58}$$

The first way is to use one CNOT to transmit 1 classical bit from Alice to Bob and the other CNOT to transmit 1 classical bit from Bob to Alice, i.e.

$$1\text{CNOT} => 1\text{bit}_{A\to B} \tag{59}$$

and

$$1\text{CNOT} => 1\text{bit}_{B\to A}. \tag{60}$$

Another possibility is to use first one CNOT to create 1 e-bit (48) then the other CNOT plus the e-bit to transmit the 2 classical bits (47), i.e.

$$2\text{CNOTs} => 1\text{e-bit} + 1\text{CNOT} => 1\text{bit}_{A\to B} + 1\text{bit}_{B\to A}. \tag{61}$$

IX. CATALYSING CLASSICAL COMMUNICATION

A very interesting phenomenon is that of "catalysing" classical communication. This phenomenon is similar in its spirit to that of "catalysing entanglement manipulation" [8,4]. An example is the following.

On its own, the SWAP can only send one bit in each direction at the same time, and cannot be used for Alice to send 2 bits to Bob, even if Bob sends no information whatsoever. That is,

$$1\text{SWAP} \not=> 2\text{bits}_{A\to B}. \tag{62}$$

However, if Alice and Bob share 1 e-bit, Alice can send 2 bits to Bob *without destroying* the e-bit, i.e.

$$1\text{SWAP} + 1\text{e-bit} => 2\text{bits}_{A\to B} + 1\text{e-bit}. \tag{63}$$

This may be done as follows. Initially Alice and Bob share a non-local singlet; Bob also prepares a second singlet locally. Alice encodes the two bits she wishes to send to Bob by performing one of the four rotations 1, σ_x, σ_y, σ_z on her half of the non-local singlet. By performing the SWAP operation on Alice's particle from the non-local singlet and one particle of the singlet that Bob has prepared locally, Alice and Bob end up with a non-local singlet held between them; also Bob can find out the two bits by measurements on the local singlet he now holds. Specifically, we begin with the state:

$$(\uparrow_A\uparrow_{b1} + \downarrow_A\downarrow_{b1})(\uparrow_B\uparrow_{b2} + \downarrow_B\downarrow_{b2}), \tag{64}$$

where A is Alice's particle, and B, $b1$ and $b2$ are Bob's particles. Alice performs one of the rotations 1, σ_x, σ_y, σ_z on her particle. They then perform the SWAP on particles A and B, and get (if Alice performed 1):

$$(\uparrow_B\uparrow_{b1} + \downarrow_B\downarrow_{b1})(\uparrow_A\uparrow_{b2} + \downarrow_A\downarrow_{b2}) \tag{65}$$

If Alice performed one of the other rotations, Bob will get one of the other Bell states in system (B, $b1$). Bob now measures that system in the Bell basis to extract the information, and Alice and Bob are left with a singlet between systems A and $b2$.

In effect the SWAP acts as a double teleportation; one from Alice to Bob and one from Bob to Alice. Teleporting Alice's qubit, in conjunction with the e-bit, implements a transmission of two bits from Alice to Bob using super-dense coding; it destroys the e-bit in the process. Simultaneously, the Bob to Alice teleportation restores the e-bit.

X. TRADING ONE TYPE OF ACTIONS FOR ANOTHER

An interesting question is the following. There are cases in which two different actions require the same resources. For example the resources needed for 1 SWAP are the same as for 2 CNOTs, i.e., $2\text{e-bits} + 2\text{bits}_{A\to B} + 2\text{bits}_{B\to A}$. Now, suppose we had already used the resources to build 2 CNOTs, but we wanted to change our mind and we wanted to do 1 SWAP instead. Due to the irreversibility discussed above, we cannot simply get back the original resources and use them to construct the SWAP. Is it however possible to go *directly* from 2 CNOTs to 1 SWAP, without going back to the original resources? As far as we are aware, the answer is "No".

It turns out however that if we have many CNOTs it is nevertheless useful to build a SWAP from CNOTs directly rather than going back to the original resources. Indeed, to obtain the entanglement and classical communication resources needed for 1 SWAP, i.e. $2\text{e-bits} + 2\text{bits}_{A\to B} + 2\text{bits}_{B\to A}$ we need 4 CNOTs. However, it is well-known that one can construct 1 SWAP directly from 3 CNOTs. Indeed, we don't even need 3 CNOTs, but can realize a SWAP by

$$2\text{CNOTs} + 1\text{bit}_{A\to B} + 1\text{bit}_{B\to A} => 1\text{SWAP} \tag{66}$$

which uses less non-local resources than 3 CNOTs. To see this, it suffices to note that

$$1\text{CNOT} + 1\text{bit}_{A\to B} => 1\text{teleportation}_{A\to B} \tag{67}$$

and similarly

$$1\text{CNOT} + 1\text{bit}_{B\to A} => 1\text{teleportation}_{B\to A} \qquad (68)$$

To implement (67) Alice starts with her qubit in the state $\Psi = \alpha \uparrow + \beta \downarrow$ which has to be teleported and Bob with his qubit in the state $\uparrow$. After CNOT the state becomes:

$$\Psi \uparrow = (\alpha \uparrow + \beta \downarrow) \uparrow \mapsto \alpha \uparrow\uparrow + \beta \downarrow\downarrow \qquad (69)$$

Alice then measures her qubit in the $|+ >= \frac{1}{\sqrt{2}}(\uparrow + \downarrow)$ and $|- >= \frac{1}{\sqrt{2}}(\uparrow - \downarrow)$ basis and communicates the result to Bob. If $(+)$ then Bob's qubit is already in the required state $\Psi = \alpha \uparrow + \beta \downarrow$; if $(-)$ then Bob's qubit is in the state $\Psi' = \alpha \uparrow - \beta \downarrow$ and Bob can obtain Ψ by changing the relative phase between $\uparrow$ and $\downarrow$ by π.

Note added. While completing this work we became aware of closely related work by J. Eisert, K. Jacobs, P. Papadopoulos and M. Plenio [9].

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