

Continued Fractions and the d -dimensional Gauss Transformation

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HP L-BRIMS-2000-15
27th June, 2000*

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multi-
dimensional
Gauss
transformation,
natural
extension,
invariant
measure

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* Internal Accession Date Only

Approved for External Publication

Continued fractions and the d -dimensional Gauss transformation

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Abstract

In this paper we study a multidimensional continued fraction algorithm which is related to the Modified Jacobi-Perron algorithm considered by Podsypanin and Schweiger. We demonstrate that this algorithm has many important properties which are natural generalisations of properties of one-dimensional continued fractions. For this reason, we call the transformation associated to the algorithm the *d-dimensional Gauss transformation*. We construct a coordinate system for the natural extension which reveals its symmetries and allows one to give an explicit formula for the density of its invariant measure. We also discuss the ergodic properties of this invariant measure.

1 Introduction

The theory of one-dimensional continued fractions has a rich and long history. They originated in Euclid's algorithm and their theory was later developed by Gauss, Hurwitz, Legendre, Lagrange and many others. One of the most important contributions made by Gauss was the discovery of an explicit formula for the invariant measure of the transformation associated to one-dimensional continued fractions; this measure is now known as the Gauss measure.

A generalisation of the one-dimensional continued fraction algorithm to two dimensions was first considered by Jacobi in the 1830s. This work was published posthumously in 1868 [10]. Perron later performed a detailed study of Jacobi's algorithm in arbitrary dimension [17]; for this reason, the algorithm is now known as the Jacobi-Perron algorithm (JPA). In fact, the study of the JPA inspired Perron to develop his famous theory of positive matrices. The JPA has been widely studied since then and in particular F. Schweiger has considered its ergodic and metrical properties [21]. The ergodic properties of the

Jacobi-Perron transformation and other similar maps have also been studied by Gordin [5], Mayer [15], and Ito and Yuri [9].

Since the development of the JPA, many other multidimensional continued fraction algorithms have been proposed, in particular we mention the algorithms of Poincaré [19], Brun [3] and Selmer [23].

Podsypanin introduced a two-dimensional algorithm which is closely related to the algorithm of Brun [18]. Later, Schweiger considered a multidimensional modification of Podsypanin's algorithm called the Modified Jacobi-Perron algorithm and gave an explicit formula for its invariant density [22]. In this paper we study an algorithm which is equivalent to the modified JPA. We demonstrate that this algorithm has many properties which are natural generalisations of properties of one-dimensional continued fractions. To the best of our knowledge, it is the only algorithm which possesses these properties. We find it natural to call it the d -dimensional Gauss algorithm, especially since the invariant measure is a generalisation of the Gauss measure.

The structure of the paper is as follows. In Section 2 we give a geometrical description of the one-dimensional continued fraction algorithm and briefly discuss some of its most important properties. In Section 3 we describe two different geometrical schemes for producing a sequence of vectors of rational numbers simultaneously approximating an irrational vector. These two schemes are based on the concepts of time-ordering and space-ordering. We briefly describe the Jacobi-Perron algorithm, which is based on the time-ordering concept, and two other algorithms which are related to the idea of space-ordering. One of these algorithms leads to the d -dimensional Gauss transformation which is the subject of the rest of the paper. Section 4 is concerned with finding a good coordinate system for the natural extension of the d -dimensional Gauss transformation. In Section 5, various important properties of the natural extension are proved. In particular, using the symmetries of the natural extension, we find an explicit formula for the density of its invariant measure.

Acknowledgements. The authors are grateful to the European Science Foundation for the opportunity to participate in their PRODYN (Probabilistic methods in non-hyperbolic dynamics) programme. The first author also wishes to thank the Engineering and Physical Sciences Research Council of the U.K. for financial support.

2 One-dimensional continued fractions

In this section we discuss the approximation of irrationals by rationals in the classical one-dimensional case.

The theory of one-dimensional continued fractions is one of the most beautiful examples of the applications of ergodic theory. We realise that the theory of continued fractions is classical (see [12]) and that the reader is well aware of this theory. Nevertheless we wish to spend some time on a formal introduction of the Gauss transformation and a discussion of its ergodic properties and its connection with the theory of one-dimensional continued fractions. This introduction will be useful in the next section where we will discuss multi-dimensional generalisations of this theory. In this section we will also formulate the most important properties of the one-dimensional case. We will see later that only one of the

multidimensional generalisations inherits these nice properties.

We will start with a geometrical approach to the problem of finding a sequence of rational approximations to an irrational number $\omega \in [0, 1]$. This geometrical scheme is based on the following picture. A point ω is approximated by a sequence of nested intervals which contain ω . These intervals are constructed inductively. Suppose that on the n^{th} step one has an interval Δ_n which contains ω and which has rational end points $\frac{p_n}{q_n}, \frac{p'_n}{q'_n}$. The point ω , like any point in the interval Δ_n , can be written in the form

$$\omega = \alpha_0 \left(\frac{p_n}{q_n} \right) \oplus \alpha_1 \left(\frac{p'_n}{q'_n} \right) \quad (1)$$

for some $\alpha_0 \geq \alpha_1 \geq 0$. Here \oplus denotes Farey addition, i.e.

$$\alpha \left(\frac{p_1}{q_1} \right) \oplus \beta \left(\frac{p_2}{q_2} \right) = \frac{\alpha p_1 + \beta p_2}{\alpha q_1 + \beta q_2}.$$

Note that we can regard (α_0, α_1) as an element of $\mathbb{R}P^1$ since the representation (1) is unique up to multiplication by a scalar factor. Also note that the order of the end points $\frac{p_n}{q_n}, \frac{p'_n}{q'_n}$ in (1) is governed by the relation $\alpha_0 \geq \alpha_1$, rather than by the natural order of the end points on the real line. In the next step of the scheme, we produce an interval Δ_{n+1} which has end points $\frac{p_{n+1}}{q_{n+1}}$ and

$$m_{n+1} \left(\frac{p_n}{q_n} \right) \oplus \left(\frac{p'_n}{q'_n} \right), \quad \text{where } m_{n+1} \in \mathbb{N}.$$

To consider how to choose the integer m_{n+1} we rewrite (1) as

$$\omega = \left(\frac{p_n}{q_n} \right) \oplus \omega^{(n)} \left(\frac{p'_n}{q'_n} \right) \quad \text{where } \omega^{(n)} = \frac{\alpha_1}{\alpha_0}.$$

Then

$$\omega = \frac{1}{\omega^{(n)}} \left(\frac{p_n}{q_n} \right) \oplus \left(\frac{p'_n}{q'_n} \right) = \left[\frac{1}{\omega^{(n)}} \right] \left(\frac{p_n}{q_n} \right) \oplus \left(\frac{p'_n}{q'_n} \right) \oplus \left\{ \frac{1}{\omega^{(n)}} \right\} \left(\frac{p_n}{q_n} \right),$$

where $[x]$ and $\{x\}$ denote the integer and fractional parts of a real number x respectively. We let $m_{n+1} = \left[\frac{1}{\omega^{(n)}} \right]$,

$$\frac{p_{n+1}}{q_{n+1}} = m_{n+1} \left(\frac{p_n}{q_n} \right) \oplus \left(\frac{p'_n}{q'_n} \right) = \frac{m_{n+1}p_n + p'_n}{m_{n+1}q_n + q'_n} \quad \text{and} \quad \frac{p'_{n+1}}{q'_{n+1}} = \frac{p_n}{q_n}.$$

Then Δ_{n+1} , which is the closed interval with end points $\frac{p_{n+1}}{q_{n+1}}$ and $\frac{p'_{n+1}}{q'_{n+1}}$, contains ω and

$$\omega = \left(\frac{p_{n+1}}{q_{n+1}} \right) \oplus \omega^{(n+1)} \left(\frac{p'_{n+1}}{q'_{n+1}} \right) \quad \text{where } \omega^{(n+1)} = \left\{ \frac{1}{\omega^{(n)}} \right\}.$$

We see that our geometric scheme has led to the Gauss transformation $T(\omega) = \left\{ \frac{1}{\omega} \right\}$. To make this scheme work it is necessary to specify the interval Δ_0 . Take $p_0 = 0, q_0 = 1, p'_0 = 1$ and $q'_0 = 0$ so that

$$\omega = \left(\frac{p_0}{q_0} \right) \oplus \omega^{(0)} \left(\frac{p'_0}{q'_0} \right) \quad \text{where } \omega^{(0)} = \omega.$$

Hence Δ_0 is associated with the semi-infinite interval $[0, \infty)$. Notice that $\omega^{(n)} = T^n \omega^{(0)}$ gives the projective coordinate of the point ω inside the interval Δ_n . If we chose m_{n+1} to be an integer greater than $\lceil \frac{1}{\omega^{(n)}} \rceil$ then the interval Δ_{n+1} would not contain ω . This guarantees that the approximation given by continued fractions is the best possible. One can show that for any rational $\frac{p}{q} \in \text{Int}(\Delta_n)$, $q > \max\{q_n, q'_n\}$ and in fact $q \geq q_n + q'_n$.

The rational approximations $\frac{p_n}{q_n}$ defined above are called the *convergents* of the irrational number ω . We now describe an easy way to calculate them.

The map T is expanding and has infinitely many inverse branches. Each of these is characterised by an integer $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, let T_m^{-1} denote the branch of T^{-1} given by

$$T_m^{-1}(\omega) = \frac{1}{m + \omega}.$$

Notice that $\omega \in T_m^{-1}[0, 1]$ if and only if $\lceil \frac{1}{\omega} \rceil = m$. The trajectory of ω under T gives the sequence of integers produced in the continued fraction expansion:

$$m_n = \left\lceil \frac{1}{T^{n-1}\omega} \right\rceil.$$

Take $0 = \frac{0}{1}$ as an approximation to $\omega^{(n)}$. One can easily show that

$$\frac{p_n}{q_n} = T_{m_1}^{-1} \circ \dots \circ T_{m_n}^{-1}(0).$$

It is convenient to present this using matrix multiplication. Firstly, note that

$$\frac{\tilde{p}}{\tilde{q}} = T_{m_n}^{-1}\left(\frac{p}{q}\right) \text{ if and only if } \begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} m_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.$$

Thus

$$\begin{pmatrix} q_n \\ p_n \end{pmatrix} = \begin{pmatrix} m_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} m_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For $n \in \mathbb{N}$, let

$$A_n = \begin{pmatrix} m_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$q_n = \langle A_1 \cdots A_n \mathbf{e}_1, \mathbf{e}_1 \rangle = \langle \mathbf{e}_1, A_n \cdots A_1 \mathbf{e}_1 \rangle$$

and

$$p_n = \langle A_1 \cdots A_n \mathbf{e}_1, \mathbf{e}_2 \rangle = \langle \mathbf{e}_1, A_n \cdots A_1 \mathbf{e}_2 \rangle,$$

where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis of \mathbb{R}^2 : $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Notice that

$$\begin{pmatrix} q'_n \\ p'_n \end{pmatrix} = \begin{pmatrix} q_{n-1} \\ p_{n-1} \end{pmatrix} = A_1 \cdots A_{n-1} \mathbf{e}_1 = A_1 \cdots A_n \mathbf{e}_2.$$

We next discuss the Gauss automorphism, which is the natural extension of the Gauss transformation. Each irrational $\omega \in [0, 1]$ has a unique symbolic representation (m_1, m_2, \dots) , where $m_n = \lfloor \frac{1}{T^{n-1}\omega} \rfloor$. We write $[m_1, m_2, \dots]$ for the point ω corresponding to (m_1, m_2, \dots) . The Gauss transformation $T : [0, 1] \rightarrow [0, 1]$ is conjugate to the unit shift U on the space of one-sided sequences in \mathbb{N} :

$$U((m_1, m_2, \dots)) = (m_2, m_3, \dots).$$

The Gauss measure

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1 + \omega} d\omega, \quad (2)$$

which is the unique absolutely continuous T -invariant probability measure, is transformed by this conjugacy to an invariant Gibbs measure ν on the space $\mathbb{N}^{\mathbb{N}}$ of one-sided sequences in \mathbb{N} . The natural extension \tilde{T} of T is metrically isomorphic to the unit shift \tilde{U} on the space $\mathbb{N}^{\mathbb{Z}}$ of two-sided sequences with an invariant measure $\tilde{\nu}$. The measure $\tilde{\nu}$ is the unique measure on $\mathbb{N}^{\mathbb{Z}}$ which is \tilde{U} -invariant and whose projection onto $\mathbb{N}^{\mathbb{N}}$ coincides with ν .

However, there is a better coordinate system for the natural extension \tilde{T} of T . Given a two-sided sequence $(m_n)_{n \in \mathbb{Z}}$ we can produce $(y, x) \in [0, 1]^2$ by defining

$$y = [m_0, m_{-1}, \dots], \quad x = [m_1, m_2, \dots].$$

In 1977 Nakada, Ito and Tanaka [16] observed that the shift $\tilde{U} : \mathbb{N}^{\mathbb{Z}} \rightarrow \mathbb{N}^{\mathbb{Z}}$ is conjugate to the map $\tilde{T} : [0, 1]^2 \rightarrow [0, 1]^2$ given by

$$\tilde{T}(y, x) = \left(\frac{1}{\lfloor \frac{1}{x} \rfloor + y}, \left\{ \frac{1}{x} \right\} \right)$$

and that \tilde{T} has an invariant measure given by the density

$$\frac{1}{\log 2} \frac{1}{(1 + xy)^2}. \quad (3)$$

Clearly, projection onto x produces the Gauss measure (2):

$$\int_0^1 \frac{1}{(1 + xy)^2} dy = \frac{1}{1 + x}.$$

The transformation \tilde{T} has the important property of reversibility. One can readily see that

$$\tilde{T}^{-1} = S\tilde{T}S$$

where S is the involution $S(y, x) = (x, y)$. Notice that S corresponds to the reversing of the orientation of a two-sided sequence $(m_n)_{n \in \mathbb{Z}}$. We will see below that in the d -dimensional case, both the d -dimensional Gauss transformation and its natural extension have invariant measures which generalise the above formulae.

The y coordinate for \tilde{T} was obtained through the continued fraction expansion written in reverse order. These reverse order continued fractions appear quite naturally in the theory of continued fractions. Consider the sequence $\rho_n = \frac{q_{n-1}}{q_n}$, $n \geq 1$. It is easy to see that

$$\frac{1}{\rho_n} = \frac{q_n}{q_{n-1}} = m_n + \frac{q_{n-2}}{q_{n-1}} = m_n + \rho_{n-1}.$$

This means that

$$T(\rho_n) = \rho_{n-1} \quad \text{and} \quad m_n = \left[\frac{1}{\rho_n} \right]. \quad (4)$$

Iterating one has

$$\rho_n = \frac{q_{n-1}}{q_n} = [m_n, m_{n-1}, \dots, m_1]. \quad (5)$$

This formula expresses the ratio of the denominators in terms of the first n entries of the continued fraction of ω written in reverse order. The numbers ρ_n are important since the quality of approximation can be expressed through them. Indeed

$$\frac{1}{2q_{n+1}} \leq |\omega q_n - p_n| \leq \frac{1}{q_{n+1}}$$

and

$$\frac{1}{q_{n+1}} = \prod_{i=1}^{n+1} \rho_i.$$

We will see later that (4) and (5) can be generalised to higher dimensions. In some sense they give a basic insight into what sort of coordinates one should use for the natural extension.

We end this section with a simple result for one-dimensional continued fractions. We will see later that this result has a multidimensional generalisation which is much less trivial. Consider the finite sequence m_1, m_2, \dots, m_n . When we read it in the forward direction it corresponds to the matrix

$$C_n = \begin{pmatrix} m_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} m_1 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the opposite direction the fraction $[m_n, \dots, m_1]$ produces

$$\tilde{C}_n = \begin{pmatrix} m_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} m_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Obviously, $C_n^t = \tilde{C}_n$ since the matrices A_n are symmetric. It follows from this trivial observation that $[m_1, \dots, m_n]$ and $[m_n, \dots, m_1]$ have the same denominator. We will see later that in the d -dimensional case the matrices C_n^t and \tilde{C}_n coincide up to a change in the order of the rows and columns.

3 Multidimensional Jacobi-Perron type algorithms

In this section we describe a geometric approach to the construction of rational approximations to an irrational vector. This approach leads to many different generalisations of one-dimensional continued fractions. These algorithms can be called Jacobi-Perron type algorithms, since the transformations which are used are similar to the Jacobi-Perron transformation. We will see later that only one Jacobi-Perron type algorithm inherits the nice properties which we discussed in Section 2. We will call the corresponding transformation the d -dimensional Gauss transformation.

Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d) \in [0, 1]^d$. A geometrical scheme for approximating $\boldsymbol{\omega}$ is based on a nested sequence of d -dimensional simplices, each of which contains $\boldsymbol{\omega}$. Each simplex in the sequence has vertices which are given by rational vectors of the form

$$\left(\frac{p_1}{q}, \dots, \frac{p_d}{q} \right).$$

Given a simplex in the sequence, one forms the next simplex by deleting one of the vertices and replacing it by a Farey combination of the existing vertices. In this Farey combination, each vertex has an integer coefficient. Moreover, the deleted vertex has coefficient 1.

Let Δ_n denote the simplex which was obtained at the n^{th} step. In order to define an algorithm for producing the next simplex Δ_{n+1} , it is necessary to order the vertices of Δ_n . The vertices can be ordered in two different ways. They can be put into time-order or space-order.

We consider the time-ordering approach first. The $d + 1$ vertices of Δ_n are denoted

$$\frac{\mathbf{p}_n^{(0)}}{q_n^{(0)}}, \frac{\mathbf{p}_n^{(1)}}{q_n^{(1)}}, \dots, \frac{\mathbf{p}_n^{(d)}}{q_n^{(d)}}$$

where, for $0 \leq i \leq d$, $\mathbf{p}_n^{(i)} \in \mathbb{Z}_+^d$ and $q_n^{(i)} \in \mathbb{N}$. We order the vertices according to the times of their appearance in the nested sequence of simplices. So $\frac{\mathbf{p}_n^{(d)}}{q_n^{(d)}}$ is the vertex which appeared at the n^{th} step, $\frac{\mathbf{p}_n^{(d-1)}}{q_n^{(d-1)}}$ is the vertex which appeared at the $(n-1)^{\text{st}}$ step, and so on down to $\frac{\mathbf{p}_n^{(0)}}{q_n^{(0)}}$. The Jacobi-Perron algorithm is based on the following procedure. One deletes the oldest vertex $\frac{\mathbf{p}_n^{(0)}}{q_n^{(0)}}$ and adds the vertex

$$\frac{\mathbf{p}_{n+1}^{(d)}}{q_{n+1}^{(d)}} = \left(\frac{\mathbf{p}_n^{(0)}}{q_n^{(0)}} \right) \oplus \bigoplus_{i=1}^d m_i \left(\frac{\mathbf{p}_n^{(i)}}{q_n^{(i)}} \right)$$

where the m_i are integers which will be specified later. In the formula above, \oplus stands for Farey addition:

$$\alpha \left(\frac{\mathbf{p}}{q} \right) \oplus \beta \left(\frac{\mathbf{p}'}{q'} \right) = \frac{\alpha \mathbf{p} + \beta \mathbf{p}'}{\alpha q + \beta q'}.$$

Since $\boldsymbol{\omega} \in \Delta_n$ we can write

$$\boldsymbol{\omega} = \alpha_{d+1} \left(\frac{\mathbf{p}_n^{(d)}}{q_n^{(d)}} \right) \oplus \alpha_d \left(\frac{\mathbf{p}_n^{(d-1)}}{q_n^{(d-1)}} \right) \oplus \dots \oplus \alpha_1 \left(\frac{\mathbf{p}_n^{(0)}}{q_n^{(0)}} \right)$$

where $\alpha_{d+1}, \alpha_d, \dots, \alpha_1 \geq 0$. The representation of $\boldsymbol{\omega}$ by $(\alpha_{d+1}, \alpha_d, \dots, \alpha_1)$ is unique up to multiplication by a scalar, i.e. $(\alpha_{d+1}, \alpha_d, \dots, \alpha_1) \in \mathbb{R}P^d$. It is convenient to take a representative of $(\alpha_{d+1}, \alpha_d, \dots, \alpha_1)$ which has first coordinate 1. So

$$\boldsymbol{\omega} = \left(\frac{\mathbf{p}_n^{(d)}}{q_n^{(d)}} \right) \oplus \omega_d^{(n)} \left(\frac{\mathbf{p}_n^{(d-1)}}{q_n^{(d-1)}} \right) \oplus \dots \oplus \omega_1^{(n)} \left(\frac{\mathbf{p}_n^{(0)}}{q_n^{(0)}} \right) \quad \text{where } \omega_i^{(n)} = \frac{\alpha_i}{\alpha_{d+1}}.$$

Then

$$\begin{aligned} \boldsymbol{\omega} &= \left(\frac{\mathbf{p}_n^{(d)}}{q_n^{(d)}} \right) \oplus \bigoplus_{i=0}^{d-1} \omega_{i+1}^{(n)} \left(\frac{\mathbf{p}_n^{(i)}}{q_n^{(i)}} \right) \\ &= \frac{1}{\omega_1^{(n)}} \left(\frac{\mathbf{p}_n^{(d)}}{q_n^{(d)}} \right) \oplus \bigoplus_{i=0}^{d-1} \frac{\omega_{i+1}^{(n)}}{\omega_1^{(n)}} \left(\frac{\mathbf{p}_n^{(i)}}{q_n^{(i)}} \right) \\ &= \left[\frac{1}{\omega_1^{(n)}} \right] \left(\frac{\mathbf{p}_n^{(d)}}{q_n^{(d)}} \right) \oplus \bigoplus_{i=1}^{d-1} \left[\frac{\omega_{i+1}^{(n)}}{\omega_1^{(n)}} \right] \left(\frac{\mathbf{p}_n^{(i)}}{q_n^{(i)}} \right) \oplus \left(\frac{\mathbf{p}_n^{(0)}}{q_n^{(0)}} \right) \\ &\quad \oplus \left\{ \frac{1}{\omega_1^{(n)}} \right\} \left(\frac{\mathbf{p}_n^{(d)}}{q_n^{(d)}} \right) \oplus \bigoplus_{i=1}^{d-1} \left\{ \frac{\omega_{i+1}^{(n)}}{\omega_1^{(n)}} \right\} \left(\frac{\mathbf{p}_n^{(i)}}{q_n^{(i)}} \right). \end{aligned}$$

The first three terms define the new vertex $\frac{\mathbf{p}_{n+1}^{(d)}}{q_{n+1}^{(d)}}$, i.e.

$$\frac{\mathbf{p}_{n+1}^{(d)}}{q_{n+1}^{(d)}} = \left[\frac{1}{\omega_1^{(n)}} \right] \left(\frac{\mathbf{p}_n^{(d)}}{q_n^{(d)}} \right) \oplus \bigoplus_{i=1}^{d-1} \left[\frac{\omega_{i+1}^{(n)}}{\omega_1^{(n)}} \right] \left(\frac{\mathbf{p}_n^{(i)}}{q_n^{(i)}} \right) \oplus \left(\frac{\mathbf{p}_n^{(0)}}{q_n^{(0)}} \right),$$

and

$$\frac{\mathbf{p}_{n+1}^{(i)}}{q_{n+1}^{(i)}} = \frac{\mathbf{p}_n^{(i+1)}}{q_n^{(i+1)}}, \quad 0 \leq i \leq d-1.$$

We get

$$\boldsymbol{\omega} = \left(\frac{\mathbf{p}_{n+1}^{(d)}}{q_{n+1}^{(d)}} \right) \oplus \bigoplus_{i=0}^{d-1} \omega_{i+1}^{(n+1)} \left(\frac{\mathbf{p}_{n+1}^{(i)}}{q_{n+1}^{(i)}} \right),$$

where

$$\omega_i^{(n+1)} = \left\{ \frac{\omega_{i+1}^{(n)}}{\omega_1^{(n)}} \right\}, \quad 1 \leq i \leq d-1, \quad \text{and } \omega_d^{(n+1)} = \left\{ \frac{1}{\omega_1^{(n)}} \right\}.$$

Denote $\boldsymbol{\omega}^{(n+1)} = (\omega_1^{(n+1)}, \dots, \omega_d^{(n+1)})$. Then

$$\boldsymbol{\omega}^{(n+1)} = JP_d(\boldsymbol{\omega}^{(n)}) = \left(\left\{ \frac{\omega_2^{(n)}}{\omega_1^{(n)}} \right\}, \dots, \left\{ \frac{\omega_d^{(n)}}{\omega_1^{(n)}} \right\}, \left\{ \frac{1}{\omega_1^{(n)}} \right\} \right).$$

Clearly, JP_d is a map from I^d into itself. This map is an exact endomorphism which has a unique absolutely continuous invariant probability measure [21, 15, 9].

We now consider the space-ordering approach. We denote the $d + 1$ vertices of the n^{th} step by

$$\frac{\mathbf{p}(n, 0)}{q(n, 0)}, \frac{\mathbf{p}(n, 1)}{q(n, 1)}, \dots, \frac{\mathbf{p}(n, d)}{q(n, d)},$$

where $\mathbf{p}(n, i) \in \mathbb{Z}_+^d$ and $q(n, i) \in \mathbb{N}$ for $0 \leq i \leq d$. In this approach we order the vertices according to their contribution to the expansion

$$\boldsymbol{\omega} = \bigoplus_{i=0}^d \alpha_i \left(\frac{\mathbf{p}(n, i)}{q(n, i)} \right). \quad (6)$$

More precisely, the ordering in (6) is such that

$$\alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_d.$$

Again we will normalise the representation of $\boldsymbol{\omega}$ so that

$$\boldsymbol{\omega} = \left(\frac{\mathbf{p}(n, 0)}{q(n, 0)} \right) \oplus \bigoplus_{i=1}^d \omega_i^{(n)} \left(\frac{\mathbf{p}(n, i)}{q(n, i)} \right) \quad \text{where } \omega_i^{(n)} = \frac{\alpha_i}{\alpha_0}.$$

Notice that $1 \geq \omega_1^{(n)} \geq \omega_2^{(n)} \geq \dots \geq \omega_d^{(n)} \geq 0$. In order to produce the next approximation one must decide which vertex to delete. This vertex may be any one except the first. We will consider two extreme cases. The first case is when we delete the vertex $\frac{\mathbf{p}(n, d)}{q(n, d)}$ and the second case is when we delete $\frac{\mathbf{p}(n, 1)}{q(n, 1)}$. In the first case we have

$$\begin{aligned} \boldsymbol{\omega} &= \left(\frac{\mathbf{p}(n, 0)}{q(n, 0)} \right) \oplus \bigoplus_{i=1}^d \omega_i^{(n)} \left(\frac{\mathbf{p}(n, i)}{q(n, i)} \right) \\ &= \frac{1}{\omega_d^{(n)}} \left(\frac{\mathbf{p}(n, 0)}{q(n, 0)} \right) \oplus \bigoplus_{i=1}^d \frac{\omega_i^{(n)}}{\omega_d^{(n)}} \left(\frac{\mathbf{p}(n, i)}{q(n, i)} \right) \\ &= \left[\frac{1}{\omega_d^{(n)}} \right] \left(\frac{\mathbf{p}(n, 0)}{q(n, 0)} \right) \oplus \left(\frac{\mathbf{p}(n, d)}{q(n, d)} \right) \oplus \bigoplus_{i=1}^{d-1} \left[\frac{\omega_i^{(n)}}{\omega_d^{(n)}} \right] \left(\frac{\mathbf{p}(n, i)}{q(n, i)} \right) \\ &\quad \oplus \left\{ \frac{1}{\omega_d^{(n)}} \right\} \left(\frac{\mathbf{p}(n, 0)}{q(n, 0)} \right) \oplus \bigoplus_{i=1}^{d-1} \left\{ \frac{\omega_i^{(n)}}{\omega_d^{(n)}} \right\} \left(\frac{\mathbf{p}(n, i)}{q(n, i)} \right). \end{aligned}$$

We get the new vertex

$$\frac{\mathbf{p}(n+1, 0)}{q(n+1, 0)} = \left[\frac{1}{\omega_d^{(n)}} \right] \left(\frac{\mathbf{p}(n, 0)}{q(n, 0)} \right) \oplus \left(\frac{\mathbf{p}(n, d)}{q(n, d)} \right) \oplus \bigoplus_{i=1}^{d-1} \left[\frac{\omega_i^{(n)}}{\omega_d^{(n)}} \right] \left(\frac{\mathbf{p}(n, i)}{q(n, i)} \right)$$

and the transformation

$$\boldsymbol{\omega}^{(n+1)} = (\omega_1^{(n+1)}, \dots, \omega_d^{(n+1)}) = \text{ord} \left(\left\{ \frac{1}{\omega_d^{(n)}} \right\}, \left\{ \frac{\omega_1^{(n)}}{\omega_d^{(n)}} \right\}, \dots, \left\{ \frac{\omega_{d-1}^{(n)}}{\omega_d^{(n)}} \right\} \right). \quad (7)$$

Here $\text{ord}(\alpha_1, \dots, \alpha_d)$ is an ordering of $(\alpha_1, \dots, \alpha_d)$. In other words $\text{ord}(\alpha_1, \dots, \alpha_d) = (\alpha_{\pi(1)}, \dots, \alpha_{\pi(d)})$ where π is a permutation of $\{1, 2, \dots, d\}$ such that

$$\alpha_{\pi(1)} \geq \alpha_{\pi(2)} \geq \dots \geq \alpha_{\pi(d)}.$$

Obviously, the permutation π depends on $(\alpha_1, \dots, \alpha_d)$. The vertices of the simplex Δ_{n+1} have to be ordered according to the ordering in (7). Notice that (7) defines a transformation of the simplex

$$\Delta^d = \{(\omega_1, \dots, \omega_d) \in [0, 1]^d : \omega_1 \geq \omega_2 \geq \dots \geq \omega_d\}$$

into itself.

We now consider the second choice of the vertex which is to be deleted, namely $\frac{\mathbf{p}(n,1)}{q(n,1)}$. We have

$$\begin{aligned} \boldsymbol{\omega} &= \left(\frac{\mathbf{p}(n,0)}{q(n,0)} \right) \oplus \bigoplus_{i=1}^d \omega_i^{(n)} \left(\frac{\mathbf{p}(n,i)}{q(n,i)} \right) \\ &= \frac{1}{\omega_1^{(n)}} \left(\frac{\mathbf{p}(n,0)}{q(n,0)} \right) \oplus \bigoplus_{i=1}^d \frac{\omega_i^{(n)}}{\omega_1^{(n)}} \left(\frac{\mathbf{p}(n,i)}{q(n,i)} \right) \\ &= \left[\frac{1}{\omega_1^{(n)}} \right] \left(\frac{\mathbf{p}(n,0)}{q(n,0)} \right) \oplus \left(\frac{\mathbf{p}(n,1)}{q(n,1)} \right) \oplus \left\{ \frac{1}{\omega_1^{(n)}} \right\} \left(\frac{\mathbf{p}(n,0)}{q(n,0)} \right) \oplus \bigoplus_{i=2}^d \frac{\omega_i^{(n)}}{\omega_1^{(n)}} \left(\frac{\mathbf{p}(n,i)}{q(n,i)} \right). \end{aligned}$$

The formula above gives a new vertex

$$\frac{\mathbf{p}(n+1,0)}{q(n+1,0)} = \left[\frac{1}{\omega_1^{(n)}} \right] \left(\frac{\mathbf{p}(n,0)}{q(n,0)} \right) \oplus \left(\frac{\mathbf{p}(n,1)}{q(n,1)} \right)$$

and a transformation $T_d : \Delta^d \rightarrow \Delta^d$ such that $T_d(\boldsymbol{\omega}^{(n)}) = \boldsymbol{\omega}^{(n+1)}$. In coordinates T_d is given by

$$T_d(\omega_1, \dots, \omega_d) = \text{ord} \left(\left\{ \frac{1}{\omega_1} \right\}, \frac{\omega_2}{\omega_1}, \dots, \frac{\omega_d}{\omega_1} \right). \quad (8)$$

The transformation T_d is the main subject of the rest of this paper.

Definition 1. The transformation $T_d : \Delta^d \rightarrow \Delta^d$ is called the *d-dimensional Gauss transformation*.

Strictly speaking, for all geometric schemes one has to specify the initial simplex Δ_0 . For both the space-ordering schemes above the initial simplex is given by

$$\begin{aligned} \mathbf{p}(0,0) &= (0, \dots, 0), \\ q(0,0) &= 1, \\ \mathbf{p}(0,1) &= (1, 0, \dots, 0), \\ q(0,1) &= 0, \\ &\vdots \\ \mathbf{p}(0,d) &= (0, \dots, 0, 1), \\ q(0,d) &= 0. \end{aligned}$$

By interpreting $\frac{0}{0}$ as 0 and $\frac{1}{0}$ as infinity, we can regard Δ_0 as a semi-infinite simplex which coincides with the positive quadrant of \mathbb{R}^d

$$\{\boldsymbol{\omega} = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d : \omega_i \geq 0\}.$$

In a well-defined number of steps one reaches a bounded simplex. This happens when all the vertices with a 0 denominator are removed.

We now describe a straightforward algebraic method of calculating the vectors

$$\frac{\mathbf{p}(n, 0)}{q(n, 0)}, \dots, \frac{\mathbf{p}(n, d)}{q(n, d)}$$

produced by the d -dimensional Gauss transformation T_d . From now on we will write T instead of T_d .

Define $m : \Delta^d \rightarrow \mathbb{N}$ by $m(\boldsymbol{\omega}) = \lfloor \frac{1}{\omega_1} \rfloor$ where $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d) \in \Delta^d$. The ordering in (8) consists of placing $\{\frac{1}{\omega_1}\}$ in the correct position. Let $j(\boldsymbol{\omega})$ denote this position, i.e. $j(\boldsymbol{\omega}) = i$ where the i^{th} coordinate of $T(\boldsymbol{\omega})$ is $\{\frac{1}{\omega_1}\}$. For each pair $(m, j) \in \mathbb{N} \times \{1, 2, \dots, d\}$ there is a corresponding branch of T^{-1} . The branch of T^{-1} associated to (m, j) , denoted $T_{(m,j)}^{-1}$, is given by

$$T_{(m,j)}^{-1}(\omega_1, \dots, \omega_d) = \left(\frac{1}{m + \omega_j}, \frac{\omega_1}{m + \omega_j}, \dots, \frac{\omega_{j-1}}{m + \omega_j}, \frac{\omega_{j+1}}{m + \omega_j}, \dots, \frac{\omega_d}{m + \omega_j} \right).$$

For each pair $(m, j) \in \mathbb{N} \times \{1, 2, \dots, d\}$ we define a matrix $\tilde{A}_{(m,j)} \in GL(d+1, \mathbb{Z})$. The first row of $\tilde{A}_{(m,j)}$ has only two nonzero entries:

$$\tilde{a}_{1,1} = m, \quad \tilde{a}_{1,j+1} = 1.$$

All other rows have only one nonzero entry, which is equal to 1. More precisely, $\tilde{a}_{i,i-1} = 1$ for $i = 2, \dots, j+1$ and $\tilde{a}_{i,i} = 1$ for $i = j+2, \dots, d+1$. In short,

$$\tilde{A}_{(m,j)} = \begin{pmatrix} m & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \quad (9)$$

It is easy to check that

$$T_{(m,j)}^{-1} \left(\frac{p_1}{q}, \dots, \frac{p_d}{q} \right) = \left(\frac{\tilde{p}_1}{\tilde{q}}, \dots, \frac{\tilde{p}_d}{\tilde{q}} \right) \text{ if and only if } \begin{pmatrix} \tilde{q} \\ \tilde{p}_1 \\ \vdots \\ \tilde{p}_d \end{pmatrix} = \tilde{A}_{(m,j)} \begin{pmatrix} q \\ p_1 \\ \vdots \\ p_d \end{pmatrix}.$$

We also define $A_{(m,j)} = \tilde{A}_{(m,j)}^t$. Notice that in the one-dimensional case $A_m = \tilde{A}_m$ since \tilde{A}_m is symmetric.

Let $\boldsymbol{\omega}$ be the point of Δ^d that we wish to approximate. We can produce the vectors

$$\frac{\mathbf{p}(n,i)}{q(n,i)} = \frac{1}{q(n,i)}(p_1(n,i), \dots, p_d(n,i))$$

by the method described above. These vertices $\mathbf{p}(n,i)/q(n,i)$ form a matrix $D(\Delta_n) \in GL(d+1, \mathbb{Z})$, namely

$$D(\Delta_n) = \begin{pmatrix} q(n,0) & p_1(n,0) & \dots & p_d(n,0) \\ q(n,1) & p_1(n,1) & \dots & p_d(n,1) \\ \vdots & \vdots & & \vdots \\ q(n,d) & p_1(n,d) & \dots & p_d(n,d) \end{pmatrix}.$$

Consider the trajectory of $\boldsymbol{\omega}$ under T :

$$\boldsymbol{\omega} = \boldsymbol{\omega}^{(0)} \xrightarrow{T} \boldsymbol{\omega}^{(1)} \xrightarrow{T} \dots \xrightarrow{T} \boldsymbol{\omega}^{(n)}.$$

Associated to this trajectory is the sequence $(m_1, j_1), \dots, (m_n, j_n)$ where

$$m_i = m(T^{i-1}\boldsymbol{\omega}), \quad j_i = j(T^{i-1}\boldsymbol{\omega}).$$

Let

$$\tilde{C}_n = (\tilde{c}_{i,k}^{(n)})_{1 \leq i,k \leq d+1} = \tilde{A}_{(m_1, j_1)} \tilde{A}_{(m_2, j_2)} \dots \tilde{A}_{(m_n, j_n)}$$

and

$$C_n = (c_{i,k}^{(n)})_{1 \leq i,k \leq d+1} = \tilde{C}_n^t = A_{(m_n, j_n)} \dots A_{(m_2, j_2)} A_{(m_1, j_1)}.$$

It can be shown that $C_n = D(\Delta_n)$ (see [6]). This obviously implies that

$$\frac{\mathbf{p}(n,i)}{q(n,i)} = \left(\frac{c_{i+1,2}^{(n)}}{c_{i+1,1}^{(n)}}, \dots, \frac{c_{i+1,d+1}^{(n)}}{c_{i+1,1}^{(n)}} \right), \quad 0 \leq i \leq d.$$

Also, for the first vertex

$$\frac{\mathbf{p}(n,0)}{q(n,0)} = T_{(m_1, j_1)}^{-1} \circ T_{(m_2, j_2)}^{-1} \circ \dots \circ T_{(m_n, j_n)}^{-1} \left(\frac{0}{1}, \dots, \frac{0}{1} \right).$$

Remark. In the case $d = 1$, the three geometric schemes considered above all lead to the same transformation, namely the Gauss transformation. This is because, in the one-dimensional case, the earlier vertex always gives a smaller contribution to the decomposition (6). It seems to be natural to get rid of the vertex which gives the smallest contribution to (6). However, the natural generalisation of the Gauss transformation arises from a different strategy: one has to delete the vertex which gives the second largest contribution to the decomposition (6).

4 The d -dimensional Gauss transformation and its natural extension

It was shown by Schweiger [22] that the d -dimensional Gauss transformation T has an ergodic invariant probability measure μ given by

$$\mu(d\boldsymbol{\omega}) = \frac{1}{K} \rho(\boldsymbol{\omega}) d\boldsymbol{\omega},$$

$$\rho(\boldsymbol{\omega}) = \sum_{\pi \in S_d} \frac{1}{1 + \omega_{\pi(1)}} \frac{1}{1 + \omega_{\pi(1)} + \omega_{\pi(2)}} \cdots \frac{1}{1 + \omega_{\pi(1)} + \omega_{\pi(2)} + \cdots + \omega_{\pi(d)}}$$

where $K = \int_{\Delta^d} \rho(\boldsymbol{\omega}) d\boldsymbol{\omega}$ and S_d is the group of permutations of $\{1, 2, \dots, d\}$. It can also be shown that, for almost all $\boldsymbol{\omega}$, the approximations generated by the d -dimensional Gauss transformation are exponentially convergent to $\boldsymbol{\omega}$ in the weak or directional sense (see [6]). This means that for μ -almost every $\boldsymbol{\omega} \in \Delta^d$ the diameter of Δ_n tends to 0 exponentially fast as $n \rightarrow \infty$. Weak convergence implies that, after the removal of a set of measure 0 from Δ^d , the map Φ which associates to $\boldsymbol{\omega}$ the sequence $(m_n, j_n)_{n \in \mathbb{N}} = (m(T^{n-1}\boldsymbol{\omega}), j(T^{n-1}\boldsymbol{\omega}))_{n \in \mathbb{N}}$ is a bijection. We will write $[(m_1, j_1), (m_2, j_2), \dots]$ for the vector $\boldsymbol{\omega}$ corresponding to the sequence $((m_1, j_1), (m_2, j_2), \dots)$. The invariant measure μ is projected by the transformation Φ onto a stationary measure ν on the space of one-sided sequences in $\mathbb{N} \times \{1, 2, \dots, d\}$. Clearly, the dynamical system (Δ^d, T, μ) is metrically isomorphic to the unit shift on the space of one-sided sequences in $\mathbb{N} \times \{1, 2, \dots, d\}$ with stationary measure ν . There exists a unique stationary extension of ν onto the space of two-sided sequences. We denote this extension by $\tilde{\nu}$ as in the one-dimensional case. The natural extension of (Δ^d, T, μ) is isomorphic to the unit shift on the space of two-sided sequences with the invariant measure $\tilde{\nu}$. However, our aim is to find a good coordinate system for the natural extension. One can naively try to mimic the one-dimensional strategy by defining

$$\boldsymbol{x} = [(m_1, j_1), (m_2, j_2), \dots], \quad \boldsymbol{y} = [(m_0, j_0), (m_{-1}, j_{-1}), \dots].$$

It turns out that this is not a good way of proceeding. Before we give a formal definition of the right coordinates we offer the following motivation.

4.1 The backwards Gauss transformation

In the one-dimensional case we had the important property that the ratios of the denominators are connected by the backward Gauss transformation with the same integer entry m as for the forward Gauss transformation. More precisely,

$$T(\omega^{(n-1)}) = \omega^{(n)}, \quad T\left(\frac{q_{n-1}}{q_n}\right) = \frac{q_{n-2}}{q_{n-1}} \quad \text{and} \quad m(\omega^{(n-1)}) = m\left(\frac{q_{n-1}}{q_n}\right).$$

We will see below that a similar property holds in the d -dimensional case. The vectors generated by the ratios of the denominators of the vertices are related by the d -dimensional Gauss transformation. However, in the d -dimensional case there are two numbers related

to the Gauss transformation, namely $m(\boldsymbol{\omega})$ and $j(\boldsymbol{\omega})$. It turns out that while the parameter m for the forward and backward transformation is the same, the parameter j changes. This change of j leads to the appearance of additional discrete structure in the natural extension.

Consider the simplex Δ_n which is the n^{th} approximation to $\boldsymbol{\omega}$. Each vertex of Δ_n is a rational vector with a certain denominator. Thus there are $d+1$ denominators associated to Δ_n . We put these denominators into their chronological order $q_n^{(0)}, q_n^{(1)}, \dots, q_n^{(d)}$ where $q_n^{(i)}$ appeared more recently than $q_n^{(i+1)}$. It is easy to see that the denominator of a new vertex is greater than or equal to all previous denominators. Hence

$$q_n^{(0)} \geq q_n^{(1)} \geq \dots \geq q_n^{(d)}.$$

It is natural to compare the sequence of denominators in chronological order with the sequence in its space order. Recall that

$$\boldsymbol{\omega} = \left(\frac{\mathbf{p}(n, 0)}{q(n, 0)} \right) \oplus \bigoplus_{i=1}^d \omega_i^{(n)} \left(\frac{\mathbf{p}(n, i)}{q(n, i)} \right).$$

It follows from the construction that $q(n, 0)$ corresponds to the most recent vertex, i.e.

$$q(n, 0) = q_n^{(0)}.$$

However, $q(n, 1), \dots, q(n, d)$ appear in an arbitrary order. Let $\epsilon_n \in S_d$ be the permutation which reflects this order, i.e.

$$q(n, i) = q_n^{(\epsilon_n(i))} \quad \text{for } 1 \leq i \leq d.$$

Denote

$$\boldsymbol{\phi}_n = \left(\frac{q_n^{(1)}}{q_n^{(0)}}, \dots, \frac{q_n^{(d)}}{q_n^{(0)}} \right) \in \Delta^d.$$

Lemma 4.1. $\boldsymbol{\phi}_{n-1} = T(\boldsymbol{\phi}_n)$, $m(\boldsymbol{\phi}_n) = m(\boldsymbol{\omega}^{(n-1)}) = m_n$, $j(\boldsymbol{\phi}_n) = \epsilon_{n-1}(1)$.

Proof. The space-ordering of the denominators of the vertices of Δ_{n-1} is connected to their chronological order by the permutation ϵ_{n-1} . More precisely,

$$q(n-1, i) = q_{n-1}^{(\epsilon_{n-1}(i))} \quad \text{for } 1 \leq i \leq d.$$

Clearly,

$$q_n^{(0)} = q(n, 0) = m_n q(n-1, 0) + q(n-1, 1) = m_n q_{n-1}^{(0)} + q_{n-1}^{(\epsilon_{n-1}(1))},$$

and

$$q_n^{(i)} = \begin{cases} q_{n-1}^{(i-1)} & \text{if } 1 \leq i \leq \epsilon_{n-1}(1); \\ q_{n-1}^{(i)} & \text{if } i > \epsilon_{n-1}(1). \end{cases}$$

Note that

$$\left\{ \frac{q_n^{(0)}}{q_n^{(1)}} \right\} = \left\{ \frac{m_n q_{n-1}^{(0)} + q_{n-1}^{(\epsilon_{n-1}(1))}}{q_{n-1}^{(0)}} \right\} = \frac{q_{n-1}^{(\epsilon_{n-1}(1))}}{q_{n-1}^{(0)}}$$

and

$$\left[\frac{q_n^{(0)}}{q_n^{(1)}} \right] = \left[\frac{m_n q_{n-1}^{(0)} + q_{n-1}^{(\epsilon_{n-1}(1))}}{q_{n-1}^{(0)}} \right] = m_n.$$

This implies that

$$\begin{aligned} T(\phi_n) &= T\left(\frac{q_n^{(1)}}{q_n^{(0)}}, \dots, \frac{q_n^{(d)}}{q_n^{(0)}}\right) \\ &= \text{ord}\left(\frac{q_{n-1}^{(\epsilon_{n-1}(1))}}{q_{n-1}^{(0)}}, \frac{q_{n-1}^{(1)}}{q_{n-1}^{(0)}}, \dots, \frac{q_{n-1}^{(\epsilon_{n-1}(1)-1)}}{q_{n-1}^{(0)}}, \frac{q_{n-1}^{(\epsilon_{n-1}(1)+1)}}{q_{n-1}^{(0)}}, \dots, \frac{q_{n-1}^{(d)}}{q_{n-1}^{(0)}}\right) \\ &= \phi_{n-1}, \end{aligned}$$

and $j(\phi_n) = \epsilon_{n-1}(1)$, $m(\phi_n) = m_n$. □

Lemma 4.1 demonstrates that

$$\phi_n, \phi_{n-1}, \phi_{n-2}, \dots$$

is indeed a trajectory of the d -dimensional Gauss transformation T . However, $j(\phi_n) = \epsilon_{n-1}(1)$ and in general $j(\phi_n) \neq j_n$. This means that the inverse branches connecting ϕ_{n-1} and ϕ_n , and $\omega^{(n)}$ and $\omega^{(n-1)}$ are different. Instead of j_n one has to use $l_n = \epsilon_{n-1}(1)$. Then

$$\phi_n = T_{(m_n, l_n)}^{-1} \phi_{n-1} \quad \text{and} \quad \omega^{(n-1)} = T_{(m_n, j_n)}^{-1} \omega^{(n)}.$$

Remark. Notice that the permutations ϵ_n are not defined for small n . This is because, for sufficiently small n , several vertices of Δ_n have the same denominator. However, there exists a random variable $n(\omega)$ such that, for $n \geq n(\omega)$, the denominators are ordered and ϵ_n is defined (see [6]).

4.2 Combinatorial properties and symmetry

In the previous section we introduced three variables: $j_n, l_n \in \{1, 2, \dots, d\}$ and $\epsilon_n \in S_d$. In this section we discuss the connections between them. We have already seen that $l_n = \epsilon_{n-1}(1)$.

For $1 \leq i \leq d$, let $\sigma_i = (\sigma_i(1), \dots, \sigma_i(d))$ denote the permutation $(2, 3, \dots, i, 1, i+1, \dots, d)$. Let $S_d(1) = \{\epsilon \in S_d : \epsilon(1) = 1\}$. Define $P : S_d \rightarrow S_d(1)$ by

$$(P\epsilon)(i) = \begin{cases} 1 & \text{if } i = 1; \\ \epsilon(i) & \text{if } i > 1 \text{ and } \epsilon(i) > \epsilon(1); \\ \epsilon(i) + 1 & \text{if } i > 1 \text{ and } \epsilon(i) < \epsilon(1). \end{cases} \quad (10)$$

It is easy to check that P can be represented as multiplication by the permutation $\sigma_{\epsilon(1)}$, namely

$$P\epsilon = \sigma_{\epsilon(1)} \cdot \epsilon.$$

Here we adopt the convention that permutations are to be composed from right to left. More precisely, if $\hat{\epsilon}$ is a bijection from $\{1, 2, \dots, d\}$ to itself associated to the permutation ϵ , i.e. $\hat{\epsilon} : i \mapsto \epsilon(i)$, then $\widehat{\epsilon_1 \cdot \epsilon_2} = \hat{\epsilon}_1 \circ \hat{\epsilon}_2 = \hat{\epsilon}_1(\hat{\epsilon}_2)$.

Define a permutation valued function

$$E(\epsilon, j) = (P\epsilon) \cdot \sigma_j = \sigma_{\epsilon(1)} \cdot \epsilon \cdot \sigma_j.$$

Notice that multiplication by σ_j transforms $P\epsilon$ in the following way: the entry 1 moves from the first to the j^{th} position.

Lemma 4.2. (i) $\epsilon_n = E(\epsilon_{n-1}, j_n)$, $l_n = \epsilon_{n-1}(1)$.
(ii) Let $\bar{\epsilon} = E(\epsilon, j)$. Then j is uniquely determined by $\bar{\epsilon}$, in fact

$$j = (\bar{\epsilon})^{-1}(1).$$

(iii) Let $\bar{\epsilon} = E(\epsilon, j)$ where $j = (\bar{\epsilon})^{-1}(1)$. Denote $\tau = \epsilon^{-1}$, $\bar{\tau} = (\bar{\epsilon})^{-1}$. Then $\tau = E(\bar{\tau}, l)$ where $l = \epsilon(1) = \tau^{-1}(1)$.

(iv) For all fixed $\bar{\epsilon}$ and l there exists a unique ϵ such that $\epsilon(1) = l$ and $\bar{\epsilon} = E(\epsilon, j)$ where $j = (\bar{\epsilon})^{-1}(1)$. Moreover,

$$\epsilon = (E((\bar{\epsilon})^{-1}, l))^{-1}.$$

Proof. (i) Notice that ϵ_{n-1} and j_n uniquely determine ϵ_n . It is easy to see that the definition of the function $E(\epsilon, j)$ exactly corresponds to the process of determining ϵ_n from ϵ_{n-1} and j_n . The permutation $P\epsilon$ corresponds to the order of the denominators when the new denominator is added and the denominator $q(n-1, 1)$ is deleted. The permutation $(P\epsilon_{n-1}) \cdot \sigma_{j_n}$ appears after the denominator $q(n-1, 0)$ is placed in the j_n^{th} position. Hence the first relation holds. The second relation is a trivial consequence of Lemma 4.1.

(ii) Obviously $\bar{\epsilon}(j) = 1$. Hence $j = (\bar{\epsilon})^{-1}(1)$.

(iii) Note that

$$\bar{\epsilon} = \sigma_{\epsilon(1)} \cdot \epsilon \cdot \sigma_j \quad \text{and} \quad \bar{\tau} = (\bar{\epsilon})^{-1} = \sigma_j^{-1} \cdot \epsilon^{-1} \cdot \sigma_{\epsilon(1)}^{-1}.$$

Hence $\tau = \epsilon^{-1} = \sigma_j \cdot \bar{\tau} \cdot \sigma_{\epsilon(1)}$. Since $j = \bar{\tau}(1)$, it follows that $\tau = E(\bar{\tau}, l)$, where $l = \epsilon(1) = \tau^{-1}(1)$.

(iv) The uniqueness of ϵ follows from (iii). Indeed $\bar{\epsilon} = E(\epsilon, j)$ implies that $\epsilon^{-1} = E((\bar{\epsilon})^{-1}, l)$ where $l = \epsilon(1)$. Hence

$$\epsilon = (E((\bar{\epsilon})^{-1}, l))^{-1}. \tag{11}$$

It is easy to check that ϵ given by (11) satisfies $\bar{\epsilon} = E(\epsilon, j)$, $\epsilon(1) = l$. □

Consider a two-sided sequence $(m_n, j_n)_{n \in \mathbb{Z}}$. We will suppose that this sequence is typical with respect to the invariant measure of the natural extension of T . In particular this means that for any finite sequence $(m^{(1)}, j^{(1)}), (m^{(2)}, j^{(2)}), \dots, (m^{(k)}, j^{(k)})$ there are infinitely many positive and negative integers n such that

$$(m_{n+s}, j_{n+s}) = (m^{(s)}, j^{(s)}), \quad s = 1, \dots, k.$$

This property is a consequence of Birkhoff's Ergodic Theorem, since for any finite sequence $(m^{(1)}, j^{(1)}), (m^{(2)}, j^{(2)}), \dots, (m^{(k)}, j^{(k)})$,

$$\tilde{\nu}(\{(m_n, j_n)_{n \in \mathbb{Z}} : (m_s, j_s) = (m^{(s)}, j^{(s)}), s = 1, \dots, k\}) > 0.$$

Denote a two-sided sequence $(j_n)_{n \in \mathbb{Z}}$ by J and let \mathcal{E} denote a two-sided sequence of permutations $(\epsilon_n)_{n \in \mathbb{Z}}$.

Definition 2. The sequence \mathcal{E} is said to be *compatible with J* if, for any $n \in \mathbb{Z}$, $\epsilon_n = E(\epsilon_{n-1}, j_n)$.

To establish the existence and uniqueness of a sequence \mathcal{E} which is compatible with J we will need the following Lemma:

Lemma 4.3. *Suppose that $n_0 < n$ and the finite sequence $j_{n_0+1}, j_{n_0+2}, \dots, j_{n-1}, j_n$ contains at least $d - 1$ entries d . For an arbitrary permutation ϵ_{n_0} define*

$$\epsilon_s = E(\epsilon_{s-1}, j_s) \quad \text{for } n_0 + 1 \leq s \leq n. \quad (12)$$

Then ϵ_n depends only on the sequence $(j_s)_{n_0+1 \leq s \leq n}$ and it does not depend on ϵ_{n_0} .

Proof. The Lemma has a purely combinatorial nature. We shall consider (12) as the iteration of a sequence of mappings $E(\cdot, j_s)$ acting on permutations with initial point ϵ_{n_0} . Each entry of the permutation ϵ_s except the first one gets mapped into some entry of ϵ_{s+1} which is either to the left of it or just above it (see Figure 1). $E(\cdot, j_s)$ also produces one entry 1 in the j_s position of ϵ_s and terminates the first entry of ϵ_{s-1} .

The whole process of iteration produces itineraries which originate either at one of the entries of the original permutation ϵ_{n_0} or at one of the new ones. Notice that the itinerary of every newly produced element is independent of ϵ_{n_0} and depends only on the future sequence of j_s 's. Hence the resulting permutation ϵ_n is independent of the original permutation ϵ_{n_0} if all the itineraries which start at the 0th level (i.e. the entries of ϵ_{n_0}) get terminated before n . Notice that if $j_s = d$ then all existing itineraries move one unit to the left, except the one which gets terminated. Thus after $d - 1$ iterations of $E(\cdot, d)$, all the itineraries which start at the 0th level will reach their left most position and will be terminated. Notice that because of monotonicity the last itinerary which will be terminated is the one which starts in the right most element of ϵ_{n_0} . \square

Let \mathcal{D} denote the set of two-sided sequences $J = (j_n)_{n \in \mathbb{Z}}$ for which there are infinitely many positive and negative integers n such that $j_n = d$.

Proposition 4.4. *If $J \in \mathcal{D}$ then there exists a unique sequence $\mathcal{E} = (\epsilon_n)_{n \in \mathbb{Z}}$ which is compatible with J .*

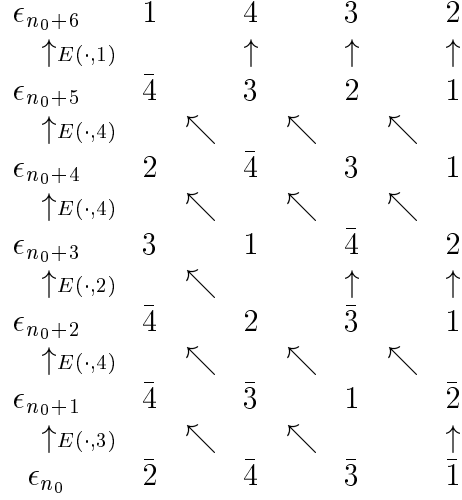


Figure 1: The orbit of the permutation $\epsilon_{n_0} = (2, 4, 3, 1)$ under repeated applications of $E(\cdot, j_s)$. Numbers with a bar over them denote elements of the itineraries of the elements of ϵ_{n_0} .

Proof. Uniqueness follows immediately from the previous Lemma. To prove existence we consider a sequence of one-sided sequences $(\epsilon_{-s}^{(s)}, \epsilon_{-s+1}^{(s)}, \dots)$ where $\epsilon_{-s}^{(s)}$ is an arbitrary permutation and $\epsilon_n^{(s)} = E(\epsilon_{n-1}^{(s)}, j_n)$, $n > -s$. It follows from the Lemma that for any $s \in \mathbb{Z}$

$$\epsilon_n^{(s)} \rightarrow \epsilon_n \text{ as } s \rightarrow \infty,$$

which simply means that $\epsilon_n^{(s)} = \epsilon_n$ for s large enough. Obviously, $\mathcal{E} = (\epsilon_n)_{n \in \mathbb{Z}}$ is a sequence which is compatible with J . \square

We can now give the definition of the compatibility of a sequence $L = (l_n)_{n \in \mathbb{Z}}$ with J . This definition follows from the relation $l_n = \epsilon_{n-1}(1)$.

Definition 3. A sequence $L = (l_n)_{n \in \mathbb{Z}}$ is said to be *compatible with J* if there exists a sequence $\mathcal{E} = (\epsilon_n)_{n \in \mathbb{Z}}$ which is compatible with J and for which $l_n = \epsilon_{n-1}(1)$ for all $n \in \mathbb{Z}$.

Proposition 4.5. *For an arbitrary sequence $J \in \mathcal{D}$ the sequence L which is compatible with J also belongs to \mathcal{D} .*

Proof. Consider the itinerary of the entry of ϵ_{n_0} which is equal to d (see Lemma 4.3). It does not change its value, but can only change its position. It moves one unit to the left every time we apply $E(\cdot, d)$. After at most $d - 1$ applications of $E(\cdot, d)$ the itinerary of the entry d will reach the left-most position. Hence for any finite sequence $j_s, n_0 \leq s \leq n$, which contains at least $d - 1$ entries d , there is at least one s for which $\epsilon_s(1) = d$. This implies that $l_{s+1} = d$. \square

Recall that the sequence $L = (l_n)_{n \in \mathbb{Z}}$ labels a backward sequence of the d -dimensional Gauss transformation. We can give the definition of the compatibility of a sequence of

permutations $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ with the sequence L , and hence with J . This definition is analogous to Definition 2.

Definition 4. (i) The sequence \mathcal{T} is *compatible with L* if, for any $n \in \mathbb{Z}$,

$$\tau_n = E(\tau_{n+1}, l_{n+1}).$$

(ii) The sequence \mathcal{T} is *compatible with J* if there exists a sequence L which is compatible with J such that \mathcal{T} is compatible with L .

If $J \in \mathcal{D}$ then $L \in \mathcal{D}$ and hence by Proposition 4.4 there exist unique \mathcal{E}, \mathcal{T} which are compatible with J . The compatibility of J, \mathcal{E}, L and \mathcal{T} is presented graphically in Figure 2.

$$\begin{array}{ccccccc} \dots & \xrightarrow{E(\cdot, j_n)} & \epsilon_n & \xrightarrow{E(\cdot, j_{n+1})} & \epsilon_{n+1} & \xrightarrow{E(\cdot, j_{n+2})} & \dots \\ & & \bullet & & \bullet & & \\ \dots & \xleftarrow{E(\cdot, l_n)} & \tau_n & \xleftarrow{E(\cdot, l_{n+1})} & \tau_{n+1} & \xleftarrow{E(\cdot, l_{n+2})} & \dots \\ & & \bullet & & \bullet & & \end{array}$$

Figure 2: The compatibility of (\mathcal{E}, J) and (\mathcal{T}, L) .

Theorem 1. Suppose $J \in \mathcal{D}$. Let $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ and $\mathcal{E} = (\epsilon_n)_{n \in \mathbb{Z}}$ be compatible with J . Then, for any $n \in \mathbb{Z}$, $\tau_n = \epsilon_n^{-1}$.

Proof. The sequence \mathcal{T} is uniquely defined by the condition of compatibility and thus it is enough to check that the sequence $(\epsilon_n^{-1})_{n \in \mathbb{Z}}$ is indeed compatible with L . Since

$$\epsilon_{n+1}^{-1} = (E(\epsilon_n, j_{n+1}))^{-1} = (\sigma_{\epsilon_n(1)} \cdot \epsilon_n \cdot \sigma_{j_{n+1}})^{-1} = \sigma_{j_{n+1}}^{-1} \cdot \epsilon_n^{-1} \cdot \sigma_{\epsilon_n(1)}^{-1}$$

we have

$$\epsilon_n^{-1} = \sigma_{j_{n+1}} \cdot \epsilon_{n+1}^{-1} \cdot \sigma_{\epsilon_n(1)}.$$

Notice that $j_{n+1} = \epsilon_{n+1}^{-1}(1)$ and $l_{n+1} = \epsilon_n(1)$. Thus

$$\epsilon_n^{-1} = \sigma_{\epsilon_{n+1}^{-1}(1)} \cdot \epsilon_{n+1}^{-1} \cdot \sigma_{l_{n+1}} = E(\epsilon_{n+1}^{-1}, l_{n+1}).$$

□

Consider a representation of the group S_d by permutation matrices. Namely, for any $\epsilon \in S_d$ consider a d -dimensional permutation matrix $V(\epsilon)$ which has 1 in the positions $(\epsilon(1), 1), (\epsilon(2), 2), \dots, (\epsilon(d), d)$ and 0's elsewhere. Let $Q(\epsilon)$ be the $(d+1)$ -dimensional matrix

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & V(\epsilon) \end{array} \right).$$

Notice that $Q(\epsilon)$ gives a $(d+1)$ -dimensional representation of the group S_d , i.e. $Q(\epsilon \cdot \bar{\epsilon}) = Q(\epsilon)Q(\bar{\epsilon})$ and $Q(\epsilon^{-1}) = (Q(\epsilon))^{-1}$. Since the matrices $Q(\epsilon)$ are orthogonal, we also have $Q^t(\epsilon) = (Q(\epsilon))^{-1} = Q(\epsilon^{-1})$. If $\epsilon(1) = 1$ then $V(\epsilon)$ is of the form

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & W(\epsilon) \end{array} \right)$$

where $W(\epsilon)$ is a $(d-1)$ -dimensional permutation matrix which has entries 1 in the positions $(\epsilon(i+1) - 1, i)$, $1 \leq i \leq d-1$. Again we have $W(\epsilon \cdot \bar{\epsilon}) = W(\epsilon)W(\bar{\epsilon})$ and $(W(\epsilon))^{-1} = W(\epsilon^{-1}) = W^t(\epsilon)$ assuming that $\epsilon(1) = \bar{\epsilon}(1) = 1$.

Recall the definition of the matrix $\tilde{A}_{(m,j)}$ (see equation (9) in Section 3), and that $A_{(m,j)} = \tilde{A}_{(m,j)}^t$.

Proposition 4.6. (i) For arbitrary m and ϵ

$$\tilde{A}_{(m,\epsilon(1))}Q(\epsilon) = \left(\begin{array}{cc|c} m & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & W(P\epsilon) \end{array} \right) \quad (13)$$

where $P\epsilon = \sigma_{\epsilon(1)} \cdot \epsilon$.

(ii) For arbitrary ϵ and j ,

$$P\epsilon = (P\bar{\tau})^{-1} \quad (14)$$

where $\bar{\tau} = (\bar{\epsilon})^{-1}$ and $\bar{\epsilon} = E(\epsilon, j)$.

(iii) For arbitrary m , j and ϵ

$$A_{(m,j)} = Q^{-1}(\bar{\epsilon})\tilde{A}_{(m,l)}Q(\epsilon) \quad (15)$$

where $\bar{\epsilon} = E(\epsilon, j)$ and $l = \epsilon(1)$.

Proof. (i) Clearly the first column of $\tilde{A}_{(m,\epsilon(1))}Q(\epsilon)$ coincides with the first column of $\tilde{A}_{(m,\epsilon(1))}$. For $2 \leq i \leq d+1$, the i^{th} column of $\tilde{A}_{(m,\epsilon(1))}Q(\epsilon)$ is equal to the $(\epsilon(i-1) + 1)^{\text{th}}$ column of $\tilde{A}_{(m,\epsilon(1))}$. This immediately implies that (13) is correct for the second column. Also, if $i > 2$ then the i^{th} column has only one non-zero entry, which is the entry 1 in the row $(\epsilon(i-1) + 2)$ if $\epsilon(i-1) < \epsilon(1)$ or in the row $(\epsilon(i-1) + 1)$ if $\epsilon(i-1) > \epsilon(1)$. Using (10) we find that the entry 1 is located in the $((P\epsilon)(i-1) + 1)^{\text{th}}$ row. Now consider the minor corresponding to the last $d-1$ rows and columns of $\tilde{A}_{(m,\epsilon(1))}Q(\epsilon)$. Take $k = i-2$ and consider the k^{th} column. The entry 1 is located in the $((P\epsilon)(k+1) + 1 - 2) = ((P\epsilon)(k+1) - 1)^{\text{th}}$ row. This implies (13).

(ii) Using $j = (\bar{\epsilon})^{-1}(1)$ we get

$$\begin{aligned} (P\bar{\tau})^{-1} &= (\sigma_{\bar{\tau}(1)} \cdot \bar{\tau})^{-1} = (\bar{\tau})^{-1} \cdot \sigma_{\bar{\tau}(1)}^{-1} = \bar{\epsilon} \cdot \sigma_{\bar{\tau}(1)}^{-1} \\ &= E(\epsilon, j) \cdot \sigma_{\bar{\tau}(1)}^{-1} = \sigma_{\epsilon(1)} \cdot \epsilon \cdot \sigma_j \cdot \sigma_{\bar{\tau}(1)}^{-1} = \sigma_{\epsilon(1)} \cdot \epsilon \cdot \sigma_j \cdot \sigma_{(\bar{\epsilon})^{-1}(1)}^{-1} \\ &= \sigma_{\epsilon(1)} \cdot \epsilon = P\epsilon. \end{aligned}$$

(iii) It is enough to show that $Q(\bar{\epsilon})A_{(m,j)} = \tilde{A}_{(m,l)}Q(\epsilon)$. Using (13) we get

$$\tilde{A}_{(m,l)}Q(\epsilon) = \tilde{A}_{(m,\epsilon(1))}Q(\epsilon) = \left(\begin{array}{cc|c} m & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & W(P\epsilon) \end{array} \right).$$

We also have

$$\begin{aligned} Q(\bar{\epsilon})A_{(m,j)} &= \left(\tilde{A}_{(m,j)}Q^t(\bar{\epsilon}) \right)^t = \left(\tilde{A}_{(m,j)}Q((\bar{\epsilon})^{-1}) \right)^t = \left(\tilde{A}_{(m,(\bar{\epsilon})^{-1}(1))}Q((\bar{\epsilon})^{-1}) \right)^t \\ &= \left(\begin{array}{cc|c} m & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & W(P((\bar{\epsilon})^{-1})) \end{array} \right)^t = \left(\begin{array}{cc|c} m & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & W^t(P((\bar{\epsilon})^{-1})) \end{array} \right) \\ &= \left(\begin{array}{cc|c} m & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & W((P((\bar{\epsilon})^{-1}))^{-1}) \end{array} \right). \end{aligned}$$

Using (14) we have $P\epsilon = (P((\bar{\epsilon})^{-1}))^{-1}$ which implies (15). \square

We now formulate a Theorem which relates the product of the matrices $A_{(m_n,j_n)}$ to the product of the matrices $A_{(m_n,l_n)}$.

Theorem 2. *Suppose \mathcal{E} and L are compatible with J . Then for an arbitrary sequence $M = (m_n)_{n \in \mathbb{Z}}$ and arbitrary $a < b$ we have*

$$(A_{(m_a,l_a)} \cdots A_{(m_b,l_b)})^t = Q(\epsilon_b)A_{(m_b,j_b)} \cdots A_{(m_a,j_a)}Q^{-1}(\epsilon_{a-1}). \quad (16)$$

Proof. It follows from Proposition 4.6 and the compatibility of \mathcal{E}, L and J that for any n

$$Q(\epsilon_n)A_{(m_n,j_n)}Q^{-1}(\epsilon_{n-1}) = \tilde{A}_{(m_n,l_n)}.$$

Taking the product over $a \leq n \leq b$ we get (16). \square

Remark. As we have seen above, the product of the matrices $A_{(m_n,j_n)}$ produces the approximations corresponding to the d -dimensional Gauss transformation. Theorem 2 says that forward iteration of the d -dimensional Gauss transformation and backward iteration produce the same matrix up to transposition and a change in the order of the rows and the columns. Notice that $Q^{-1}(\epsilon_{a-1}) = Q(\tau_{a-1})$. One can say that ϵ_{a-1} determines the correct order of the rows and τ_b the correct order of the columns.

Let us give one more definition which we shall use below. Let N be an arbitrary subset of \mathbb{Z} . Denote $\mathcal{E}_N = (\epsilon_n)_{n \in N}$, $L_N = (l_n)_{n \in N}$, $\mathcal{T}_N = (\tau_n)_{n \in N}$.

Definition 5. A configuration \mathcal{E}_N (respectively L_N, \mathcal{T}_N) is said to be *compatible with J* if there exists \mathcal{E} (respectively L, \mathcal{T}) which is compatible with J and is such that $\mathcal{E}|_N = \mathcal{E}_N$ (respectively $L|_N = L_N, \mathcal{T}|_N = \mathcal{T}_N$).

4.3 Coordinates for the natural extension

The aim of this section is to define new coordinates for the natural extension of the d -dimensional Gauss transformation. Instead of using a two-sided sequence $(m_n, j_n)_{n \in \mathbb{Z}}$, we use a two-sided sequence $M = (m_n)_{n \in \mathbb{Z}}$ and two one-sided sequences $L_- = (l_n)_{n \leq 0}$ and $J_+ = (j_n)_{n \geq 1}$ where L_- is a subsequence of the unique sequence L which is compatible with J . We also use a discrete coordinate $\epsilon_0 \in S_d$ which is the 0^{th} entry of the sequence \mathcal{E} which is compatible with J . As we have seen above, L_- and ϵ_0 are uniquely determined by J if $J \in \mathcal{D}$. The converse is also true: for arbitrary (L_-, J_+, ϵ_0) there exists a unique J such that ϵ_0 and L_- are compatible with J .

Let \mathcal{D}_+ (respectively \mathcal{D}_-) denote the set of one-sided sequences $J_+ = (j_n)_{n \geq 1}$ (respectively $L_- = (l_n)_{n \leq 0}$) which contain infinitely many entries equal to d .

Proposition 4.7. *If $L_- \in \mathcal{D}_-$ and $J_+ \in \mathcal{D}_+$ then for any ϵ_0 there exists a unique sequence $J = (j_n)_{n \in \mathbb{Z}} \in \mathcal{D}$ which coincides with J_+ for $n \geq 1$ and is such that ϵ_0 and L_- are compatible with J .*

Proof. It follows from Theorem 1 that

$$\epsilon_{n-1}^{-1} = E(\epsilon_n^{-1}, l_n).$$

Applying this formula repeatedly to ϵ_0 and the sequence $(l_0, l_{-1}, l_{-2}, \dots)$ we can define the sequence $(\epsilon_{-1}, \epsilon_{-2}, \dots)$. Hence we can determine $(j_0, j_{-1}, j_{-2}, \dots)$ using $j_n = \epsilon_n^{-1}(1)$. Obviously, J is the only sequence that can be compatible with L_- , J_+ and ϵ_0 . To see that it is indeed compatible it is enough to show that $J \in \mathcal{D}$. This easily follows from the argument used in Proposition 4.5. \square

Propositions 4.4, 4.5 and 4.7 imply that the mapping from $\{(M, J) : J \in \mathcal{D}\}$ into $\{(M, L_-, J_+, \epsilon_0) : L_- \in \mathcal{D}_-, J_+ \in \mathcal{D}_+\}$ is a bijection. Let $\tilde{\nu}_{-,+}$ denote the measure on $\{(M, L_-, J_+, \epsilon_0) : L_- \in \mathcal{D}_-, J_+ \in \mathcal{D}_+\}$ which is the image of the natural extension's invariant measure $\tilde{\nu}$ under this bijection. Denote the projection of $\tilde{\nu}_{-,+}$ onto $\{(M, L_-, J_+) : L_- \in \mathcal{D}_-, J_+ \in \mathcal{D}_+\}$ by $\bar{\nu}_{-,+}$.

Next we associate two vectors $\mathbf{x} = (x_1, \dots, x_d), \mathbf{y} = (y_1, \dots, y_d) \in \Delta^d$ to the sequences $(M_+, J_+), (M_-, L_-)$ (where $M_- = (m_n)_{n \leq 0}, M_+ = (m_n)_{n \geq 1}$). We do this by regarding the sequences as symbolic representations of \mathbf{y} and \mathbf{x} corresponding to the d -dimensional Gauss transformation. More precisely,

$$\mathbf{y} = [(m_0, l_0), (m_{-1}, l_{-1}), (m_{-2}, l_{-2}), \dots], \quad \mathbf{x} = [(m_1, j_1), (m_2, j_2), \dots].$$

We will show that this mapping from $\{(M, L_-, J_+) : L_- \in \mathcal{D}_-, J_+ \in \mathcal{D}_+\}$ into $\{(\mathbf{y}, \mathbf{x}) : \mathbf{y}, \mathbf{x} \in \Delta^d\}$ is well-defined on a set of full $\bar{\nu}_{-,+}$ -measure. Let $\tilde{\Phi}$ denote the inverse mapping which associates $((M_-, L_-), (M_+, J_+)) = (M, L_-, J_+)$ to (\mathbf{y}, \mathbf{x}) :

$$\tilde{\Phi}(\mathbf{y}, \mathbf{x}) = ((M_-, L_-), (M_+, J_+)) = (M, L_-, J_+).$$

Clearly, $\tilde{\Phi}$ is well-defined if \mathbf{x} and \mathbf{y} and their orbits $(T^n \mathbf{x})_{n \geq 1}, (T^n \mathbf{y})_{n \geq 1}$ under the Gauss transformation do not belong to the boundary of Δ^d .

Proposition 4.8. $\tilde{\Phi}$ is a bijection between a set of full Lebesgue measure in $\Delta^d \times \Delta^d$ and a set of full $\bar{\nu}_{-,+}$ -measure in $\{(M, L_-, J_+) : L_- \in \mathcal{D}_-, J_+ \in \mathcal{D}_+\}$.

Proof. Let \mathcal{M} denote the set of (M, L_-, J_+) for which there are infinitely many positive n 's such that

$$(m_{n+s}, j_{n+s}) = (1, d), \quad 0 \leq s \leq 2d - 1,$$

and infinitely many negative n 's such that

$$(m_{n+s}, l_{n+s}) = (1, d), \quad 0 \leq s \leq 2d - 1.$$

Let $\mathcal{Z} = \tilde{\Phi}^{-1}(\mathcal{M})$, i.e. \mathcal{Z} is the preimage of \mathcal{M} under $\tilde{\Phi}$. It follows from [6] that \mathcal{Z} has full Lebesgue measure and that $\tilde{\Phi}$ is a bijection between \mathcal{Z} and \mathcal{M} . To show that \mathcal{M} has full $\bar{\nu}_{-,+}$ measure, consider a set \mathcal{N} of sequences (M, J) such that for infinitely many positive and negative n 's

$$(m_{n+s}, j_{n+s}) = (1, d), \quad 0 \leq s \leq 3d - 1.$$

Clearly $\tilde{\nu}(\mathcal{N}) = 1$. Notice that if (M, J) has a piece of length $3d$ consisting of $(1, d)$'s then (M, L) has a corresponding piece of length $2d$ which consists of $(1, d)$'s. This implies that $\bar{\nu}_{-,+}(\mathcal{M}) \geq \tilde{\nu}(\mathcal{N})$ and hence $\bar{\nu}_{-,+}(\mathcal{M}) = 1$. \square

We are now ready to define an automorphism \tilde{T} which is metrically isomorphic to the natural extension of T . The definition below describes the unit shift on the space of two-sided sequences in terms of the coordinates $(\mathbf{y}, \epsilon, \mathbf{x})$. For $\mathbf{x}, \mathbf{y} \in \Delta^d$ and $\epsilon \in S_d$ let

$$\tilde{T}(\mathbf{y}, \epsilon, \mathbf{x}) = (\mathbf{y}', \epsilon', \mathbf{x}')$$

where

- (i) $\mathbf{x}' = T(\mathbf{x})$,
- (ii) $\epsilon' = E(\epsilon, j(\mathbf{x}))$,
- (iii) $\mathbf{y}' = T_{(m(\mathbf{x}), l)}^{-1}(\mathbf{y})$ where $l = \epsilon(1)$, i.e.

$$\mathbf{y}' = \left(\frac{1}{m(\mathbf{x}) + y_l}, \frac{y_1}{m(\mathbf{x}) + y_l}, \dots, \frac{y_{l-1}}{m(\mathbf{x}) + y_l}, \frac{y_{l+1}}{m(\mathbf{x}) + y_l}, \dots, \frac{y_d}{m(\mathbf{x}) + y_l} \right).$$

Although this definition appears to be a bit complicated, it does indeed correspond to the forward and backward dynamics of the d -dimensional Gauss transformation.

We hope that the properties which we will describe in the next section will provide ample motivation for our definition of \tilde{T} .

Remarks. (i) The transformation \tilde{T} is well-defined when \mathbf{x} does not belong to the boundary of Δ^d .

(ii) In the one-dimensional case, the transformation \tilde{T} coincides with the natural extension defined in Section 2. In this one-dimensional setting, the discrete coordinate ϵ is absent.

(iii) In the two-dimensional case, ϵ can be only one of two permutations: $(1, 2)$ and $(2, 1)$. We will say that $\epsilon = 1$ in the first case and $\epsilon = 2$ in the second case. With this notation,

Lemma 4.2 can be simplified. It is easy to see that $\epsilon_n = j_n$ and $l_n = \epsilon_{n-1}$. Thus $l_n = j_{n-1}$, i.e. the sequence of j 's is just the unit shift of the sequence of l 's. In this two-dimensional case, it is especially easy to see that L_-, J_+ and ϵ_0 allow one to construct the whole sequence of j 's. Indeed $j_0 = \epsilon_0$ and $j_n = l_{n+1}$ for $n \leq -1$.

(iv) Let $\mathbf{x}^{(0)} \in \Delta^d$ and $\epsilon_0 \in S_d$ be arbitrary, and let $\mathbf{y}^{(0)} = (0, \dots, 0)$. Define

$$(\mathbf{y}^{(n)}, \epsilon_n, \mathbf{x}^{(n)}) = \tilde{T}^n(\mathbf{y}^{(0)}, \epsilon_0, \mathbf{x}^{(0)}).$$

Then

$$\mathbf{y}^{(n)} = \left(\frac{q_n^{(1)}}{q_n^{(0)}}, \dots, \frac{q_n^{(d)}}{q_n^{(0)}} \right).$$

5 Properties of the d -dimensional Gauss transformation

In this section we formulate and prove the most important properties of \tilde{T} .

Let us denote the cardinality of a set \mathcal{S} by $\#(\mathcal{S})$. We define a measure χ on $\Delta^d \times S_d \times \Delta^d$ by

$$\chi(A_1 \times \mathcal{S} \times A_2) = \left(\int_{A_1} d\mathbf{y} \right) \left(\int_{A_2} d\mathbf{x} \right) (\#(\mathcal{S}))$$

for Borel subsets A_1 and A_2 of Δ^d and $\mathcal{S} \subset S_d$, i.e. χ is the direct product of Lebesgue measure on each copy of Δ^d and the counting measure on S_d .

Denote

$$\Omega^d = \{\mathbf{x} \in \Delta^d : T^n \mathbf{x} \in \text{Int}(\Delta^d) \text{ for all } n \geq 0\}.$$

Obviously, $\chi(\Omega^d \times S_d \times \Omega^d) = \chi(\Delta^d \times S_d \times \Delta^d)$, i.e. $\Omega^d \times S_d \times \Omega^d$ is a set of full χ -measure in $\Delta^d \times S_d \times \Delta^d$.

Proposition 5.1. \tilde{T} is a bijection from $\Omega^d \times S_d \times \Omega^d$ to itself.

Proof. It is easy to see that $\mathbf{y}' = T\mathbf{y} \in \text{Int} \Delta^d$ whenever $\mathbf{y} \in \text{Int} \Delta^d$. Hence, \tilde{T} maps $\Omega^d \times S_d \times \Omega^d$ into itself. The invertibility of \tilde{T} on $\Omega^d \times S_d \times \Omega^d$ follows immediately from its definition. Indeed, given $(\mathbf{y}', \epsilon', \mathbf{x}') \in \Omega^d \times S_d \times \Omega^d$ define $\mathbf{y} = T\mathbf{y}'$, $\epsilon = (E((\epsilon')^{-1}, l))^{-1}$ and $\mathbf{x} = T_{(m,j)}^{-1} \mathbf{x}'$ where $l = j(\mathbf{y}')$, $m = m(\mathbf{y}')$ and $j = (\epsilon')^{-1}(1)$. Then it is easy to check that $(\mathbf{y}, \epsilon, \mathbf{x})$ is the unique point in $\Omega^d \times S_d \times \Omega^d$ such that $\tilde{T}(\mathbf{y}, \epsilon, \mathbf{x}) = (\mathbf{y}', \epsilon', \mathbf{x}')$. \square

We now consider an invariant measure for \tilde{T} . Let $\tilde{\mu}$ be the probability measure on $\Delta^d \times S_d \times \Delta^d$ which, with respect to χ , has the density

$$\frac{d\tilde{\mu}}{d\chi}(\mathbf{y}, \epsilon, \mathbf{x}) = \frac{1}{C} \frac{1}{(1 + \sum_{i=1}^d x_i y_{\epsilon(i)})^{d+1}}$$

where C is a normalising constant:

$$C = \int_{\Delta^d \times S_d \times \Delta^d} \frac{1}{(1 + \sum_{i=1}^d x_i y_{\epsilon(i)})^{d+1}} \chi(d\mathbf{y}, d\epsilon, d\mathbf{x}).$$

We will also use the notation

$$f_\epsilon(\mathbf{y}, \mathbf{x}) = \frac{1}{(1 + \sum_{i=1}^d x_i y_{\epsilon(i)})^{d+1}}.$$

Theorem 3. $\tilde{\mu}$ is an invariant measure for \tilde{T} .

Proof. Consider a set $A_{\mathbf{y}} \times \{\epsilon\} \times A_{\mathbf{x}}$ where $A_{\mathbf{y}}, A_{\mathbf{x}} \subset \Delta^d$ and $\epsilon \in S_d$. Denote

$$A_{\mathbf{x}}^j = \{\mathbf{x} \in A_{\mathbf{x}} : j(\mathbf{x}) = j\}, \quad 1 \leq j \leq d.$$

Then

$$\tilde{\mu}(A_{\mathbf{y}} \times \{\epsilon\} \times A_{\mathbf{x}}) = \frac{1}{C} \sum_{j=1}^d \iint_{A_{\mathbf{y}} \times A_{\mathbf{x}}^j} f_\epsilon(\mathbf{y}, \mathbf{x}) d\mathbf{y} d\mathbf{x}.$$

Let $\tilde{T}_\epsilon : \Delta^d \times \Delta^d \rightarrow \Delta^d \times \Delta^d$ be the restriction of \tilde{T} on to the variables (\mathbf{y}, \mathbf{x}) with ϵ fixed. Then the measure of $\tilde{T}(A_{\mathbf{y}} \times \{\epsilon\} \times A_{\mathbf{x}})$ is given by

$$\begin{aligned} \tilde{\mu}(\tilde{T}(A_{\mathbf{y}} \times \{\epsilon\} \times A_{\mathbf{x}})) &= \frac{1}{C} \sum_{j=1}^d \iint_{\tilde{T}_\epsilon(A_{\mathbf{y}} \times A_{\mathbf{x}}^j)} f_{\epsilon'_j}(\mathbf{y}', \mathbf{x}') d\mathbf{y}' d\mathbf{x}' \\ &= \frac{1}{C} \sum_{j=1}^d \iint_{A_{\mathbf{y}} \times A_{\mathbf{x}}^j} f_{\epsilon'_j}(\tilde{T}_\epsilon(\mathbf{y}, \mathbf{x})) |\text{Jac}_\epsilon(\mathbf{y}, \mathbf{x})| d\mathbf{y} d\mathbf{x} \end{aligned}$$

where $\epsilon'_j = E(\epsilon, j)$ and Jac_ϵ denotes the Jacobian of the transformation \tilde{T}_ϵ . In order to prove that $\tilde{\mu}$ is an invariant measure one has to show that

$$f_\epsilon(\mathbf{y}, \mathbf{x}) = f_{\epsilon'_j}(\tilde{T}_\epsilon(\mathbf{y}, \mathbf{x})) |\text{Jac}_\epsilon(\mathbf{y}, \mathbf{x})|$$

for all \mathbf{x} such that $j(\mathbf{x}) = j$. This can be shown directly. Indeed a simple calculation shows that

$$|\text{Jac}_\epsilon(\mathbf{y}, \mathbf{x})| = \left(\frac{1}{y_{\epsilon(1)} + m(\mathbf{x})} \frac{1}{x_1} \right)^{d+1}.$$

Thus

$$f_{\epsilon'_j}(\mathbf{y}', \mathbf{x}') |\text{Jac}_{\epsilon'_j}(\mathbf{y}, \mathbf{x})| = \frac{1}{(1 + \sum_{i=1}^d x'_i y'_{\epsilon'_j(i)})^{d+1}} \frac{1}{((y_{\epsilon(1)} + m(\mathbf{x})) x_1)^{d+1}}$$

where $(\mathbf{y}', \mathbf{x}') = \widetilde{T}_\epsilon(\mathbf{y}, \mathbf{x})$. Since $x'_j = \frac{1}{x_1} - m(\mathbf{x})$, $\epsilon'_j(j) = 1$ and $y'_1 = \frac{1}{y_{\epsilon(1)} + m(\mathbf{x})}$ we have

$$\begin{aligned}
& \frac{1}{\left(1 + \sum_{i=1}^d x'_i y'_{\epsilon'_j(i)}\right)^{d+1}} \frac{1}{\left((y_{\epsilon(1)} + m(\mathbf{x})) x_1\right)^{d+1}} \\
&= \left(y_{\epsilon(1)} x_1 + m(\mathbf{x}) x_1 + x'_j y'_{\epsilon'_j(j)} (y_{\epsilon(1)} + m(\mathbf{x})) x_1 + \sum_{\substack{i=1 \\ i \neq j}}^d x'_i y'_{\epsilon'_j(i)} (y_{\epsilon(1)} + m(\mathbf{x})) x_1 \right)^{-(d+1)} \\
&= \left(y_{\epsilon(1)} x_1 + m(\mathbf{x}) x_1 + \left(\frac{1}{x_1} - m(\mathbf{x}) \right) y'_1 (y_{\epsilon(1)} + m(\mathbf{x})) x_1 \right. \\
&\quad \left. + \sum_{\substack{i=1 \\ i \neq j}}^d x'_i y'_{\epsilon'_j(i)} (y_{\epsilon(1)} + m(\mathbf{x})) x_1 \right)^{-(d+1)} \\
&= \left(1 + x_1 y_{\epsilon(1)} + \sum_{\substack{i=1 \\ i \neq j}}^d x'_i y'_{\epsilon'_j(i)} (y_{\epsilon(1)} + m(\mathbf{x})) x_1 \right)^{-(d+1)}.
\end{aligned}$$

Notice that for $i < j$

$$x'_i = \frac{x_{i+1}}{x_1} \quad \text{and} \quad y'_{\epsilon'_j(i)} = \frac{y_{\epsilon(i+1)}}{y_{\epsilon(1)} + m(\mathbf{x})}$$

and for $i > j$

$$x'_i = \frac{x_i}{x_1} \quad \text{and} \quad y'_{\epsilon'_j(i)} = \frac{y_{\epsilon(i)}}{y_{\epsilon(1)} + m(\mathbf{x})}.$$

Hence

$$\left(1 + x_1 y_{\epsilon(1)} + \sum_{\substack{i=1 \\ i \neq j}}^d x'_i y'_{\epsilon'_j(i)} (y_{\epsilon(1)} + m(\mathbf{x})) x_1 \right)^{-(d+1)} = \left(1 + \sum_{i=1}^d x_i y_{\epsilon(i)} \right)^{-(d+1)}. \quad \square$$

Remarks. (i) It is easy to see that for the probability measure $\widetilde{\mu}$ the conditional distributions on $\Delta^d \times \Delta^d$ under ϵ fixed are given by

$$\widetilde{\mu}(d\mathbf{y}, d\mathbf{x} | \epsilon) = \frac{1}{C(\epsilon)} \frac{1}{\left(1 + \sum_{i=1}^d x_i y_{\epsilon(i)}\right)^{d+1}} d\mathbf{y} d\mathbf{x}$$

where the $C(\epsilon)$ are normalising constants:

$$C(\epsilon) = \iint_{\Delta^d \times \Delta^d} \frac{1}{\left(1 + \sum_{i=1}^d x_i y_{\epsilon(i)}\right)^{d+1}} d\mathbf{y} d\mathbf{x}.$$

Obviously,

$$\sum_{\epsilon \in S_d} C(\epsilon) = C.$$

(ii) Let κ denote the marginal distribution of the measure $\tilde{\mu}$ on S_d , i.e. $\tilde{\mu}|_{S_d} = \kappa$. Then

$$\kappa(\epsilon) = \iint_{\Delta^d \times \Delta^d} \frac{1}{C} \frac{1}{(1 + \sum_{i=1}^d x_i y_{\epsilon(i)})^{d+1}} d\mathbf{y} d\mathbf{x} = \frac{C(\epsilon)}{C}.$$

Theorem 4. (i) *The automorphism $(\tilde{T}, \tilde{\mu})$ on $\Delta^d \times S_d \times \Delta^d$ is metrically isomorphic to the natural extension of the d -dimensional Gauss transformation.*

(ii) *$(\tilde{T}, \tilde{\mu})$ is a K -automorphism.*

(iii) *\tilde{T} is reversible with respect to the involution $S(\mathbf{y}, \epsilon, \mathbf{x}) = (\mathbf{x}, \epsilon^{-1}, \mathbf{y})$, i.e.*

$$\tilde{T}^{-1} = S\tilde{T}S.$$

Proof. (i) By Proposition 4.8, for Lebesgue almost all $\mathbf{x}, \mathbf{y} \in \Delta^d$ there exists a unique symbolic representation of (\mathbf{y}, \mathbf{x}) :

$$((M_-, L_-), (M_+, J_+)) = (M, L_-, J_+) = \tilde{\Phi}(\mathbf{y}, \mathbf{x})$$

where $L_- \in \mathcal{D}_-$ and $J_+ \in \mathcal{D}_+$. From Proposition 4.7, there exists a unique $J \in \mathcal{D}$ such that J_+ is the restriction of J onto $n \geq 1$, and L_-, ϵ_0 are compatible with J . Moreover, the transformation of (L_-, ϵ_0, J_+) onto $J \in \mathcal{D}$ is also one-to-one. Together these two facts imply that there is a one-to-one transformation ψ from a set of $(\mathbf{y}, \epsilon, \mathbf{x})$ of full χ -measure onto the space of two-sided sequences $(m_n, j_n)_{n \in \mathbb{Z}}$ where $J = (j_n)_{n \in \mathbb{Z}} \in \mathcal{D}$. It follows easily from the construction that \tilde{T} is conjugated by ψ to the unit shift of the sequence $(m_n, j_n)_{n \in \mathbb{Z}}$.

Denote the image of $\tilde{\mu}$ under ψ by $\tilde{\nu}$, i.e. $\tilde{\nu} = \psi\tilde{\mu}$. Obviously, $\tilde{\nu}$ is the measure corresponding to the natural extension; indeed it is translation invariant and its projection onto the space of one-sided sequences $(m_n, j_n)_{n \in \mathbb{N}}$ coincides with ν , which proves (i).

(ii) T is an exact endomorphism with respect to the invariant measure μ (see [9] or [15]). Hence its natural extension is a K -automorphism [20].

(iii) The property of reversibility easily follows from (i) and Theorem 1. Indeed, the unit shift of a two-sided sequence is always reversible with respect to the involution corresponding to the reflection $n \mapsto -n$. Using Theorem 1 it is easy to see that this involution gives S in the coordinates $(\mathbf{y}, \epsilon, \mathbf{x})$. However, we will give another proof of (iii) which is based on a direct calculation. Suppose $\mathbf{x}, \mathbf{y} \in \text{Int}(\Delta^d)$. Then $\tilde{T}(\mathbf{y}, \epsilon, \mathbf{x}) = (\mathbf{y}', \epsilon', \mathbf{x}')$ where $\mathbf{y}' \in \text{Int}(\Delta^d)$, $\mathbf{y} = T\mathbf{y}'$, $j(\mathbf{y}') = l = \epsilon(1)$, $m(\mathbf{y}') = m(\mathbf{x}) = m$ and $\epsilon' = \sigma_{\epsilon(1)} \cdot \epsilon \cdot \sigma_j = \sigma_l \cdot \epsilon \cdot \sigma_j$. Hence

$$\begin{aligned} S\tilde{T}S\tilde{T}(\mathbf{y}, \epsilon, \mathbf{x}) &= S\tilde{T}S(\mathbf{y}', \epsilon', \mathbf{x}') \\ &= S\tilde{T}(\mathbf{x}', (\epsilon')^{-1}, \mathbf{y}') \\ &= S(T_{(m,j)}^{-1}(\mathbf{x}'), \sigma_j \cdot \sigma_j^{-1} \cdot \epsilon^{-1} \cdot \sigma_l^{-1} \cdot \sigma_l, \mathbf{y}) \\ &= S(\mathbf{x}, \epsilon^{-1}, \mathbf{y}) \\ &= (\mathbf{y}, \epsilon, \mathbf{x}). \end{aligned}$$

Similarly,

$$\tilde{T}S\tilde{T}S(\mathbf{y}, \epsilon, \mathbf{x}) = (\mathbf{y}, \epsilon, \mathbf{x}).$$

This proves (iii). □

Corollary 1. *The involution S preserves the invariant measure $\tilde{\mu}$.*

Proof. It follows from the reversibility of \tilde{T} that $S\tilde{\mu}$ is also an invariant measure for \tilde{T} . Since \tilde{T} is ergodic and both $\tilde{\mu}$ and $S\tilde{\mu}$ are absolutely continuous with respect to χ we get that $S\tilde{\mu} = \tilde{\mu}$. \square

Consider the trajectory $(T^n \mathbf{x})_{n \geq 0}$ of an arbitrary $\mathbf{x} \in \Delta^d$ under the endomorphism T and the corresponding sequence of permutations $\epsilon_n(\mathbf{x})$. We have seen in Section 4 that, for almost all \mathbf{x} , $\epsilon_n(\mathbf{x})$ is well-defined for n large enough. We shall show that the stationary distribution for $\epsilon_n(\mathbf{x})$ is given by κ .

Corollary 2. *For any $\epsilon_0 \in S_d$ and Lebesgue almost every $\mathbf{x} \in \Delta^d$*

$$\frac{\#\{1 \leq n \leq N : \epsilon_n(\mathbf{x}) = \epsilon_0\}}{N} \rightarrow \kappa(\epsilon_0) \text{ as } N \rightarrow \infty.$$

Proof. Consider an observable $g_{\epsilon_0}(\mathbf{y}, \epsilon, \mathbf{x}) = \delta_{\epsilon_0}(\epsilon) = \begin{cases} 1 & \text{if } \epsilon = \epsilon_0; \\ 0 & \text{if } \epsilon \neq \epsilon_0. \end{cases}$ For $\tilde{\mu}$ -almost all $(\mathbf{y}, \epsilon, \mathbf{x})$ we have

$$\frac{1}{N} \sum_{n=0}^{N-1} g_{\epsilon_0}(\tilde{T}^n(\mathbf{y}, \epsilon, \mathbf{x})) \rightarrow \int_{\Delta^d \times S_d \times \Delta^d} g_{\epsilon_0} d\tilde{\mu} = \kappa(\epsilon_0).$$

Let $\chi_{\epsilon_0}(\epsilon)$ denote the characteristic function of ϵ_0 . Since for Lebesgue almost all \mathbf{x} , the sequence $g_{\epsilon_0}(\tilde{T}^n(\mathbf{y}, \epsilon, \mathbf{x}))$ does not depend on \mathbf{y} or ϵ for N large enough, and is equal to $\chi_{\epsilon_0}(\epsilon_n(\mathbf{x}))$, we get the required result. \square

6 Conclusions

We have shown that the d -dimensional Gauss transformation and its natural extension have many of the important ergodic and dynamic properties which are valid in the one-dimensional situation. We summarise these similarities below:

- (i) The invariant measures for both the d -dimensional Gauss transformation and its natural extension are given by explicit formulae. It is quite obvious that the explicit formula for the invariant measure of the natural extension is a generalisation of the corresponding formula in the one-dimensional case.
- (ii) The vectors $\phi_n = \left(\frac{q_n^{(1)}}{q_n^{(0)}}, \dots, \frac{q_n^{(d)}}{q_n^{(0)}}\right)$ are connected by the backwards d -dimensional Gauss transformation, i.e. $\phi_{n-1} = T^{-1}\phi_n$.
- (iii) The matrix $C_n(\mathbf{x}) = A_{(m_n, j_n)} \cdots A_{(m_1, j_1)}$ gives the vertices of the simplex $\Delta_n(\mathbf{x})$ which is the n^{th} approximation to \mathbf{x} , and also after taking its transpose and a suitable rearrangement of the rows and columns, the vertices of the simplex $\Delta_n(\phi_n)$.

Although there are many multidimensional generalisations of continued fractions, the d -dimensional Gauss transformation is the only one we know which enjoys the properties (i)–(iii). We believe that there is a connection between the existence of explicit formulas for the invariant density and the symmetry of the natural extension. These symmetries are “hidden”, i.e. non-obvious, in the d -dimensional case. One of the manifestations of the symmetry is the existence of an “almost” first integral. Define $F(\mathbf{y}, \epsilon, \mathbf{x}) = y_{\epsilon(1)} + \frac{1}{x_1}$. It is easy to see that

$$F(S\tilde{T}(\mathbf{y}, \epsilon, \mathbf{x})) = F(\mathbf{y}, \epsilon, \mathbf{x}).$$

Hence F is a first integral for $S\tilde{T}$. In the one-dimensional case, the existence of F allows one to construct an S -symmetric invariant absolutely continuous measure for $S\tilde{T}$ in the regular way, which gives a unique absolutely invariant measure for \tilde{T} (see [11]). We believe that such a construction can be carried out in the d -dimensional case as well.

Despite the many similarities between one-dimensional continued fractions and the d -dimensional Gauss algorithm, there do exist significant differences. The main difference is the presence of a discrete coordinate ϵ in the natural extension and the non-trivial dependence of the sequences L and J . The sequences \mathcal{E} , J and L are completely absent in the one-dimensional case. In fact the first really non-trivial case is $d = 3$. In the case $d = 2$, ϵ belongs to the commutative group \mathbb{Z}_2 and as a consequence the sequences J and L are related in an elementary way: L is the unit shift of the sequence J .

Another beautiful and important aspect of the classical theory of continued fractions is a deep connection between the one-dimensional Gauss automorphism and the geodesic flow on a surface of constant negative curvature. This connection was studied by R. Adler and L. Flatto [1], C. Series [24, 25] and recently by M. Kontsevich and Yu. Suhov [13]. It would be very interesting to find a similar geometrical interpretation in the d -dimensional case.

In this paper we have not discussed the convergence of the approximations provided by the d -dimensional Gauss algorithm. In fact the explicit forms for the invariant measure make it possible to give computer assisted proofs of almost everywhere strong convergence in dimensions 2 and 3 [7, 4, 6]. However we hope that the hidden symmetries which we have discussed here will eventually contribute to a conceptual proof of almost everywhere strong convergence which is currently an open problem.

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