A Law of Large Numbers for Random Hierarchical Sequences

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We generate a hierarchical sequence of random variables from a map taking k copies of a Banach space to itself. With a simple assumption on the map and an integrability condition for the initial random variables we prove that the renormalized sequence converges almost surely to a constant. We apply this result to the homogenization of Laplace operators on fractals.

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Abstract

We generate a hierarchical sequence of random variables from a map taking k copies of a Banach space to itself. With a simple assumption on the map and an integrability condition for the initial random variables we prove that the renormalized sequence converges almost surely to a constant. We apply this result to the homogenization of Laplace operators on fractals.

1 Introduction

Let $\{X_i^{(0)}; i \in \mathbb{N}\}$ be a sequence of independent and identically distributed random variables taking values in a Banach space, B. Consider a map $f : B^k \to B$, a function of k Banach space valued variables. We can use this map to generate a sequence of random variables by setting

$$X_i^{(n+1)} = f(X_{(i-1)k+1}^{(n)}, \ldots, X_{ik}^{(n)}), \quad i \in \mathbb{N}, \quad \forall n \geq 0.$$

The question of interest to us is a simple version of renormalization group theory concerning the convergence of the hierarchical sequence given by $\{X_1^{(n)}; n \in \mathbb{N}\}$.

The renormalization group is a widely used technique of mathematical physics which is often difficult to make mathematically rigorous. Hierarchical models provide a setting where exact computations are possible and they are also interesting models in their own right. In this paper we will establish an abstract renormalization theorem, giving a condition on the map f which guarantees almost sure convergence of the sequence to a constant, under a $1 + \delta$ moment condition on $\|X_1^{(0)}\|$. Note that the usual strong law of large numbers can be thought of as a special case by setting $f(x_1, \ldots, x_k) = (x_1 + \cdots + x_k)/k$. Our condition on f will be satisfied for concave homogeneous functions with a non-trivial fixed point. The hierarchical sequence generated by a concave homogeneous function occurs in the renormalization of electrical networks, where the effective resistance is a concave function of the individual resistors in the network. Previous work on this problem can be found in [17, 18, 10, 7] in the case of real valued random variables.

A motivating example for this work is the homogenization problem for finitely ramified fractals. The homogenization problem is to determine the effective macroscopic properties of a medium which has microscopic irregularities. There are well developed methods for tackling such problems in translation invariant settings such as $\mathbb{R}^d$, [15] and $\mathbb{Z}^d$, [3],

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however new techniques are required for fractals. We will consider graphs whose large scale structure is that of a finitely ramified fractal and with random resistors on each edge. We establish a homogenization result by proving that the rescaled effective conductivity across the fractal graph converges to a non-trivial limit.

A finitely ramified fractal is a fractal subset of $\mathbb{R}^d$ with the property that any connected subset can be disconnected by the removal of only a finite number of points. The Sierpinski gasket provides a basic example. The construction of a Laplace operator on a finitely ramified fractal already involves a deterministic renormalization problem. The Laplacian is obtained from a compatible sequence of electrical networks which approximate the fractal. For the networks to be compatible they must be constructed from the scaled fixed point of a renormalization map from one level to the next. The scaling is called the conductance scale factor. For general finitely ramified self-similar sets there is a deterministic fixed point problem which needs to be solved in order to deduce the existence of an operator. As the electrical network can be viewed as a Dirichlet form, the fixed point problem can be regarded as an eigenvalue problem on the cone of Dirichlet forms. Conditions for the existence and uniqueness of a fixed point have been found in [16].

We will consider a random version of this fixed point problem. We take an infinite fractal lattice, consisting of translates of a basic cell, and place random resistors on each such basic cell, chosen as random elements of the cone of Dirichlet forms on the cell. We rescale the lattice by the natural length scale factor and consider the convergence of the induced effective conductance across the unit cell rescaled by the conductance scale factor. This problem was first considered in [9] in which it was shown that for nested fractals, under some assumptions corresponding to the uniqueness of the fixed point and boundedness of the conductors, that there was L^1 convergence of the conductance. In [5] the Vicsek set, a nested fractal for which there is not a unique fixed point for the renormalized conductivity map was discussed. However the Vicsek set is sufficiently simple that the renormalization map is effectively linear and the strong law of large numbers can be used to give the almost sure convergence of the effective conductance under a finite first moment condition on the conductors. Here we will prove that, for a more general map allowing weights on different parts of the fractal and only assuming a finite $1+\delta$ moment condition, the rescaled effective conductance converges almost surely. However, to ensure positivity of the limit in general, we require our conductors to be bounded away from 0.

The paper will begin with some preliminary results giving the necessary background for the convergence of reversed submartingales in Banach spaces. We will state and prove our main result in Section 3 and in the final section apply it to the homogenization problem on nested fractals.

2 Preliminaries

In order to prove our results we introduce a suitable setting for understanding convergence of hierarchical sequences. The main result we will need is the reversed submartingale convergence theorem for Banach spaces. We will also record a version of the Hahn-Banach theorem needed for our application.

Let B be a σ-complete Banach lattice with the property that the norm is continuous in the order topology. (For the properties of Banach lattices that we use here we refer the reader to [1], [19]). A sequence of strongly measurable random variables $\{X_n\}_{n\geq 0}$ with respect to a decreasing filtration $\{\mathcal{F}_n\}_{n\geq 0}$ is a reversed submartingale if for each $n \leq 0$, $E|X_n| \in B$ and

$$X_{n+1} \leq E(X_n|\mathcal{F}_{n+1}).$$

We will denote by P the positive cone in the Banach space B. Unless stated otherwise convergence in B will be in the metric topology.

Theorem 2.1 *Let $\{X_n\}$ be a reversed submartingale in P with respect to a decreasing filtration $\mathcal{F}_n$. Then X_n converges almost surely (in the metric topology) to a random variable X_∞.*

Proof: We prove this using a martingale convergence theorem for Banach spaces and a Doob decomposition. Observe that we can write the submartingale in terms of a martingale M_n null at 0, and a decreasing process A_n as $X_n = M_n + A_n$ where

$$\begin{aligned} M_n &= \sum_{i=1}^{n} E(X_{i-1}|\mathcal{F}_i) - X_{i-1}, \\ A_n &= X_0 + \sum_{i=1}^{n} E(X_i - X_{i-1}|\mathcal{F}_i). \end{aligned}$$

The martingale part M_n is a reversed martingale and hence converges (in the metric topology) by the usual reversed martingale convergence theorem for Banach spaces, [2] Theorem 4.

For the decreasing part we have

$$0 \leq E(A_n) = E(X_n) \leq E(X_0) = X_0.$$

We want to prove that the sequence A_n will converge. In order to do this we will show that $0 \leq A_n \leq X_0$. The upper bound is obvious. For the lower bound we assume that the sequence A_n leaves the positive cone. Thus there exists an element $\xi = A_{p+1}$ where $p = \sup\{n : A_n \in P\}$, the first point at which the sequence was not in the cone. As the sequence A_n is decreasing we see that $a_n = P(A_n \leq \xi) \to 1$ as $n \to \infty$. Now

$$0 \leq E(A_n) \leq a_n\xi + (1 - a_n)X_0 \to \xi,$$

as $n \to \infty$. Thus we have a contradiction and the sequence must remain in the positive cone.

As we have a monotone decreasing sequence which is order bounded below it converges in the order topology and hence in the norm. Thus $X_n \to X_\infty$ almost surely as required. ∎

Corollary 2.2 *A reversed submartingale in B which is bounded below converges almost surely.*

We will be studying concave functions on the k-fold products of the Banach space and the following lemma gives us control on such functions.

Lemma 2.3 *Let $f : P^k \to P$ be a concave homogeneous function for which there is a fixed point $x^0 \in P \backslash \{0\}$, that is $f(x^0, \ldots, x^0) = x^0$. Then there exists $w \in \mathbb{R}_+^k$ such that $\sum_{i=1}^k w_i = 1$ and*

$$f(x_1, \ldots, x_k) \leq \sum_{i=1}^k w_i x_i.$$

Proof: We use the geometric version of the Hahn-Banach Theorem as given in [4] p117. Consider the Banach space $E = B^{k+1}$ and let

$$A = \{\mathbf{x} \in (P^\circ)^{k+1} : f(x_1, \ldots, x_k) > x_{k+1}\}.$$

It is easy to see by the concavity of f that the set A is convex and open. Let $\mathbf{x}^0$ be the point containing $k+1$ copies of the fixed point and let $M = \{\alpha \mathbf{x}^0 : \alpha \in \mathbb{R}\}$. Observe that, by the homogeneity of f, $M \cap A = \emptyset$. Applying the Hahn-Banach Theorem, there exists a hyperplane H such that $M \subset H$ and $H \cap A = \emptyset$. Since the hyperplane H in E contains the origin it can be written as $\sum_{i=1}^{k+1} u_i x_i = 0$. As the point $\mathbf{x}^0$ lies in H and is non-zero we see that $\sum_{i=1}^{k+1} u_i = 0$ and we can write the hyperplane as

$$x_{k+1} = \sum_{i=1}^k w_i x_i,$$

where $\sum_{i=1}^k w_i = 1$. Now as $H \subset A^c$ we have

$$f(x_1, \ldots, x_k) \leq x_{k+1} = \sum_{i=1}^k w_i x_i.$$

To see that the $w_i \geq 0$ we observe that, if not, we get a contradiction to the positivity of f. ∎

3 Convergence under Renormalization

The results of this section extend those of [17] and [18] in the real valued case. As in Section 2, B is a σ-complete Banach lattice with the property that the norm is continuous in the order topology. We also assume that B is a Banach space of type p for some $p > 1$ and recall the definition. Let

$$C(B) = \{\{x_n\} : \sum_n \epsilon_n x_n \text{ converges in probability}\},$$

and write $\{\epsilon_n\}$, for a sequence of independent Bernoulli random variables, taking the value ± 1 with equal probability. A Banach space is said to be of type p (for $1 \leq p \leq 2$) if there exists $A \in \mathbb{R}_+$ such that

$$E\|\sum_n \epsilon_n x_n\|^p \leq A \sum_n \|x_n\|^p,$$

for every sequence $\{x_n\} \in C(B)$.

Theorem 3.1 *Let $f : B^k \to B$ be a function bounded below with the property that there exists $w \in \mathbb{R}_+^k$ such that $w_1 + \cdots + w_k = 1$ and*

$$f(x_1, \ldots, x_k) \leq w_1 x_1 + \cdots + w_k x_k$$

for all $x \in B^k$. Let $X_1, X_2, \ldots$ be a sequence of iid integrable random variables in B with $E\|X_1\|^p < \infty$. For each j, set $X_j^{(0)} = X_j$, and for $n \geq 0$,

$$X_i^{(n+1)} = f(X_{(i-1)k+1}^{(n)}, \ldots, X_{ik}^{(n)}), \quad i \in \mathbb{N}. \tag{3.1}$$

Then the sequence $X_1^{(n)}$ converges almost surely to a constant in B.

We will prove this via a series of Lemmas.

Let $\tilde{X}_i^{(n)}$ be the sequence of random variables constructed as in (3.1) with the linear function $f(x_1, \ldots, x_k) = w_1 x_1 + \cdots + w_k x_k$. Note that we can write $X_1^{(n)} = \sum_{i=1}^{k^n} a_i^{(n)} X_i$.

Lemma 3.2 *There exist constants $b > 1, c > 0$, such that*

$$E\|\tilde{X}_1^{(n)} - E(\tilde{X}_1^{(n)})\|^p \leq c b^{-n} E\|X^{(0)}\|^p.$$

Proof: We first show that

$$\sum_{i=1}^{k^n} (a_i^{(n)})^p \leq b^{-n}. \tag{3.2}$$

Note that the $a_i^{(n)}$ are the terms in the expansion of $(w_1 + \ldots + w_k)^n$. Hence $\sum_{i=1}^{k^n} (a_i^{(n)})^p = (w_1^p + \ldots + w_k^p)^n$. As x^p is strictly convex for $p > 1$ and $w_i > 0$ for $i = 1, \ldots, k$,

$$b^{-1} = \sum_{i=1}^{k} w_i^p < \left(\sum_{i=1}^{k} w_i \right)^p = 1.$$

Thus we have the inequality (3.2).

As B is a type p Banach space we have the following ([6] Theorem 2.1), that there exists an $A > 0$ such that

$$E\|\sum_{i=1}^{k^n} U_i\|^p \leq A \sum_{i=1}^{k^n} E\|U_i\|^p, \tag{3.3}$$

for any sequence of independent zero mean random variables $\{U_i\}$ with finite p-th moment. Without loss of generality we can assume that $EX_i^{(0)} = 0$, and applying (3.3) we have

$$\begin{aligned} E\|\tilde{X}_1^{(n)}\|^p &= E\|\sum_{i=1}^{k^n} a_i^{(n)} X_i^{(0)}\|^p \\ &\leq A \sum_{i=1}^{k^n} (a_i^{(n)})^p E\|X_i^{(0)}\|^p. \end{aligned}$$

Finally, using (3.2), we have the result. ∎

Lemma 3.3 *If $E(\|X_1^{(0)}\|^p) < \infty$, then $E(\|X_1^{(n)}\|^p)$ is uniformly bounded.*

Proof: Firstly we observe that as f is bounded below there is an $l \in B$ such that $l \leq X_1^{(n)} \leq \tilde{X}_1^{(n)}$ almost surely. It is a standard fact about Banach lattices that if $0 \leq x \leq y$, then $\|x\| \leq \|y\|$. Thus, combining this with the triangle inequality we have

$$\begin{aligned} 2^{1-p} E\|X_1^{(n)}\|^p &\leq E\|X_1^{(n)} + l\|^p + \|l\|^p \\ &\leq E\|\tilde{X}_1^{(n)} + l\|^p + \|l\|^p \\ &\leq 2^{p-1}\left(E\|\tilde{X}^{(n)} - EX_1\|^p + \|l - EX_1\|^p\right) + \|l\|^p. \end{aligned}$$

Combining this with Lemma 3.2 yields the result. ∎

Lemma 3.4 *The sequence $\{X_1^{(n)}\}$ converges in probability.*

Proof: By the σ-lattice property and the fact that f is bounded below we have $E(X_1^{(n)}) \downarrow v$ for some v.

By hypothesis we have $X_1^{(n)} \leq \tilde{X}_1^{(n)}$ almost surely. It follows from this that

$$\limsup_{n\to\infty} X_1^{(n)} \leq EX_1^{(0)}, \text{ almost surely.}$$

The sequence $\{\tilde{X}_1^{(n)}\}$ can be defined by replacing $\{X_j^{(0)}; j \geq 1\}$ by $\{X_j^{(n_0)}; j \geq 1\}$ for any n_0. Thus

$$\limsup_{n\to\infty} X_1^{(n)} \leq EX_1^{(n_0)}, \text{ almost surely.}$$

Letting $n_0 \to \infty$ gives

$$\limsup_{n\to\infty} X_1^{(n)} \leq v, \text{ almost surely.} \tag{3.4}$$

We now assume that $v = 0$. Let $\epsilon \in P$, $m > 0$, and consider the following events

$$\begin{aligned} A_0 &= \{X_1^{(n)} < \epsilon\}, \\ A_1 &= A_0 \cap \{\|X_1^{(n)} - \epsilon\| \leq m + 2\|\epsilon\|\}, \\ C &= A_0 \cap A_1^c. \end{aligned}$$

Now

$$EX_1^{(n)} = EX_1^{(n)} I_{A_0^c} + EX_1^{(n)} I_{A_1} + EX_1^{(n)} I_C.$$

By Lemma 3.3, (3.4) and Hölder's inequality,

$$\|E(X_1^{(n)} I_{A_0^c})\| \leq (E\|X_1^{(n)}\|^p)^{1/p} (P(A_0^c))^{1-1/p} \to 0. \tag{3.5}$$

Now

$$\begin{aligned} \|\epsilon\| + \|E(X_1^{(n)} I_{A_0})\| &\geq \|E(\epsilon - X_1^{(n)}) I_{A_0}\| \\ &= \|E(\epsilon - X_1^{(n)}) I_{A_1} + E(\epsilon - X_1^{(n)}) I_C\| \\ &\geq \|E(\epsilon - X_1^{(n)}) I_C\| \\ &\geq mP(C). \end{aligned}$$

By (3.5) and the fact that $\|E(X_1^{(n)})\| \to 0$ we have that $\|E(X_1^{(n)} I_{A_0})\| \to 0$. Thus for any $\delta > 0$, we can choose ϵ with $\|\epsilon\| = \delta m/2$; moreover there exists an n_0 such that $\|E(X_1^{(n)} I_{A_0})\| < \delta m/2$ for all $n > n_0$. Hence $P(C) \leq \delta$ for $n > n_0$.

Thus there is an n_1 such that $P(X_1^{(n)} \in A_1) > 1 - 2\delta$, and hence $P(\|X_1^{(n)}\| < m + 3\|\epsilon\|) = P(\|X_1^{(n)}\| < m(1+3\delta/2)) > 1 - 2\delta$ for $n > n_1$. Since m and δ are arbitrary we are done. ∎

To prove almost-sure convergence we will use a randomization technique introduced in [7]. Define random variables $Y_i^{(n)}$ as follows. For each j, set $Y_j^{(0)} = X_j$. For permutations $\sigma \in S_k$, define σf by

$$(\sigma f)(x_1, \ldots, x_k) = f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}).$$

Now let $\sigma_i^{(n)}$ be iid, uniformly distributed on S_k, and set $F_i^{(n)} = \sigma_i^{(n)} f$. For $n \geq 0$, set

$$Y_i^{(n+1)} = F_i^{(n)}(Y_{(i-1)k+1}^{(n)}, \ldots, Y_{ik}^{(n)}), \quad i \in \mathbb{N}.$$

Let $Z_n = \sum_{i=1}^k Y_i^{(n)}$.

Lemma 3.5 *The sequence $\{Z_n\}$ converges almost surely.*

Proof: We will show that Z_n is a reversed submartingale with respect to the decreasing filtration $\mathcal{F}_n = \sigma(Z^{(l)},\ l \geq n)$, and apply Corollary 2.2.

By symmetry, $Ew_{\sigma_i^{(n)}(j)} = 1/k$, for each j and the sequence $\sigma^{(n)}$ is independent of $\mathcal{F}_{n+1}$. Thus,

$$\begin{aligned}
E(Z_n|\mathcal{F}_{n+1}) &= k^{-1}E(Y_1^{(n)} + \cdots Y_k^{(n)} + \cdots + Y_{k^2-k+1}^{(n)} + \cdots + Y_{k^2}^{(n)}|\mathcal{F}_{n+1}) \\
&= E(w_{\sigma_1^{(n)}(1)} Y_1^{(n)} + \cdots + w_{\sigma_1^{(n)}(k)} Y_k^{(n)} + \cdots \\
&\qquad \cdots + w_{\sigma_k^{(n)}(1)} Y_{k^2-k+1}^{(n)} + \cdots + w_{\sigma_k^{(n)}(k)} Y_{k^2}^{(n)}|\mathcal{F}_{n+1}) \\
&\geq E(F_1^{(n)}(Y_1^{(n)}, \ldots, Y_k^{(n)}) + \cdots + F_k^{(n)}(Y_{k^2-k+1}^{(n)}, \ldots, Y_{k^2}^{(n)})|\mathcal{F}_{n+1}) \\
&= Z_{n+1},
\end{aligned}$$

as required. ∎

Proof of Theorem: We show that the sequence $\{X_1^{(n)}\}$ converges almost surely to v, the limit arising in the proof of Lemma 3.4.

Observe that for each n, $X^{(n)}$ has the same law as $Y^{(n)}$. It follows that Z_n has the same law as $\sum_{i=1}^k X_i^{(n)}$ for each n. By Lemma 3.4 $\{X_i^{(n)}\}$ converges in probability to v for each i and hence, as $\{X_i^{(n)}; i \geq 1\}$ is an iid sequence, $\{Z_n\}$ converges in probability to kv. Combining this with Lemma 3.5 we see that $\{Z_n\}$ converges almost surely to kv.

We will now deduce that $Y_k^{(n)}$ converges almost surely to v. This will follow from the fact that $\{Y_i^{(n)}; i \geq 1\}$ is an iid sequence.

Assume without loss of generality that $v = 0$. Fix $\epsilon > 0$ and suppose $\sum_{i=1}^{k-1} Y_i^{(n)}$ does not converge almost surely to 0. Then we can define a sequence of finite stopping times

$$\begin{aligned} T_0 &= \inf\{n \geq 0 : \| \sum_{i=1}^{k-1} Y_i^{(n)} \| > \epsilon\} \\ T_i &= \inf\{n > T_{i-1} : \| \sum_{i=1}^{k-1} Y_i^{(n)} \| > \epsilon\}. \end{aligned}$$

Since $Z_n \to 0$ almost surely there exists an n_0 such that $\|Z_n\| < \epsilon/2$ for all $n > n_0$ and hence $\|Y_k^{(T_i)}\| > \epsilon/2$ for all i sufficiently large. Note that $Y_k^{(T_i)}$ is independent of $\mathcal{G}_{T_i} = \sigma(\sum_{i=1}^{k-1} Y_i^{(n)} : n \leq T_i)$.

Lemma 3.6 *Let $\{U_n\}$ be a sequence of random variables in B and $\{S_n\}$ a strictly increasing sequence of stopping times with respect to an increasing filtration $\mathcal{F}_n$. Assume $U_{n-1} \in \mathcal{F}_n$, U_n is independent of $\mathcal{F}_n$ and $U_n \to 0$ in probability, then $U_{S_n} \to 0$ in probability.*

Proof: Let $\epsilon > 0$. By independence,

$$\begin{aligned} P(\|U_{S_n}\| > \epsilon) &= \sum_j P(\|U_{S_n}\| > \epsilon | S_n = j) P(S_n = j) \\ &= \sum_j P(\|U_j\| > \epsilon) P(S_n = j). \end{aligned}$$

Since $U_n \to 0$ in probability, there exists a J such that

$$P(\|U_j\| > \epsilon) < \epsilon/2, \quad j > J.$$

Furthermore, since $S_n \to \infty$ there exists an N such that

$$\sum_{j \leq J} P(S_n = j) < \epsilon/2 \quad n > N.$$

Hence $P(\|U_{S_n}\| > \epsilon) < \epsilon$ for $n > N$ as required. ∎

Our sequence $Y_k^{(n)}$, stopping times T_n and filtration $\mathcal{G}_n$ satisfy the conditions of the above Lemma and hence $Y_k^{(T_i)} \to 0$ in probability as $i \to \infty$.

Now, using the triangle inequality, we have

$$\begin{aligned} P(\|Y_k^{(T_i)}\| > \epsilon/2) &= P(\|Y_k^{(T_i)}\| > \epsilon/2 | \| \sum_{j=1}^{k-1} Y_j^{(T_i)} \| > \epsilon) \\ &\geq P(\| \sum_{j=1}^{k} Y_j^{(T_i)} \| < \epsilon/2 | \| \sum_{j=1}^{k-1} Y_j^{(T_i)} \| > \epsilon) \\ &= P(\| \sum_{j=1}^{k} Y_j^{(T_i)} \| < \epsilon/2). \end{aligned}$$

By our choice of stopping times we have a contradiction as the right hand side converges to 1 while the left hand side converges to 0. Thus $\sum_{i=1}^{k-1} Y_i^{(n)} \to 0$ almost surely and hence $Y_k^{(n)} \to 0$ almost surely.

Using the mutual independence of the sequence of random variables $\{Y_k^{(n)} : n \geq 0\}$ (as $k > 1$) and Borel-Cantelli

$$\sum_n P(\|Y_k^{(n)}\| > \epsilon) < \infty.$$

Hence

$$\sum_n P(\|X_1^{(n)}\| > \epsilon) < \infty,$$

so that $X_1^{(n)}$ converges almost surely. ∎

Combining the above Theorem with Lemma 2.3 we have

Corollary 3.7 *Let $f : P^k \to P$ be a concave homogeneous function with a non-zero fixed point. Then $X_1^{(n)}$ converges almost surely to a constant in P.*

3.1 On the positivity of the limit

We have established convergence for our hierarchical sequence but there is still the possibility that the limit is 0. We give some conditions which will ensure strict positivity.

Let $p_n = P(X_1^{(n)} > 0)$. Following [14] we define

$$\mathcal{U} = \{A \subset \{1, \ldots, k\} : x_j > 0, \forall j \in A \Rightarrow f(x_1, \ldots, x_k) > 0, \ \forall x_i \text{ with } i \in A^c\},$$

and

$$g(p) = \sum_{A \in \mathcal{U}} p^{|A|}(1-p)^{k-|A|}.$$

Lemma 3.8 *If $f : P^k \to P$ is concave, then $p_{n+1} = g(p_n)$ for all n.*

Proof: It suffices to show that

$$x_j > 0, \forall j \in A \Rightarrow f(x_1, \ldots, x_k) > 0, \ \forall x_i \text{ with } i \in A^c$$

if and only if

$$\exists x_j > 0, j \in A \text{ such that } f(x_1, \ldots, x_k) > 0, \ \forall x_i \text{ with } i \in A^c.$$

We will use the positivity and concavity of the function f. Fix A and label its elements $i_1, \ldots, i_m$. Now define $f_A : P^{|A|} \to P$ by

$$f_A(x_1, \ldots, x_m) = \inf\{f(y_1, \ldots, y_k) : y_{i_j} = x_j, j = 1, \ldots, m\}.$$

Note that f_A is concave.

We will show that if f_A is positive at any point in the interior of $K = P^{|A|}$, then f_A is positive throughout the interior. Suppose otherwise: Suppose $f_A(x) > 0$ for some $x \in K^\circ$

and $f_A(y) = 0$ for some $y \in K^\circ$. Let $l(t) = x + t(y-x)$ for $t \geq 0$. There exists $\epsilon > 0$ such that $z = l(1+\epsilon) \in K^\circ$. Note that

$$y = \frac{\epsilon}{1+\epsilon}x + \frac{1}{1+\epsilon}z,$$

and hence by concavity,

$$0 = f_A(y) \geq \frac{\epsilon}{1+\epsilon}f_A(x) + \frac{1}{1+\epsilon}f_A(z).$$

Thus $f_A(z) < 0$ contradicting the positivity. ∎

We state some properties of g. Firstly $g(0) = 0, g(1) = 1$, g is an increasing function and $g'(0)$ is the number of $A \in \mathcal{U}$ with exactly one element. It is shown in [14] that g has a unique repulsive fixed point p_g in $[0,1]$. Also

$$\lim_{n\to\infty} p_n = \begin{cases} 0, & p_0 < p_g, \\ 1, & p_0 > p_g. \end{cases}$$

We summarize the above in a Lemma.

Lemma 3.9 *If $f : P^k \to P$ is concave, then $X^{(n)} \to 0$ if $p_0 < p_g$.*

It is more difficult to establish strict positivity.

Lemma 3.10 *In the context of Corollary 3.7, a sufficient condition for a strictly positive limit is that there exists a neighbourhood U of the origin such that $x_i \notin U$ for all i implies that $f(x) \notin U$ and $P(X^{(0)} \notin \epsilon U) = 1$ for some $\epsilon > 0$.*

Proof: The conditions ensure that $P(X^{(n)} \notin \epsilon U) = 1$ for all n and hence the limit is strictly positive. ∎

3.2 Conjugacy

An alternative approach to this problem which seems to have some potential is to use the idea of conjugacy. We begin with an example. Let $P = \mathbb{R}_+^2$ and $f(x_1, x_2) = x_1x_2/(x_1 + x_2)$. If we define $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\phi(x) = 1/x$, then

$$f(\phi(x_1), \phi(x_2)) = \phi(g(x_1, x_2)), \tag{3.6}$$

where $g(x_1, x_2) = (x_1 + x_2)/2$. Thus

$$X^{(n)} = \phi\left(2^{-n}\sum_{i=1}^{2^n} X_i^{(0)}\right),$$

and $X^{(n)}$ converges almost surely by the usual strong law of large numbers provided $\phi(E(X^{(0)})) < \infty$. Note that the electrical network generating this map is a pair of resistors in parallel and hence the map ϕ examines the effective conductivity instead of the effective resistance across the network.

This extends the usual notion of conjugacy to our setting: We will say that a non-linear function $f : P^k \to P$ is conjugate to a linear function $g : P^k \to P$ if there exists a one-to-one function ϕ satisfying the extension of (3.6) to k variables. (Note that this idea also works in the case where ϕ is not one-to-one; this is the natural analogue of semi-conjugacy.) As in the usual dynamical systems context conjugacy is a powerful tool for reducing non-linear problems to linear ones even if it is not possible to determine the conjugating function explicitly; there are abstract theorems giving sufficient conditions for the existence of conjugating functions (see for example [8]).

In the case of a homogeneous function $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ with a fixed point at $(1,1)$, we can apply the standard results in [8] directly. More generally (as far as we know) there are no abstract theorems on this form of conjugacy.

4 The homogenization problem for nested fractals

The homogenization problem for fractals has been treated in [9] and [5]. We consider a fractal graph, such as the Sierpinski gasket graph in which we take unit equilateral triangles and extend them to infinity using the structure of the Sierpinski gasket. The problem is to assign a random resistor to each edge of the graph and then rescale the graph according to the appropriate length scale factor. As we iterate this process, does the effective conductivity, renormalized by the appropriate factor, converge? The previous work on this problem made either strong assumptions about the random variables or looked at sufficiently simple fractals.

We will consider the class of nested fractals first defined by [11]. For $l > 1$, an *l-similitude* is a map $\psi : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\psi(x) = l^{-1}U(x) + x_0, \tag{4.1}$$

where U is a unitary, linear map and $x_0 \in \mathbb{R}^d$. Let $\psi = \{\psi_1, \ldots, \psi_m\}$ be a finite family of maps where ψ_i is an l-similitude. For $B \subset \mathbb{R}^d$, define

$$\Psi(B) = \bigcup_{i=1}^{m} \psi_i(B),$$

and let

$$\Psi_n(B) = \Psi \circ \ldots \circ \Psi(B).$$

The map Ψ on the set of compact subsets of $\mathbb{R}^d$ has a unique fixed point F, which is a self-similar set satisfying $F = \Psi(F)$.

As each ψ_i is a contraction, it has a unique fixed point. Let F_0' be the set of fixed points of the mappings ψ_i, $1 \leq i \leq m$. A point $x \in F_0'$ is called an essential fixed point if there exist $i, j \in \{1, \ldots, m\}$, $i \neq j$ and $y \in F_0'$ such that $\psi_i(x) = \psi_j(y)$. We write F_0 for the set of essential fixed points. Now define

$$\psi_{i_1,\ldots,i_n}(B) = \psi_{i_1} \circ \ldots \circ \psi_{i_n}(B), \quad B \subset \mathbb{R}^D.$$

The set $F_{i_1,\ldots,i_n} = \psi_{i_1,\ldots,i_n}(F_0)$ is called an n-cell. The lattice of fixed points F_n is defined by

$$F_n = \Psi_n(F_0), \tag{4.2}$$

and the set F can be recovered from the essential fixed points by setting

$$F = cl(\cup_{n=0}^{\infty} F_n).$$

We can now define a nested fractal as follows.

Definition 4.1 The set F is a nested fractal if $\{\psi_1, \ldots, \psi_m\}$ satisfy:
(A1) *(Connectivity)* For any 1-cells C and C', there is a sequence $\{C_i : i = 0, \ldots, n\}$ of 1-cells such that $C_0 = C, C_n = C'$ and $C_{i-1} \cap C_i \neq \emptyset, \quad i = 1, \ldots, n$.
(A2) *(Symmetry)* If $x, y \in F_0$, then reflection in the hyperplane $H_{xy} = \{z : |z - x| = |z - y|\}$ maps F_n to itself.
(A3) *(Nesting)* If $\{i_1, \ldots, i_n\}, \{j_1, \ldots, j_n\}$ are distinct sequences, then

$$\psi_{i_1,\ldots,i_n}(F) \bigcap \psi_{j_1,\ldots,j_n}(F) = \psi_{i_1,\ldots,i_n}(F_0) \bigcap \psi_{j_1,\ldots,j_n}(F_0).$$

(A4) *(Open set condition)* There is a non-empty, bounded, open set V such that the $\psi_i(V)$ are disjoint and $\cup_{i=1}^{m} \psi_i(V) \subset V$.

Let ψ_1 denote the similitude which has the origin as a fixed point. We can construct the pre-fractal, an infinite graph, by setting $G^{(n)} = \psi_1^{-n}(G_0)$, where G_0 is the complete graph on the essential fixed points F_0. The pre-fractal graph is then

$$G^{(\infty)} = \bigcup_{n=0}^{\infty} G^{(n)}.$$

Let $\mathcal{T}_n$ denote the set of translations of G_0 in $G^{(n)}$ and define $\mathcal{T} = \cup_n \mathcal{T}_n$. Then the infinite fractal lattice can be written, in terms of translations of the complete graph on F_0, as

$$G^{(\infty)} = \bigcup_{\tau \in \mathcal{T}} \tau(G_0).$$

We now recall the fixed point problem for nested fractals which we set up as an eigenvalue problem for a nonlinear map on discrete Dirichlet forms. We let c_0 be a set of conductors on G_0, which we can view as an $|E_0| \times |E_0|$ matrix. Define the quadratic form $\mathcal{E}^{(0)}$ by

$$\mathcal{E}^{(0)}(f, g) = \frac{1}{2} \sum_{x,y \in G_0} (f(x) - f(y))(g(x) - g(y))c(x, y),$$

with domain $\mathcal{F}_0 = \{f | f : F_0 \to \mathbb{R}\}$.

The renormalization map we are interested in can be defined as the composition of two maps. Let positive real numbers r_i be given and set

$$\Psi(\mathcal{E}_1, \ldots, \mathcal{E}_N)(f, g) = \mathcal{E}_{(1)}(f, g) = \sum_{i=1}^{N} r_i^{-1} \mathcal{E}_i(f \circ \psi_i, g \circ \psi_i),$$

for $f, g \in \mathcal{F}_1 := \{f | f : F_1 \to \mathbb{R}\}$. Define the renormalized Dirichlet form by taking its trace on F_0,

$$\Phi(\mathcal{E}_{(1)})(f, f) = \tilde{\mathcal{E}}_0(f, f) = \inf\{\mathcal{E}_{(1)}(g, g) | g = f \text{ on } F_0\}.$$

This Dirichlet form is associated with the electrical network on F_0 with the effective conductance between vertices of F_0 induced by the weighted electrical network F_1. The operation we have described is a nonlinear map on conductors and we can write this as a map on Dirichlet forms as $\Lambda = \Phi \circ \Psi$. For a given set of r, the existence problem for the corresponding Laplace operator on the fractal is solved, if we can find a set of conductors c and a constant λ such that

$$\Lambda(\mathcal{E}_0, \ldots, \mathcal{E}_0)(f, f) = \tilde{\mathcal{E}}_0(f, f) = \lambda \mathcal{E}_0(f, f), \quad \forall f \in \mathcal{F}_0.$$

The case $r_i = 1$ for all i corresponds to that considered in [11] and for which the homogenization problem is discussed in [9]. However we can consider the more general non-symmetric framework here. Conditions for the existence of a fixed point in this case were determined by [16].

Let $\mathbb{P}$ denote the set of positive semidefinite forms on F_0 and let $\mathbb{D}$ denote the set of Dirichlet forms on F_0. The space $\mathbb{B} := \mathbb{D} - \mathbb{D}$ with norm $\|\mathcal{E}\|^2 = \sup\{\mathcal{E}(f, f) | f \in \mathcal{F}_0, < f, f >= 1\}$ is a Banach space. Let $\mathbb{P}^\circ$ be interior of $\mathbb{P}$, the set of irreducible positive semidefinite forms. A Dirichlet form is said to be irreducible if the only functions for which $\mathcal{E}(f, f) = 0$ are the constant functions. Thus we have $\mathbb{D} \subset \mathbb{P}$ and the set of irreducible forms in $\mathbb{D}$ is $\mathbb{D} \cap \mathbb{P}^\circ$. The sets $\mathbb{D}, \mathbb{P}$ are cones and we refer to [12] for more details.

We will be interested in fixed points of the map Λ which lie in $\mathbb{D} \cap \mathbb{P}^\circ$ as these correspond to non-degenerate Laplace operators on the fractal. In order to work with these fixed points we use Hilbert's projective metric on the cone $\mathbb{P}$. The projective metric is defined on comparable quadratic forms. Two elements $\mathcal{E}, \mathcal{E}'$ of our cone are comparable if there exist real constants $a, b > 0$ such that $a\mathcal{E} \le \mathcal{E}' \le b\mathcal{E}$. This is an equivalence relation and the cone can be divided into parts based on this relation. If we take $\mathcal{E}, \mathcal{E}'$ to be two non zero comparable elements of our cone, then we can define

$$h(\mathcal{E}, \mathcal{E}') = \ln \frac{M(\mathcal{E}/\mathcal{E}')}{m(\mathcal{E}/\mathcal{E}')},$$

where

$$\begin{aligned} M(\mathcal{E}/\mathcal{E}') &= \inf\{b > 0 : \mathcal{E} \le b\mathcal{E}'\}, \\ m(\mathcal{E}/\mathcal{E}') &= \sup\{a > 0 : a\mathcal{E}' \le \mathcal{E}\}. \end{aligned}$$

If $\mathcal{E}, \mathcal{E}'$ are not comparable, then $h(\mathcal{E}, \mathcal{E}') = \infty$. The function h is called Hilbert's projective metric, even though it is not a metric. It is not difficult to see that if we restrict h to a part of the cone which intersects with the surface of the unit ball in the Banach space $\mathbb{B}$, then it is a metric. The projective metric, h and the norm $\|.\|$ are locally equivalent on the surface of the unit ball in $\mathbb{B}$.

We now consider the renormalization map Λ, which we can regard as $\Lambda : \mathbb{P}^N \to \mathbb{P}$. From the definitions we have that $\Lambda(\mathbb{D}, \ldots, \mathbb{D}) \subset \mathbb{D}$. By using the fact that the unit ball in $\mathbb{B}$ is compact and Brouwer's fixed point Theorem, the eigenvalue problem can be solved on the cone of Dirichlet forms.

Lemma 4.2 *There exists $\alpha \in \mathbb{R}$ and $\mathcal{E}' \in \mathbb{D}$ such that $\Lambda(\mathcal{E}', \ldots, \mathcal{E}') = \alpha \mathcal{E}'$.*

However this solution does not necessarily correspond to an irreducible Dirichlet form. From [12], the map is non-expansive on $\mathbb{P}^{\circ}$, in Hilbert's projective metric and we can show, using this metric, that under certain conditions there exists an irreducible fixed point. ¿From this point we make the following assumption.

Assumption 4.3 *There exists $\hat{\mathcal{E}} \in \mathbb{D} \cap \mathbb{P}^{\circ}$ such that $\Lambda(\hat{\mathcal{E}}, \ldots, \hat{\mathcal{E}}) = \lambda \hat{\mathcal{E}}$.*

For nested fractals with conductors invariant under the full symmetry group the existence was first demonstrated in [11], and the uniqueness of $\mathcal{E}$ was first established by [16] (it is also shown via an alternative approach in [13]). In [16] conditions are given for existence and uniqueness in the general case. We note that for the Sierpinski gasket there are explicit conditions on the weights $r_i, i = 1, 2, 3$ which give existence and uniqueness of the fixed point in $\mathbb{D} \cap \mathbb{P}^{\circ}$.

Let $X_i^{(0)}$ be a sequence of $\mathbb{D}$ valued random variables. We can place these on the infinite nested fractal lattice $G^{(\infty)}$ to define a random Dirichlet form

$$\mathcal{E}_0^{(\infty)}(f, g) = \sum_{\tau \in \mathcal{T}} X_\tau^{(0)}(f \circ \tau, g \circ \tau),$$

for $f, g \in \{f | f : G^{(\infty)} \to \mathbb{R}\}$. Our map Λ can be extended to act on this random form. Hence we can construct the sequence $X^{(n)}$ as

$$X^{(n+1)} = \lambda^{-n} \Lambda(X_1^{(n)}, \ldots, X_N^{(n)}),$$

with associated random form

$$\mathcal{E}_n^{(\infty)}(f, g) = \sum_{\tau \in \mathcal{T}} X_\tau^{(n)}(f \circ \tau, g \circ \tau),$$

where the $X_\tau^{(n)}$ are independent copies of $X_1^{(n)}$. Note that $X_1^{(n)}$ is the Dirichlet form on the unit cell induced by rescaling the lattice by l^{-n} and weighting the cells of the fractal according to the weights r.

Theorem 4.4 *(1). If $E\|X_1^{(0)}\|^p < \infty$, for some $p > 1$, then*

$$X_1^{(n)} \to X^{(\infty)}, \;\; a.s. \;\; as \;\; n \to \infty,$$

where $X^{(\infty)} = \gamma \mathcal{E}'$, a scaled copy of a fixed point $\mathcal{E}' \in \mathbb{D}$.
(2). If $Eh(\hat{\mathcal{E}}, X_1^{(0)})^p < \infty$ and there exists $\epsilon > 0$ such that $P(X^{(0)} > \epsilon \hat{\mathcal{E}}) = 1$, then $X_1^{(n)}$ converges almost surely to $X^{(\infty)} = \gamma \hat{\mathcal{E}}$ where $\hat{\mathcal{E}} \in \mathbb{D} \cap \mathbb{P}^{\circ}$ and $\gamma > 0$. Also the sequence $\mathcal{E}_n^{(\infty)}$ of random Dirichlet forms on $G^{(\infty)}$ converges to $\hat{\mathcal{E}}^{(\infty)}$, the Dirichlet form in which each unit cell has a copy of $\gamma \hat{\mathcal{E}}$.

Proof: As the Banach space is finite dimensional all norms on $\mathbb{B}$ are equivalent and it is straightforward to show that it is of type p for any $1 \leq p \leq 2$. Using the fact that there is a one to one correspondence between the Dirichlet forms and symmetric positive definite matrices it is not difficult to show that the norm is continuous in the order topology. The map Λ is derived from an electrical network and, as shown in [12] Theorem 2.2, it is concave and homogeneous on $\mathbb{P}$.

For (1) we note that from Lemma 4.2 there is a fixed point $\mathcal{E}' \in \mathbb{D}$. By assumption we have $E\|X_1^{(0)}\|^p < \infty$ and hence we can apply Theorem 3.1 to deduce the convergence of the hierarchical sequence. As the limit is constant and satisfies the fixed point equation it must be a multiple of the fixed point (though it may be the case that $\gamma = 0$).

For (2), the assumption on the Hilbert distance ensures that our random variables lie in the same part, and on this part the norm and the Hilbert metric are locally equivalent. Thus $P(X_1^{(0)} \in \mathbb{D} \cap \mathbb{P}^\circ) = 1$ and $E\|X_1^{(0)}\|^p < \infty$. Again we can apply our result, Theorem 3.1 to obtain the almost sure convergence. In this case, by our assumption 4.3 the limit $X^{(\infty)}$ must be a multiple of the non-degenerate fixed point $\hat{\mathcal{E}}$ as the limit is a constant which satisfies the fixed point equation. In order to check that $\gamma > 0$ we note that the map Λ is monotone. Thus, if $P(X^{(0)} > \epsilon\hat{\mathcal{E}}) = 1$, then $P(X^{(n)} > \epsilon\hat{\mathcal{E}}) = 1$ for all n and hence for the limit (recall Lemma 3.10). ∎

Remarks: 1. In the case of equal weights, this theorem extends the first result obtained in [9] on the L^1 convergence of the effective conductance. We do not need the $\mathbb{D}$-valued random variables to be bounded away from ∞ and we obtain an almost sure limit. For strict positivity of the limit in general we require the same bounded below condition. In the case of the Vicsek set, a tree like fractal, [5] proved that the limit is strictly positive provided $P(X^{(0)} \in \mathbb{D} \cap \mathbb{P}^\circ) = 1$.
2. For the Sierpinski gasket, if the weights do not satisfy the condition for the existence of a unique fixed point, then the effective conductivity will converge to a multiple of a degenerate fixed point. This fixed point only has a non-zero connection between one pair of vertices.
3. If the fixed point of the map Λ is not unique, we still have convergence to a limit. In this case the set of fixed points is arcwise connected and our sequence will converge (up to a scale factor) to a point in this set. For the Vicsek set, this limit was explicitly identified in [5].
4. In case (2) the limiting Dirichlet form will correspond to a diffusion on the fractal which is a deterministic time change of the diffusion corresponding to the Laplace operator $\hat{\mathcal{E}}^{(\infty)}$. We note that more work is required to prove the weak convergence of the associated sequence of Markov chains to the diffusion.

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References

[1] Birkhoff, G. *Lattice Theory.* American Math Society colloquium publications XXV, (1940).

[2] Chatterji, S.D. Martingale convergence and the Radon-Nikodym theorem in Banach spaces. *Math. Scand.*, 22:21-41, (1968).

[3] A. De Masi, P.A. Ferrari, S. Goldstein and W.D. Wick. An invariance principle for reversible Markov processes. Applications to random motions in random environments. *J. Stat. Phys.*, 55 (1989) 787–855.

[4] Edwards, R.E. *Functional Analysis, theory and applications.* Dover, (1965).

[5] Hambly, B.M. and Metz, V. The homogenization problem for the Vicsek set. *Stoch. Proc. Applic.*, 76:167–190, (1998).

[6] Hoffmann-Jorgensen, J. and Pisier, G. The law of large numbers and the central limit theorem in Banach spaces. *Ann. Probab.*, 4:587–599, (1976).

[7] Jordan, J.A. Almost sure convergence for iterated functions of independent random variables. BRIMS technical report, HPL-BRIMS-2000-14, (2000).

[8] Kuczma, M, Choczewski, B and Ger, R. *Iterative functional equations* Cambridge University Press, (1990).

[9] Kumagai, T. and Kusuoka, S. Homogenization on nested fractals. *Probab. Theory Related Fields*, 104:375–398, (1996).

[10] Li, D. and Rogers, T.D. Asymptotic behaviour of iterated functions of random variables *Ann. Appl. Probab.*, 9:1175–1201, (1999).

[11] Lindstrøm, T. Brownian motion on nested fractals. *Memoirs Am. Math. Soc.*, No 420, 83, (1990).

[12] Metz, V. Hilbert's projective metric on cones of Dirichlet forms. *J. Funct. Anal.*, 127:438–455, (1995).

[13] Metz, V. Renormalization contracts on nested fractals. *J. Reine Angew Math.*, 480:161–175, (1996).

[14] Moore, E. and Shannon, C.E. Reliable circuits using less reliable relays, I, II. *J. Franklin Inst.*, 262, 191–208, 281–297, (1956).

[15] Papanicolaou, G. and Varadhan, S. Boundary value problems with rapidly oscillating random coefficients. In 'Random Fields', Colloq. Math. Soc. Janos Bolyai 27, Ed: Fritz, J. et al, North-Holland, pp 835–873, (1979).

[16] Sabot, C. Existence and uniqueness of diffusions on finitely ramified self-similar fractals. *Ann. Scient. Ecole Norm. Sup*, 30:605–673, (1997).

[17] Shneiberg, I.Ya. Hierarchical sequences of random variables. *Theory Probab. Applic.* 31:137–141, (1987).

[18] Wehr, J. A strong law of large numbers for iterated functions of independent random variables. *J. Stat. Phys.* 86:1373–1384, (1997).

[19] Yosida, K. *Functional Analysis* Springer, 1980.