Brownian Analogues of Burke's Theorem

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We discuss Brownian analogues of a celebrated theorem, due to Burke, which states that the output of an M/M/1 queue is Poisson, and the related notion of quasireversibility. A direct analogue of Burke's theorem for the Brownian queue was stated and proved by Harrison and Williams (1985). We present several different proofs of this and related results. We also present an analogous result for geometric functionals of Brownian motion. By considering series of queues in tandem, these theorems can be applied to a certain class of directed percolation and directed polymer models. It was recently discovered that there is a connection between this directed percolation model and the GUE random matrix ensemble. We extend and give a direct proof of this connection in the two dimensional case. In all of the above, reversibility plays a key role.

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Brownian analogues of Burke's theorem

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Abstract

We discuss Brownian analogues of a celebrated theorem, due to Burke, which states that the output of an M/M/1 queue is Poisson, and the related notion of quasireversibility. A direct analogue of Burke's theorem for the Brownian queue was stated and proved by Harrison and Williams (1985). We present several different proofs of this and related results. We also present an analogous result for geometric functionals of Brownian motion. By considering series of queues in tandem, these theorems can be applied to a certain class of directed percolation and directed polymer models. It was recently discovered that there is a connection between this directed percolation model and the GUE random matrix ensemble. We extend and give a direct proof of this connection in the two dimensional case. In all of the above, reversibility plays a key role.

1 Introduction and summary

Burke's theorem states that the output of an M/M/1 queue (that is, a single-server first-come-first-served queue with Poisson arrivals and exponential holding times) is Poisson. This fact was anticipated but not proved by O'Brien [30] and Morse [25]. Burke [6] also proved that the output up to a given instant is independent of the number of customers in the queue at that instant. This property is also called *quasi-reversibility*. Discussions on Burke's theorem and related material can be found in the books of Asmussen [2], Bremaud [5] and Kelly [22].

In this paper we discuss Brownian analogues of Burke's theorem and the related notion of quasi-reversibility. The first can be obtained by taking a 'heavy-traffic' limit of M/M/1 queues, taking care to keep a distinction between the arrivals and service processes. Heavy-traffic queueing models are well-understood (see, for example, [19] or [29, 38] for recent surveys) and in fact this variant of Burke's theorem was presented in [20] in an attempt to understand quasi-reversibility in the heavy-traffic context. The resulting queueing system is characterised as follows.

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Let B and C be two independent standard Brownian motions indexed by the entire real line, and write

$$B_{(s,t)} = B_t - B_s, \quad C_{(s,t)} = C_t - C_s.$$

Fix $m > 0$ and, for $t \in \mathbb{R}$, set

$$q(t) = \sup_{-\infty < s \leq t} \{B_{(s,t)} + C_{(s,t)} - m(t-s)\} \tag{1}$$

and

$$d(t) = B_t + q(0) - q(t). \tag{2}$$

The Brownian motions B_t and $mt - C_t$ can be thought of, respectively, as the *arrivals* and *service* processes, q as the *queue-length* process and d as the *output*, or *departure*, process. We shall refer to this as the *Brownian queue.* The analogue of Burke's theorem in this context is that d is a standard Brownian motion and, moreover, the values of d up to a given instant t are independent of $q(t)$. This result was presented in [20]. As was observed there, it is closely related to Pitman's representation of the 3-dimensional Bessel process and William's path decomposition of Brownian motion, as extended in [33] to the case of non-zero drift.

Pitman's representation theorem states that if B is a Brownian motion, and $M_t = \sup_{0<s<t} B_s$, then $2M - B$ is a 3-dimensional Bessel process. Matsumoto and Yor [26, 27] have recently obtained a version of Pitman's representation theorem for geometric Brownian motions, and a variety of related results, which essentially rely upon the observation that, if one replaces 'sup' by 'log $\int$ exp' in the statement of Pitman's theorem, the result holds true with the 3-dimensional Bessel process replaced by another Markov process. This can be thought of as a generalisation because Pitman's theorem can be recovered by rescaling and applying Laplace's method. It turns out that, if one replaces 'sup' by 'log $\int$ exp' in the definition (4) of the Brownian queue, the conclusion remains valid: d is a standard Brownian motion, and $\{d(s), s \leq t\}$ is independent of $q(t)$. This follows from results presented in [27]. We refer to this as the *generalised Brownian queue.*

By considering a sequence of n Brownian queues in tandem, we are lead to a variational formula which relates the 'total occupancy' of the system to the process

$$L_n(t) = \sup_{0 \leq s_1 \leq \cdots \leq s_{n-1} \leq t} \left\{ B^{(1)}_{(0,s_1)} + \cdots + B^{(n)}_{(s_{n-1},t)} \right\}, \tag{3}$$

where $B^{(1)}, B^{(2)}, \ldots$ is a sequence of independent standard Brownian motions. Related formulas have appeared before in [28, 36, 15]. The processes L_n were introduced in [16], in a similar context. We describe how this variational formula can be used, following a program introduced by Seppäläinen [35] (see [31] for a survey), to show that, $L_n(1)/\sqrt{n} \to 2$ almost surely, as $n \to \infty$. We remark that this limiting result can also be deduced from the recent observation, independently made by

Baryshnikov [4] and Gravner, Tracy and Widom [17], that $L_n(1)$ has the same law as the largest eigenvalue of a random GUE (Wigner) matrix of order n.

Similarly we can consider a sequence of generalised Brownian queues in tandem. This leads to a variety of large deviations results which can be interpreted in terms of a certain random polymer in a random medium. For example, we can compute the free energy density

$$f(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta),$$

where

$$Z_n(\beta) = \int_{0<s_1<\cdots<s_{n-1}<n} ds_1 \ldots ds_{n-1} \exp\left\{\beta(B^{(1)}_{(0,s_1)} + \cdots + B^{(n)}_{(s_{n-1},n)})\right\}.$$

In Section 6 we make some remarks concerning the identity in law between $L_n(1)$ and the largest eigenvalue of a random GUE (Wigner) matrix of order n. We show how, combined with a recent observation of Johansson [21], this yields a certain asymptotic relationship between the Laguerre and Wigner ensembles. We also give a direct proof of the identity in the case $n = 2$, and show that in fact there is a process-version which is closely related to Pitman's representation theorem.

In Section 7, we present a general result which states that the measure-preserving property of a certain path-transformation is equivalent to the reversibility of another; this demonstrates the key role played by reversibility.

In Section 8 we present some multidimensional extensions.

2 Burke's theorem and the Brownian queue

In this section we recall a special case of the quasi-reversibility result for Brownian queueing models which was presented in [20].

Let A and S be independent Poisson processes on the real line with respective intensities $0 < \lambda < \mu$. Then the process Q, defined by

$$Q(t) = \sup_{-\infty<s\leq t} \{A(s,t] - S(s,t]\}^+,$$

is a stationary (and reversible) birth and death process. Here, $A(s,t]$ is the Poisson measure induced by A of the half-open interval $(s,t]$. This is the classical M/M/1 queue: A is the *arrivals process*, S is the *service process* and Q is the *queue-length process*. The *departure process* D is defined by

$$D(s,t] = A(s,t] + Q(s) - Q(t).$$

The M/M/1 queue has the following remarkable property, which follows from the fact that the process Q is reversible (see, for example, [22]). This observation is originally due to Burke [6].

Theorem 1 (1) *D is a Poisson process with intensity λ.*

(2) *$\{D(s,t],\ s \leq t\}$ is independent of $\{Q(s),\ s \geq t\}$.*

Letting λ and μ tend to infinity in the right way, we obtain the following analogue of Theorem 1 for Brownian motions. Let B and C be two independent standard Brownian motions indexed by the entire real line, and write

$$B_{(s,t)} = B_t - B_s, \quad C_{(s,t)} = C_t - C_s.$$

Fix $m > 0$ and, for $t \geq 0$, set

$$q(t) = \sup_{-\infty < s \leq t} \{B_{(s,t)} + C_{(s,t)} - m(t-s)\} \tag{4}$$

and

$$d(t) = B_t + q(0) - q(t). \tag{5}$$

We shall refer to this system as the *Brownian queue*. Note that this is a standard 'heavy traffic' queueing model (see, for example, [19, 20]). The following is a special case of a result presented in [20].

Theorem 2 (1) *d is a standard Brownian motion.*

(2) *$\{d(s),\ s \leq t\}$ is independent of $\{q(s),\ s \geq t\}$.*

Theorem 2 can be obtained from Theorem 1 by weak convergence arguments: if $\mu \to \infty$ and $(\mu - \lambda)/\sqrt{\mu} \to m$, then (a sufficiently refined version of) Donsker's theorem yields the result. In [20] a more general result is proved (in a 'multiclass' context) using time-reversal arguments, but they also give a proof which demonstrates the connection with Pitman's representation theorem and Williams' path decomposition of Brownian motion. We present a variant of that proof here for completeness.

We remark that, in the Brownian storage model discussed in [29], the *storage* process is defined by (4) but the output process is defined by

$$\hat{d}(t) = B_t + C_t + q(0) - q(t). \tag{6}$$

Unlike d, the process $\hat{d}$ is *not* a Brownian motion; its law is characterised in [29].

Let $B^{(\mu)}$ be a standard Brownian motion with drift $\mu > 0$, indexed by the entire real line. For $t \geq 0$, set

$$\rho_t = \sup_{-\infty < s \leq t} \{B_s^{(\mu)} - B_t^{(\mu)}\}, \tag{7}$$

and

$$\hat{B}_t^{(\mu)} = B_t^{(\mu)} + 2(\rho_t - \rho_0). \tag{8}$$

The following result was presented in [20]. For completeness, we include a proof.

Theorem 3 (1) *The process $\hat{B}^{(\mu)}$ is standard Brownian motion with drift μ.*

(2) *$\{\hat{B}^{(\mu)}_s,\ s \le t\}$ is independent of $\{\rho_s,\ s \ge t\}$.*

Proof. (1) If we set $X^{(\mu)} = \sup_{s<0} B^{(\mu)}_s$ and $M^{(\mu)}_t = \sup_{0<s<t} B^{(\mu)}_s$, then

$$\hat{B}^{(\mu)}_t = 2(M^{(\mu)}_t - X^{(\mu)})^+ - B^{(\mu)}_t,$$

and $X^{(\mu)}$ is exponentially distributed with mean $1/2\mu$. An extension of Pitman's representation [32] for the three-dimensional Bessel process, obtained in [33], states that the process $2M^{(\mu)} - B^{(\mu)}$ is autonomously Markov; the law of this process is denoted by $\mathrm{BES}^3(0,\mu)$ (see, for example, [33]). Now recall the following path decomposition of Brownian motion with positive drift about its infimum, which extends Williams' path decomposition of Brownian motion. This is also presented in [33]. Let $B^{(-\mu)}$ be a standard Brownian motion with negative drift $-\mu$, R a $\mathrm{BES}^3(0,\mu)$ and Z an exponential random variable with mean $1/2\mu$, all independent of each other. Set

$$\tau = \inf\{t \ge 0:\ B^{(-\mu)}_t = -Z\}.$$

Then the process Y defined by

$$Y_t = \begin{cases} B^{(-\mu)}_t & t \le \tau \\ R_{t-\tau} - Z & t \ge \tau \end{cases}$$

is standard Brownian motion with drift μ. Combining these two facts, we see that $\hat{B}^{(\mu)}$ is standard Brownian motion with drift μ, as required.

(2) follows from the formula

$$\rho_t = \sup_{s \ge t}(\hat{B}^{(\mu)}_t - \hat{B}^{(\mu)}_s). \tag{9}$$

To see that this formula is valid, write

$$\sup_{s\ge t}(\hat{B}^{(\mu)}_t - \hat{B}^{(\mu)}_s) = 2S^{(\mu)}_t - B^{(\mu)}_t - \inf_{s \ge t}(2S^{(\mu)}_s - B^{(\mu)}_s),$$

where $S^{(\mu)}_t = \sup_{s \le t} B^{(\mu)}_s$, and observe that

$$\inf_{s\ge t}(2S^{(\mu)}_s - B^{(\mu)}_s) = S^{(\mu)}_t.$$

This latter identity is at the heart of Pitman's representation theorem, although it is usually stated with $S^{(\mu)}$ replaced by $M^{(\mu)}$ (the maximum over a finite time-interval). □

Proof of Theorem 2. Define two independent Brownian motions

$$\beta^{(1)} = \frac{B - C}{\sqrt{2}}, \qquad \beta^{(2)} = \frac{B + C}{\sqrt{2}}.$$

Now we can write

$$d(t) = \frac{1}{\sqrt{2}}\beta_t^{(1)} + \frac{1}{\sqrt{2}}\{-\beta_t^{(2)} + 2\nu t - 2(S_t^{(\nu)} - X^{(\nu)})^+\},$$

where $\nu = m/\sqrt{2}$,

$$S_t^{(\nu)} = \sup_{0\le s\le t}(-\beta_s^{(2)} + \nu s)$$

and

$$X^{(\nu)} = \sup_{-\infty<s\le 0}(-\beta_s^{(2)} + \nu s).$$

Set $\gamma_t^{(\nu)} = -\beta_t^{(2)} + \nu t$. We also have,

$$\rho_t := \frac{1}{\sqrt{2}}q(t) = \max(S_t^{(\nu)}, X^{(\nu)}) - \gamma_t^{(\nu)}.$$

Now, by Theorem 3,

(1) The process defined, for $t \ge 0$, by

$$\alpha_t = 2(S_t^{(\nu)} - X^{(\nu)})^+ - \gamma_t^{(\nu)}$$

is a Brownian motion with drift ν, and

(2) $\{\alpha_s,\ s \le t\}$ is independent of $\{\rho_s,\ s \ge t\}$.

Since $\beta^{(1)}$ and $\beta^{(2)}$ are independent, Theorem 2 follows. □

Note that if we define

$$e(t) = C_t + q(0) - q(t)$$

then

$$e(t) = -\frac{1}{\sqrt{2}}\beta_t^{(1)} + \frac{1}{\sqrt{2}}\{-\beta_t^{(2)} + 2\nu t - 2(S_t^{(\nu)} - X^{(\nu)})^+\},$$

and we immediately obtain the following extension of Theorem 2.

Theorem 4 (1) *d and e are independent standard Brownian motions.*

(2) *$\{(d(s), e(s))\ s \le t\}$ is independent of $q(t)$.*

In §4 we will relate the results of this section to a Brownian directed percolation problem, by considering a series of Brownian queues in tandem.

3 The generalised Brownian queue

In this section we will define a *generalised Brownian queue*, with the 'sup' in (4) replaced by '$\log \int \exp$', and show that the corresponding analogues of Theorems 2, 3 and 4 hold; in fact, they follow directly from results presented in [27]. We remark that Theorems 2, 3 and 4 can be recovered from the results of this section by rescaling and applying Laplace's method.

These results will be applied in §5 to compute the free energy density (logarithm of the partition function) of a certain directed polymer in a random medium and other large deviations results which can be related to the Brownian percolation model of §4.

As before, let B and C be standard Brownian motions indexed by the entire real line. Set

$$r(t) = \log \int_{-\infty}^{t} ds \exp\{B_{(s,t)} + C_{(s,t)} - m(t-s)\}, \tag{10}$$

$$f(t) = B_t + r(0) - r(t) \tag{11}$$

and

$$g(t) = C_t + r(0) - r(t). \tag{12}$$

We shall refer to this system as the *generalised Brownian queue.*

Theorem 5 (1) *f and g are independent standard Brownian motions.*

(2) *$\{(f(s), g(s))\ s \leq t\}$ is independent of $\{r(s),\ s \geq t\}$.*

To prove this, we will first write down an analogue of Theorem 3. Let $B^{(\mu)}$ be a standard Brownian motion with drift $\mu > 0$, indexed by the entire real line. For $t \geq 0$, set

$$\alpha_t = \log \int_{-\infty}^{t} ds \exp\{2(B_s^{(\mu)} - B_t^{(\mu)})\}, \tag{13}$$

and

$$\hat{B}_t^{(\mu)} = B_t^{(\mu)} + \alpha_t - \alpha_0. \tag{14}$$

Theorem 6 (1) *The process $\hat{B}^{(\mu)}$ is a standard Brownian motion with drift μ.*

(2) *$\{\hat{B}_s^{(\mu)},\ s \leq t\}$ is independent of $\{\alpha_s,\ s \geq t\}$.*

Proof. (1) follows from [27, Theorem 2.1]. (2) follows from the formula

$$\alpha_t = \log \int_{s\geq t} \exp\{2(\hat{B}_t^{(\mu)} - \hat{B}_s^{(\mu)})\}. \tag{15}$$

This formula is implicit in [27], but since it is not stated explicitly there we present a proof here for completeness. Set

$$A_t = \int_{s\leq t} \exp(2B_s^{(\mu)})ds.$$

From the definition of $\hat{B}^{(\mu)}$,

$$\int_{s\geq t} ds \exp\{2(\hat{B}_t^{(\mu)} - \hat{B}_s^{(\mu)})\} = \exp(-2B_t^{(\mu)})A_t^2 \int_{s\geq t} A_s^{-2} \exp(2B_s^{(\mu)})ds,$$

and (15) follows by noting that

$$dA_s^{-1} = -A_s^{-2} \exp(2B_s^{(\mu)})ds,$$

and hence

$$\int_{s\geq t} A_s^{-2} \exp(2B_s^{(\mu)})ds = A_t^{-1}.$$

□

The proof of Theorem 6 (2) given in [27] uses the method of enlargement; in Section 7 we will give an alternative proof using reversibility arguments.

Proof of Theorem 5. Define two independent Brownian motions

$$\beta^{(1)} = \frac{B-C}{\sqrt{2}}, \qquad \beta^{(2)} = \frac{B+C}{\sqrt{2}},$$

and proceed by applying Theorem 6 as in the proofs of Theorems 2 and 4. □

We conclude this section with some remarks on the process α, which is in fact the logarithm of a particular 'generalised Ornstein-Uhlenbeck process', as discussed in [7].

Theorem 7 *The process α defined by (13) is a stationary, reversible Markov process. The Markov semigroup associated with* $\exp(\alpha)$ *has infinitesimal generator*

$$2x^2\frac{d^2}{dx^2} + (1 + 2x(1-\mu))\frac{d}{dx}.$$

The stationary distribution of α (that is, the law of α_0) is the law of $-\log Z_{2\mu}$, where $Z_{2\mu}$ is gamma-distributed with parameter 2μ.

Proof. The process $\exp(\alpha)$ is stationary by construction, since $B^{(\mu)}$ has stationary increments. The Markov property is argued in [7] (this presents no difficulty and holds for any Lévy process in lieu of $B^{(\mu)}$). The reversibility of $\exp(\alpha)$ is proved in [12], where the invariant distribution is given explicitly (see the remark below). Alternatively, note that it follows from Theorem 6(1) and the symmetry inherent in the formula (15). The fact that α_0 has the same law as $-\log Z_{2\mu}$ is *Dufresne's identity* (see, for example, [7, 13, 26, 39]). □

Remark. We note that the proof of reversibility given in [12] shows that the semigroup in Theorem 7 is symmetric with respect to

$$\gamma_\mu(dx) = x^{2\mu-1} e^{-x} dx,$$

for every $\mu \in \mathbb{R}$; γ_μ is a bounded measure for $\mu > 0$, and unbounded for $\mu \leq 0$.

4 Brownian queues in tandem and directed percolation

We now turn to a related directed percolation problem. To begin with, we construct a 'tandem of Brownian queues'. Let $B, B^{(1)}, B^{(2)}, \ldots$ be a sequence of independent standard Brownian motions, each indexed by the entire real line. Set

$$q_1(t) = \sup_{-\infty < s \leq t} \{B_{(s,t)} + B^{(1)}_{(s,t)} - m(t-s)\}, \tag{16}$$

$$d_1(s,t) = B_{(s,t)} + q_1(s) - q_1(t), \tag{17}$$

and for each $k = 2, 3, \ldots$ set

$$q_k(t) = \sup_{-\infty < s \leq t} \{d_{k-1}(s,t) + B^{(k)}_{(s,t)} - m(t-s)\}, \tag{18}$$

$$d_k(s,t) = d_{k-1}(s,t) + q_k(s) - q_k(t). \tag{19}$$

It follows from Theorem 2 that $q_1(0), q_2(0), \ldots$ is a sequence of i.i.d. random variables. The distribution of $q_1(0)$ is exponential with mean $1/m$. Moreover, by construction, we have

$$\sum_{k=1}^{n} q_k(0) = \sup_{t>0} \{B_{(-t,0)} - mt + L_n(t)\}, \tag{20}$$

where

$$L_n(t) = \sup_{0 \leq s_1 \leq \cdots \leq s_{n-1} \leq t} \left\{ B^{(1)}_{(-t,-s_{n-1})} + \cdots + B^{(n)}_{(-s_1,0)} \right\}. \tag{21}$$

The variational formula (20) is obtained by iterating the recursions (18) and (19).[3]

It is easy to see, by Kingman's subadditive ergodic theorem, that the limit

$$l(x) = \lim_{n\to\infty} L_n(xn)/n$$

exists almost surely for each $x > 0$. We will now show how (20) can be used to identify the function l. See [31] for a survey on the application of this technique in the context of discrete queueing systems; the main idea originates in [35].

First note that, by Brownian scaling, $l(x)$ is proportional to $\sqrt{x}$. To identify the constant of proportionality, we normalise the variational formula (20) and let $n \to \infty$ to obtain (modulo technicalities):

$$1/m = \sup_{x>0}\{-mx + l(x)\}. \tag{22}$$

The formula (22) is valid for any $m > 0$, and is essentially a Legendre transform. Since l is concave, it can be inverted, and we obtain $l(x) = 2\sqrt{x}$. We will not prove this here because it actually follows from a recent observation, due to Baryshnikov [4] and Gravner, Tracy and Widom [17], namely that the random variable

$$M_n = \sup_{0\le s_1\le\cdots\le s_{n-1}\le 1} \left\{ B^{(1)}_{(0,s_1)} + \cdots + B^{(n)}_{(s_{n-1},1)} \right\}, \tag{23}$$

has the same law as the largest eigenvalue of an n-dimensional GUE random matrix. Therefore, in particular,

$$\lim_{n\to\infty} M_n/\sqrt{n} = 2, \tag{24}$$

almost surely. Now, by Brownian scaling, this is equivalent to the statement that $l(x) = 2\sqrt{x}$. A proof of this which uses the variational formula and also yields sharp uniform concentration will be presented in [18]. We shall return to this random matrix connection in Section 6.

5 Generalised Brownian queues in tandem and directed polymers

In this section we construct a tandem of generalised Brownian queues and apply Theorem 5 to obtain large deviations results related to the directed percolation

[3]**Historical note.** Discrete versions of this formula have appeared before in [36] and [15]; see also [3], where a similar formula is used to prove ergodic properties of infinitely many queues in tandem. The connection between queues in tandem and directed percolation was first reported in [28]. The random processes L_n were introduced in [16], in a similar context.

problem of the previous section. These results can be interpreted in terms of the partition function associated with a certain directed polymer in a random medium.

Let $B, B^{(1)}, B^{(2)}, \ldots$ be a sequence of independent standard Brownian motions, each indexed by the entire real line. Set

$$r_1(t) = \log \int_{-\infty}^{t} ds \exp\{B_{(s,t)} + B^{(1)}_{(s,t)} - m(t-s)\},$$

$$f_1(s,t) = B_{(s,t)} + r_1(s) - r_1(t),$$

and for each $k = 2, 3, \ldots$ set

$$r_k(t) = \log \int_{-\infty}^{t} ds \exp\{f_{k-1}(s,t) + B^{(k)}_{(s,t)} - m(t-s)\},$$

$$f_k(s,t) = f_{k-1}(s,t) + q_k(s) - q_k(t).$$

It follows from Theorem 5 that $r_1(0), r_2(0), \ldots$ is a sequence of i.i.d. random variables. Moreover, $r_1(0)$ has the same law as $-\log Z_m$, where Z_m is gamma-distributed with parameter m. By construction, we have

$$\sum_{k=1}^{n} r_k(0) = \log \left[\int_{-\infty}^{0} du \exp(B_{(u,0)} + mu) \right.$$
$$\left. \times \int_{u<s_1<\cdots<s_{n-1}<0} ds_1 \ldots ds_{n-1} \exp\left\{ B^{(1)}_{(u,s_1)} + \cdots + B^{(n)}_{(s_{n-1},0)} \right\} \right] \quad (25)$$

This is the $\log \int \exp$ analogue of the variational formula (20). Applying the strong law of large numbers, we obtain the following result.

Theorem 8 *We have:*

$$\lim_{n\to\infty} \frac{1}{n} \log \int_{u<s_1<\cdots<s_{n-1}<0} du ds_1 \ldots ds_{n-1} \exp\left\{ mu + B^{(1)}_{(u,s_1)} + \cdots + B^{(n)}_{(s_{n-1},0)} \right\} = -\Psi(m) \quad (26)$$

almost surely, where

$$\Psi(m) = E \log Z_m = \Gamma'(m)/\Gamma(m)$$

is the digamma function.

We defer the proof. Theorem 8 can be interpreted as follows. Let $\tau_1, \tau_2, \ldots$ be the points of a unit-rate Poisson process on $\mathbb{R}_+$, and let $\mathcal{B}$ denote the σ-field generated by the Brownian motions $B^{(1)}, B^{(2)}, \ldots$. Set

$$E_n = B^{(1)}_{(0,\tau_1)} + \cdots + B^{(n)}_{(\tau_{n-1},\tau_n)}.$$

By Brownian scaling, Theorem 8 is equivalent to:

Theorem 9

$$\lim_{n\to\infty} \frac{1}{n} \log E[\exp(\theta E_n)|\mathcal{B}] = -2\log\theta - \Psi(1/\theta^2) =: \Lambda(\theta), \tag{27}$$

almost surely.

We remark that Λ is finite and differentiable everywhere with $\Lambda(0) = \Lambda'(0) = 0$. This is a *quenched large deviation principle*, associated with the conditional law of large numbers: given $\mathcal{B}$, $E_n/n \to 0$ almost surely. For example, Theorem 9 implies that, for any $x > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log P(E_n > xn|\mathcal{B}) = -\Lambda^*(x),$$

almost surely, where

$$\Lambda^*(x) = \sup_{\theta\in\mathbb{R}} [x\theta - \Lambda(\theta)].$$

Proof of Theorem 9 (and hence Theorem 8). By (25),

$$\lim_{n\to\infty} \frac{1}{n} \log E[\exp(\theta(E_n + B_{\tau_n}))|\mathcal{B}, B] = \Lambda(\theta), \tag{28}$$

almost surely. We shall use some tools from large deviation theory—see, for example, [9]. Since Λ is finite and differentiable everywhere, and satisfies the steepness condition, we see that, conditional on $\mathcal{B}$ and B, the sequence $(E_n + B_{\tau_n})/n$ almost surely satisfies the large deviation principle in $\mathbb{R}_+$ with good convex rate function Λ^*. It therefore suffices to show that, conditional on $\mathcal{B}$ and B, the sequences $(E_n + B_{\tau_n})/n$ and E_n/n are almost surely exponentially equivalent; that is,

$$\limsup_{n\to\infty} \frac{1}{n} \log P(B_{\tau_n}) > \delta n|B) = -\infty,$$

almost surely, for any $\delta > 0$. This would imply that the sequence E_n/n almost surely satisfies the same large deviation principle and the result will follow from Varadhan's lemma. Fix $a \in (1/2, 1)$ and set $M = \sup_{t>0} B_t/(1+t^a)$. Then $M < \infty$ almost surely and

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log P(B_{\tau_n}) > \delta n|B) &\le \limsup_{n\to\infty} \frac{1}{n} \log P(\tau_n^a > \delta n M^{-1} - 1|M) &(29)\\ &= -\infty, &(30) \end{aligned}$$

almost surely, as required. □

The corresponding 'annealed' (unconditional) large deviation principle is easy to compute: for $-\sqrt{2} < \theta < \sqrt{2}$,

$$\lim_{n\to\infty} \frac{1}{n} \log E\exp(\theta E_n) = \log E\exp(\theta^2\tau_1/2) = -\log(1-\theta^2/2). \tag{31}$$

As there are two sources of randomness, there is another quenched large deviation principle which is obtained by conditioning on the σ-field $\mathcal{T} = \sigma(\tau_1, \tau_2, \dots)$; in this case the rate function is quadratic:

$$\lim_{n\to\infty} \frac{1}{n} \log E[\exp(\theta E_n)|\mathcal{T}] = \theta^2/2. \tag{32}$$

Omitting technical details, we will now show how this relates to a model of a directed polymer in a random medium, and to the directed percolation problem of the previous section. For $x < 0$, the limit

$$\gamma(x) = \lim_{n\to\infty} \frac{1}{n} \log \int_{xn<s_1<\cdots<s_{n-1}<0} ds_1 \dots ds_{n-1} \exp\left\{ B^{(1)}_{(xn,s_1)} + \cdots + B^{(n)}_{(s_{n-1},0)} \right\}$$

exists almost surely, by Kingman's subadditive ergodic theorem. It is easy to check that γ is a concave function. By Laplace's method (this would require justification),

$$-\Psi(m) = \sup_{x<0}[mx + \gamma(x)] = (-\gamma)^*(m).$$

Thus, by inversion, $\gamma = -(-\Psi)^*$.

For $\beta > 0$, set

$$Z_n(\beta) = \int_{-n<s_1<\cdots<s_{n-1}<0} ds_1 \dots ds_{n-1} \exp\left\{ \beta(B^{(1)}_{(-n,s_1)} + \cdots + B^{(n)}_{(s_{n-1},0)}) \right\}.$$

This can be thought of as a partition function, associated with a directed polymer in a random medium. Using the Brownian scaling property, we can compute the associated *free energy density*:

$$\begin{aligned} f(\beta) &= \lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta) && (33) \\ &= \gamma(-\beta^2) - 2\log\beta && (34) \\ &= -(-\Psi)^*(-\beta^2) - 2\log\beta. && (35) \end{aligned}$$

This is a continuous version of the classical two-dimensional directed polymer, where it is not known how to compute the free energy density (see, for example, [10]).

Comparing this with (24), by Laplace's method, we expect

$$\lim_{\beta\to\infty} f(\beta)/\beta = 2,$$

and this is indeed the case. Note that this provides an alternative strategy for computing the limiting constant in (24).

6 Some remarks on the random matrix connection

Let $\{s(i,j),\ (i,j) \in \mathbb{Z}^2\}$ be a collection of i.i.d. exponential random variables with mean one, and set

$$D(m,n) = \max_{\phi \in \Pi(m,n)} \sum_{(i,j)\in\phi} s(i,j),$$

where $\Pi(m,n)$ is the set of non-decreasing connected paths

$$(1,1) = (i_1, j_1) \le (i_2, j_2) \le \cdots \le (i_{m+n}, j_{m+n}) = (m,n).$$

This random variable appears in certain growth models and also has an interpretation in terms of $\cdot/M/1$ queues in tandem. It was observed by Johansson [21] that $D(m,n)$ has the same law as the largest eigenvalue $\rho(m,n)$ of the random matrix AA^*, where A is a $m \times n$ matrix with i.i.d. standard complex normal entries. This random matrix ensemble is known as the *Laguerre*, or *Wishart*, ensemble. Now, by Donsker's theorem, $[D(m,n) - m]/\sqrt{m}$ converges in law, as $m \to \infty$, to the random variable M_n defined by (23). Combining these facts with the observation of Baryshnikov [4] and Gravner, Tracy and Widom [17] that M_n has the same law as the largest eigenvalue $\omega(n)$ of an n-dimensional GUE random matrix, we see that the sequence $[\rho(m,n) - m]/\sqrt{m}$ converges in law, as $m \to \infty$, to $\omega(n)$. This complements recent work of Johnstone [23].

We now give a direct proof of the fact that M_n has the same law as $\omega(n)$, in the case $n = 2$. This turns out to be closely related to Pitman's representation theorem, and in fact there is a process version. Let $B^{(1)}$ and $B^{(2)}$ be independent standard Brownian motions, and set

$$M_2(t) = \sup_{0<s<t} [B^{(1)}_s + B^{(2)}_{(s,t)}], \tag{36}$$

$$N_2(t) = \inf_{0<s<t} [B^{(2)}_s + B^{(1)}_{(s,t)}]. \tag{37}$$

Let X_{11} and X_{22} be independent standard Brownian motions, let $\sqrt{2}X_{12}$ be a standard complex Brownian motion[4], independent of X_{11} and X_{22}, and set $X_{21} = X_{12}^*$. For each $t > 0$, denote the eigenvalues of the 2×2 matrix $X(t)$ by $\lambda_1(t) > \lambda_2(t)$. Finally, let B be a real-valued Brownian motion and R an independent 3-dimensional Bessel process.

Theorem 10 *The processes (M_2, N_2) and (λ_1, λ_2) have the same law, identical to that of $(B+R, B-R)/\sqrt{2}$.*

[4] That is, the real and imaginary parts of $\sqrt{2}X_{12}$ are independent standard real-valued Brownian motions.

Proof. Set $\beta^{(1)} = (B^{(1)} - B^{(2)})/\sqrt{2}$ and $\beta^{(2)} = (B^{(1)} + B^{(2)})/\sqrt{2}$. These are independent Brownian motions. Now observe that

$$\sqrt{2}M_2(t) = 2 \sup_{0<s<t} \beta_s^{(1)} - \beta_t^{(1)} + \beta_t^{(2)}$$

and

$$\sqrt{2}N_2(t) = -2 \sup_{0<s<t} \beta_s^{(1)} + \beta_t^{(1)} + \beta_t^{(2)},$$

so, by Pitman's theorem, $\sqrt{2}(M_2, N_2)$ has the same law as $(B+R, B-R)$.

On the other hand, the eigenvalues of X are given by

$$\sqrt{2}\lambda = \frac{X_{11}+X_{22}}{\sqrt{2}} \pm \sqrt{\left(\frac{X_{11}-X_{22}}{\sqrt{2}}\right)^2 + 2|X_{12}|^2}$$

and the result follows from the independence of $X_{11} - X_{22}$ and $X_{11} + X_{22}$, and the definition of a 3-dimensional Bessel process (as the norm of a 3-dimensional Brownian motion). □

We remark that McKean [24] has discussed the law of the eigenvalues of a 2-dimensional *Gaussian Orthogonal* matrix:

$$\begin{pmatrix} B_{11} & B_{12}/\sqrt{2} \\ B_{12}/\sqrt{2} & B_{22} \end{pmatrix},$$

where B_{11}, B_{22} and B_{12} are independent standard Brownian motions. In this case, the eigenvalues are given by

$$\sqrt{2}\lambda = \frac{B_{11}+B_{22}}{\sqrt{2}} \pm \sqrt{\left(\frac{B_{11}-B_{22}}{\sqrt{2}}\right)^2 + B_{12}^2} \tag{38}$$

$$= \beta \pm R^{(2)}, \tag{39}$$

where β is a standard Brownian motion and $R^{(2)}$ is a two-dimensional Bessel process, independent of β. This leads us to recall the following representation of $R_t^{(2)}$, presented in [8], namely that for each $t > 0$,

$$\sup_{0<r<s<t} [B_r^{(1)} + B_{(s,t)}^{(2)}]$$

has the same law as $R_t^{(2)}$. (Note that we do not have identity in law as processes, since the process

$$(\sup_{s\le t} |B_s| - |B_t|) + L_t,$$

which is distributed as

$$\sup_{0<r<s<t} [B_r^{(1)} + B_{(s,t)}^{(2)}],$$

is obviously *transient*, whereas $R^{(2)}$ is recurrent.)

We remark that in [24] it is shown that the joint distribution of the eigenvalues in the Gaussian Orthogonal case cannot be reduced by time and scale change to 2-dimensional Brownian motion. In the *Gaussian Unitary* case, which is the case we have discussed here, there is the same impossibility.

7 The role of reversibility

Let $(X_t,\ t \in \mathbb{R})$ be a real-valued stochastic process with $X_0 = 0$ almost surely, and assume that the integral

$$A_t = \int_{-\infty}^{t} ds \exp 2(X_s - X_t) \tag{40}$$

exists and is finite almost surely for each $t \in \mathbb{R}$. Define a new process $(\hat{X}_t,\ t \in \mathbb{R})$ by

$$\hat{X}_t = X_t + \log(A_t/A_0).$$

Theorem 11 *The process A is stationary and reversible if, and only if, X has stationary and reversible increments and $\hat{X}$ has the same law as X.*

Proof. This theorem is an immediate consequence of the following elementary lemma.

Lemma 12 *For each $t \in \mathbb{R}$, almost surely,*

$$A_t = \int_{t}^{\infty} ds \exp 2(\hat{X}_t - \hat{X}_s), \tag{41}$$

and

$$\exp(2X_t) = \frac{A_0}{A_t} \exp\left(\int_0^t \frac{ds}{A_s}\right). \tag{42}$$

Recall that the formula (41) was presented in Section 3 in the case where X is Brownian motion; the proof given there applies for general X. The formula (42) can be obtained by elementary calculus. If X has stationary and reversible increments and $\hat{X}$ has the same law as X, then A is stationary by construction and by the formula (41) it is reversible. We now prove the converse. Suppose that A is stationary and reversible. Then, by (42), X has stationary reversible increments. Now by (41),

$$A_{-t} = \int_{-\infty}^{t} ds \exp 2(\hat{X}_{-t} - \hat{X}_{-s}),$$

and hence, by (42),

$$\exp(2\hat{X}_{-t}) = \frac{A_0}{A_{-t}} \exp\left(\int_0^t \frac{ds}{A_{-s}}\right).$$

The fact that $\hat{X}$ has the same law as X now follows from the fact that A is reversible and X has reversible increments. □

Theorem 11 can be applied to give an alternative proof of Theorem 6, and hence Theorem 3. If X is a standard Brownian motion with positive drift, then $\hat{X}$ has the same law as X: this was proved in [27] using the method of enlargement. By Theorem 7, we see that it follows from (and is in fact equivalent to) the reversibility of A, which can be proved directly as in [12].

Note that, in the proof of Theorem 11, we use the fact that X can be recovered from A via the formula (42). If, for example, $\log \int \exp$ is replaced by sup, this is no longer the case and the statement of the theorem is false in general. The problem is that the analogue of the formula (42) does not necessarily yield a unique solution to (42). However, all we actually need is that the law of X is uniquely determined by the law of A, which is often the case. (This is certainly the case if X is Brownian motion.) Similar remarks apply if A is defined by

$$A_t = \int_{-\infty}^t d\eta_s \exp 2(X_s - X_t) \tag{43}$$

where η is a random process; in the case where η and X are independent Lévy processes, A is a generalised Ornstein-Uhlenbeck process as discussed in [7]. In this case, the analogue of (41) holds, that is,

$$A_t = \int_t^\infty d\eta_s \exp 2(\hat{X}_t - \hat{X}_s),$$

but given A, the equation (43) does not necessarily have a unique solution.

8 Multidimensional extensions

Theorem 6 has the following multi-dimensional extension (see, for example, [27]).

Theorem 13 *Let $m \in \mathbb{R}^d$, $m \neq 0$, $b > 0$. Let B be a standard Brownian motion in $\mathbb{R}^d$, indexed by the entire real line, and set $A_t = \int_{-\infty}^t ds \exp[(m, B_s) + bs]$. Then the following identity in law holds:*

$$\left\{B_t - 2\frac{m}{|m|^2} \log\left(\frac{A_t}{A_0}\right),\ t \geq 0\right\} \stackrel{(law)}{=} \left\{B_t - 2\frac{m}{|m|^2} bt,\ t \geq 0\right\}. \tag{44}$$

The corresponding multi-dimensional analogue of Theorem 3 can be deduced from this by Brownian scaling and Laplace's method.

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