Kicked Burgers Turbulence

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Burgers turbulence subject to a force $\boldsymbol{f}(x, t) = \theta_j \boldsymbol{f}_j(x) / (t - tj)$, where the t_j's are "kicking times" and the "impulses" $\boldsymbol{f}_j(x)$ have arbitrary space dependence, combines features of the purely decaying and the continuously forced cases. With large-scale forcing this "kicked" Burgers turbulence presents many of the regimes proposed by E, Khanin, Mazel and Sinai (1997) for the case of random white-in-time forcing. It is also amenable to efficient numerical simulations in the inviscid limit, using a modification of the Fast Legendre Transform method developed for decaying Burgers turbulence by Noullez and Vergassola (1994). For the kicked case, concepts such as "minimizers" and "main shock", which play crucial roles in recent developments for forced Burgers turbulence, become elementary since everything can be constructed from simple two-dimensional area-preserving Euler-Lagrange maps.

The main results are for the case of identical deterministic kicks which are periodic and analytic in space and are applied periodically in time. When the space integrals of the initial velocity and of the impulses vanish, it is proved and illustrated numerically that a space- and time-periodic solution is achieved exponentially fast. In this regime, probabilities can be defined by averaging over space and time periods. The probability densities of large negative velocity gradients and of (not-too-large) negative velocity increments follow the power law with -7/2 exponent proposed by E et al (1997) in the inviscid limit, whose existence is still controversial in the case of white-in-time forcing. This power law, which is seen very clearly in the numerical simulations, is the signature of nascent shocks (preshocks) and holds only when at least one new shock is born between successive kicks.

It is shown that the third-order structure function over a spatial separation ?x is analytic in ?x although the velocity field is generally only piecewise analytic (i.e. between shocks). Structure functions of order p ? 3 are nonanalytic at ?x = 0. For even p there is a leading-order term proportional to |?x| and for odd $p > 3$ the leading-order term a ?x has a nonanalytic correction a ?x ??x ? stemming from shock mergers.

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1 Introduction

The driven Burgers equation

$$\partial_t u + u\partial_x u = \nu\partial_x^2 u + f, \tag{1}$$
$$u(x,t_0) = u_0(x), \tag{2}$$

with a force $f(x,t)$ has a much richer structure than the decaying problem with $f = 0$. Indeed, by the Hopf (1950) and Cole (1951) transformation the latter is mapped into the heat equation, while the former goes into a kind of imaginary-time Schrödinger equation with a potential F such that $f = -\partial_x F$. Actually, the randomly forced Burgers equation appears in a number of problems in statistical mechanics. This includes the Kardar, Parisi & Zhang (1986) equation for interface dynamics (see also Barabási & Stanley 1995) and the problem of directed polymers in random media (Bouchaud, Mézard & Parisi 1995).

From the point of view of turbulence dynamics, the forced Burgers equation can be used in the same spirit as the forced Navier–Stokes equation, namely to investigate universality with respect to the forcing of various statistical properties. For Navier–Stokes turbulence, when the force is confined to large spatial scales and the Reynolds number is very high, small-scale (inertial range) statistical properties are generally conjectured not to depend on the forcing, except through overall numerical factors. Similar conjectures have been made for Burgers turbulence with large-scale forcing. For example, there is little doubt that, because of the presence of shocks, all the structure functions of order $p > 1$ have a universal exponent equal to unity (see, e.g., Bouchaud, Mézard & Parisi 1995; E *et al.* 1997). More controversial is the behavior of the probability density function (pdf) $p(\xi)$ of the velocity gradient ξ at large negative values, in the limit of zero viscosity when the force is a white-noise process in time. If it is generally believed that this pdf follows a power law,

$$p(\xi) \propto |\xi|^\alpha, \quad \text{for } \xi \to -\infty, \tag{3}$$

the conjectured values of α differ markedly. Polyakov (1995) and Boldyrev (1997), using a field-theoretical operator product expansion, predict $\alpha = -5/2$; E *et al.* (1997), using a semi-heuristic approach in which nascent shocks (preshocks) are key, predict $\alpha = -7/2$; Gotoh & Kraichnan (1998), using a Fokker–Planck equation approach, predict $\alpha = -3$; more recent work by Kraichnan (1999) favors $\alpha = -7/2$. E & Vanden Eijnden (1999a,b) develop a probabilistic formalism adapted to solutions with shocks, which gives insight into many aspects of the problem and makes a good case for $\alpha = -7/2$. The question of the correct law for the case of white-noise forcing remains however open (we shall come back to this in §5). There are simpler situations for which the arguments in favor of $\alpha = -7/2$, originally developped by E *et al.* (1997), can be made rigorous, such as decaying Burgers turbulence with smooth random initial conditions (Bec & Frisch 1999).

It may be thought that numerical experimentation on the one-dimensional forced Burgers equation should be able to easily obtain the correct scaling laws. This is actually rather difficult if one tries to use standard numerical schemes of the kind also applicable to the Navier–Stokes equation, such as spectral methods with dissipation explicitly taken into account. Indeed, it has been shown by Gotoh & Kraichnan (1998; see also Gotoh 1999) that, in the presence of a small but finite viscosity, there is a range of large negative values of the gradient for which the pdf $p(\xi) \propto \nu|\xi|^{-1}$, which decreases rather slowly at large $|\xi|$. This is a direct consequence of the hyperbolic-tangent internal structure of shocks. As a result, the behavior of the pdf at negative values of ξ smaller in absolute value is contaminated and may display power laws somewhat shallower than predicted by the theory in the inviscid limit. See, for example, figure 3 of Gotoh & Kraichnan (1998) in which a power-law range with exponent -3 is seen over a little more than one decade of ξ-values. Such artefacts will disappear if much higher resolution is used (Gotoh and Kraichnan were using between 2^{17} and 2^{20} collocation points). It is, however, much more efficient to use alternative numerical schemes where one works directly with the inviscid limit. We shall see that such schemes, which were previously used only for the decaying case can be readily extended to the forced case and are particularly well suited for the case of kicked Burgers turbulence, in which the force is concentrated at discrete times.

The paper is organized as follows. Section 2 introduces the general concept of kicked Burgers turbulence and gives an explicit representation of the solution which is a simple generalization of the so-called "minimum representation" for the decaying case. Section 2.1 presents the concepts of "minimizers" and "main shocks" for the case when the dynamics are started at $t_0 = -\infty$. Section 2.2 presents the Fast Legendre Transform numerical scheme for space-periodic kicked Burgers turbulence with spatially smooth forcing. The next two sections are about the case of deterministically kicked Burgers turbulence when the kicks are periodic in space and time. Section 3 presents numerical results on exponential convergence to a periodic solution and shows how this is related to properties of minimizers (the rigorous results on convergence to a unique periodic solution are derived in the Appendix). Section 4 presents the main results about pdf's of velocity derivatives and increments (§4.1) and about structure functions (§4.2). Section 5 presents concluding remarks and some possible extensions to Navier–Stokes turbulence.

2 The inviscid limit for kicked Burgers turbulence

We shall be concerned here with the initial-value problem for the one-dimensional Burgers equation (1) when the force is concentrated at discrete times:

$$f(x,t) = \sum_j f_j(x)\, \delta(t - t_j), \tag{4}$$

where δ is the Dirac distribution and where both the "impulses" $f_j(x)$ and the "kicking times" t_j are prescribed (deterministic or random). The kicking times are ordered and form a finite or infinite sequence. In this paper the impulses are always taken smooth, i.e. acting only at large scales. (Kraichnan (1999) has considered a simple model in which there are non-smooth impulses creating directly sawtooth profiles in the velocity.) The precise meaning we ascribe to the Burgers equation with such forcing is that at time t_j, the solution $u(x,t)$ changes discontinuously by the amount $f_j(x)$

$$u(x, t_{j+}) = u(x, t_{j-}) + f_j(x), \tag{5}$$

while, between t_{j+} and $t_{(j+1)-}$ the solution evolves according to the unforced Burgers equation. Without loss of generality, we can assume that the earliest kicking time is $t_{j_0} = t_0$, provided we set $f_{j_0} = u_0$ and $u(x,t) = 0$ for $t < t_0$.

It is clear that any force $f(x,t)$ which is continuously acting in time can be approximated in such a way by choosing the kicking times sufficiently close.

We shall also make use of the formulation in terms of the velocity potential $\psi(x,t)$ and the force potentials $F_j(x)$

$$u(x,t) = -\partial_x \psi(x,t), \qquad f_j(x) = -\frac{d}{dx} F_j(x). \tag{6}$$

The velocity potential satisfies

$$\partial_t \psi = \frac{1}{2}(\partial_x \psi)^2 + \nu \partial_{xx}\psi + \sum_j F_j(x)\,\delta(t - t_j), \tag{7}$$

$$\psi(x, t_0) = \psi_0(x), \tag{8}$$

where $\psi_0(x)$ is the initial potential.

As it is well-known, the solution to the *unforced* Burgers equation with positive viscosity ν, has an explicit integral representation obtained by Hopf (1950) and Cole (1951) which can be used to investigate the limit of vanishing viscosity. We are here exclusively interested in this limit. Generically, shocks appear then after a finite time from smooth initial data. (The correct solution may also be obtained by solving the inviscid equation with a variational formulation and the condition that, at a shock, the right velocity is less than the left velocity (Lax 1957; Oleinik 1957).) Use of Laplace's method then leads to the following "minimum representation" for the potential in the limit of vanishing viscosity (henceforth always understood) which relates the solutions at any two times $t > t'$ between which no force is applied:

$$\psi(x, t) = -\min_y \left[\frac{(x-y)^2}{2(t-t')} - \psi(y, t') \right]. \tag{9}$$

It is known that, when t' is the initial time, the position y which minimizes (9) is the Lagrangian coordinate associated to the Eulerian coordinate x. The map $y \mapsto x$ is called the Lagrangian map. By expanding the quadratic term it is easily shown that the calculation of $\psi(\cdot, t)$ from $\psi(\cdot, t')$ is equivalent to a Legendre transformation. For details on all these matters, see She, Aurell & Frisch (1992) and Vergassola *et al.* (1994).

We now turn to the *forced* case with impulses applied at the kicking times t_j. Let $t_{J(t)}$ be the last such time before t. (Henceforth we shall often just write t_J.) Using (9) iteratively between kicks and changing the potential $\psi(y, t_{j+1})$ discontinuously by the amount $F_{j+1}(y)$ at times t_{j+1}, we obtain

$$\psi(x,t) = -\min_{y_J, y_{J-1}, \ldots, y_{j_0}} \left[A(j_0; x, t; \{y_j\}) - \psi_0(y_{j_0}) \right], \tag{10}$$

$$A(j_0; x, t; \{y_j\}) \equiv \frac{(x - y_J)^2}{2(t - t_J)} + \sum_{j=j_0}^{J-1} \left[\frac{(y_{j+1} - y_j)^2}{2(t_{j+1} - t_j)} - F_{j+1}(y_{j+1}) \right], \tag{11}$$

where $A(j_0; x, t; \{y_j\})$ is called the action.

For the Burgers equation with a continuous-in-time force deriving from a potential $F(x,t)$, E *et al.* (1997, 1998) give a minimizer representation of the solution:

$$\psi(x,t) = -\min_{y(\cdot)} \left[A(t_0; x, t; y(\cdot)) - \psi_0(y(t_0)) \right], \tag{12}$$

$$A(t_0; x, t; y(\cdot)) \equiv \int_{t_0}^{t} \left[\frac{\dot{y}^2(s)}{2} - F(y(s), s) \right] ds, \tag{13}$$

where the minimum is taken over all curves $y(s)$ satisfying $y(t) = x$. Note that this representation, which actually goes back to work by Oleinik (1957) on general conservation laws, is just a continuous limit of (10)-(11), obtained by taking $t_j = j\,\Delta t$ and letting $\Delta t \to 0$.

Returning to the case of kicked Burgers turbulence, from (10), we shall now introduce the concept of "minimizers" and "main shock" (§2.1). Eq. (10) will also be our starting point for the numerical method (§2.2). For the rest of the paper, we shall assume that *the* *force potential and the initial condition are periodic in the space variable.* For convenience, the period is taken to be unity in the theory, while 2π-periodicity is assumed in numerical studies.

2.1 Minimizers and main shocks

For the case of the kicked Burgers equation with an initial condition at t_{j_0} a "minimizing sequence" associated to (x, t) is defined as a sequence of y_j's ($j = j_0, j_0+1, \ldots, J(t)$) at which the r.h.s. of (10) achieves its minimum. Differentiating the action (11) with respect to the y_j's one gets necessary conditions for such a sequence, which can be written as a sequence of (Euler–Lagrange) maps

$$v_{j+1} = v_j + f_j(y_j), \tag{14}$$
$$y_{j+1} = y_j + v_{j+1}(t_{j+1} - t_j) = y_j + (v_j + f_j(y_j))(t_{j+1} - t_j), \tag{15}$$

where

$$v_j \equiv \frac{y_j - y_{j-1}}{t_j - t_{j-1}}. \tag{16}$$

These equations must be supplemented by the initial and final conditions:

$$v_{j_0} = u_0(y_{j_0}), \tag{17}$$
$$x = y_J + v_{J+1}(t - t_J). \tag{18}$$

It is easily seen that $u(x, t) = v_{J+1} = (x - y_J)/(t - t_J)$. Observe that the "particle velocity" v_j is the velocity of the fluid particle which arrives at y_j at time t_j and which, of course, has remained unchanged since the last kick (in Lagrangian coordinates). Eq. (14) just expresses that the particle velocity changes by $f_j(y_j)$ at the the kicking time t_j.

Note that (14)-(15) define an area-preserving and (explicitly) invertible map.

The presence of a force, deterministic or random, allows a formulation of the Burgers equation in the semi-infinite time interval $]-\infty, t]$ without fully specifying the initial condition $u_0(x)$ but only its (spatial) mean value $\langle u \rangle \equiv \int_0^1 u_0(x)dx$. Heuristically, this follows from the observation that, for a force of zero spatial mean value, as assumed here, $\langle u \rangle$ is a first integral, and hence does not depend on time, while all the other information contained in the initial condition is eventually forgotten.

Actually, the construction of the solution in a semi-infinite time interval is done by extending the concept of minimizing sequence to the case of dynamics starting at $t_0 = -\infty$. For a semi-infinite sequence $\{y_j\}$ ($j \le J$), let us define the action $A(-\infty; x, t; \{y_j\})$ by (11) with $j_0 = -\infty$. Such a semi-infinite sequence will be called a "minimizer" (or "one-sided minimizer") if it minimizes this action with respect to any modification of a finite number of y_j's. Specifically, for any other sequence $\{\hat{y}_j\}$ which coincides with $\{y_j\}$ except for finitely many j's (i.e. $\hat{y}_j = y_j$, $j \le J - k, k \ge 0$), we require

$$A(J - k; x, t; \{\hat{y}_j\}) \ge A(J - k; x, t; \{y_j\}). \tag{19}$$

Of course, the Euler–Lagrange relations (14)-(15) still apply to such minimizers. Hence, if for a given x and t we know $u(x, t)$ we can recursively construct the minimizer $\{y_j\}$ backwards in time by using the inverse of (14)-(15) for all $j < J$ and the final condition – now an initial condition – (18) with $v_{J+1} = u(x, t)$. This is well defined except where $u(x, t)$ has a shock and thus more than one value. Actually, solutions in a semi-infinite time interval are constructed from minimizers and not the other way round.

One way to construct minimizers is to take a sequence of initial conditions at different times $t_0 \to -\infty$. At each such time some initial condition $u_0(x)$ is given with the only constraint that it have the same prescribed value for $\langle u \rangle$. Then, (finite) minimizing sequences extending from t_0 to t are constructed for these different initial conditions. This sequence of minimizing sequences has limiting points (sequences themselves) which are precisely minimizers (E *et al.* 1998). The uniqueness of such minimizers, which would then imply the uniqueness of a solution to Burgers equation in the time interval $]-\infty, t]$, can only be shown by using additional assumptions, for example for the case of time-periodic forcing (§3 and Appendix).

If $\langle u \rangle = 0$, the sequence $\{y_j\}$ minimizes the action $A(-\infty; x, t; \{y_j\})$ in a stronger sense. Consider any sequence $\{\hat{y}_j\}$ such that, for some integer P we have $\hat{y}_j = y_j + P, j \leq J - k, k \geq 0$ and which differs arbitrarily from $\{y_j\}$ for $j > J - k$. (In other words, in a sufficiently remote past the hatted sequence is just shifted by some integer multiple of the spatial period.) We then have

$$A(-\infty; x, t; \{\hat{y}_j\}) \geq A(-\infty; x, t; \{y_j\}). \tag{20}$$

Indeed, for $\langle u \rangle = 0$, the velocity potential for any initial condition is itself periodic. In this case a particle can be considered as moving on the circle S^1 and its trajectory is a curve on the space-time cylinder. The y_j's are now defined modulo 1 and can be coded on a representative $0 \leq y_j < 1$. The Euler–Lagrange map (14)-(15) is still valid provided (15) is defined modulo 1.

The condition of minimality implies now that y_j and y_{j+1} are connected by the shortest possible straight segment. It follows that $|v_{j+1}| = \rho(y_j, y_{j+1})/(t_{j+1} - t_j)$, where ρ is the distance on the circle between the points y_j, y_{j+1}, namely $\rho(a,b) \equiv \min\{|a-b|, 1-|a-b|\}$. Hence, the action A can be rewritten in terms of cyclic variables:

$$A(-\infty; x, t; \{y_j\}) = \frac{\rho^2(x, y_J)}{2(t - t_J)} + \sum_{j<J} \left[\frac{\rho^2(y_{j+1}, y_j)}{2(t_{j+1} - t_j)} - F_{j+1}(y_{j+1}) \right]. \tag{21}$$

We now introduce the concept of "global minimizers" (or "two-sided minimizers") limiting ourselves to the case $\langle u \rangle = 0$ for simplicity. We first observe that any minimizer $\{y_j, j \leq J\}$ can be continued for all $j \geq J$ and hence times $t' > t$ by using the system (14)-(15). However, this procedure, when extended too far in time, will not usually generate a minimizer associated to time t'. Nevertheless, for any time t there always exist positions x such that the corresponding minimizers $\{y_j, j \leq J(t)\}$ can be continued to the bilateral sequence $\{y_j, -\infty < j < +\infty\}$ while keeping the minimizing property. Such global minimizers correspond to trajectories of fluid particles which, from $t = -\infty$ to $t = +\infty$, have never been absorbed in a shock.

We then observe that any shock existing at time t can be continued for all times $s > t$: shocks can merge but they cannot otherwise disappear. However, since new shocks can be produced, it is not always possible to trace back an existing shock for arbitrary times $s < t$. A shock with this property of having always existed in the past is called a "main shock".

In E *et al.* (1998) it is shown that for the case of random forcing which is 1-periodic in space and white noise in time rather than impulsive, the solution of Burgers equation in $]-\infty, t]$ is unique for $\langle u \rangle = 0$. It is also shown that at time t, the set of points $x \in S^1$ with more than one minimizer, that is shock locations, is finite and that the main shock and the global minimizer are unique. The global minimizer forms a hyperbolic trajectory of the Euler-Lagrange equations and all other minimizers approach the global one, as $t \to -\infty$ exponentially fast. In particular the two minimizers associated to the main shock approach the global minimizer in the remote past but it may be shown that they do so from opposite directions on the circle S^1. We shall see below in §3 that the same picture holds in the case of generic time-periodic kicking.

2.2 A Fast Legendre Transform numerical method

The numerical method used here solves the kicked Burgers equation directly in the inviscid limit. The basic ideas are very simple: at each kicking time t_j the potential is changed by the amount $F_j(x)$; between two successive kicks (or between the last kick and the output time) the decaying Burgers equation is solved using the minimum representation (9); this procedure is repeated as many times as the number of kicks between the initial time and the output time.

Specifically, the space periodic interval, here taken to be $[0, 2\pi[$, is discretized on a regular grid of N collocation points $x_k \equiv 2k\pi/N$. For each of these positions, (9) is used to determine the potential at a time t in terms of the potential just before the last kicking time t_J, time at which the potential has discontinuously changed by $F_J(x)$. Hence, we have :

$$\psi_k(t) \equiv \psi(x_k, t) = -\min_{\ell} \left[\frac{(x_k - y_\ell)^2}{2(t - t_J)} - (\psi_\ell(t_{J_-}) + F_J(y_\ell)) \right], \tag{22}$$

where $\psi_\ell(t_{J-})$ is the value of the velocity potential at $x = y_\ell$, just before the kick. We note y_{ℓ_k} the minimizing position corresponding to x_k. This procedure is, in principle, applied recursively, starting from t_{j_0}.

The problem is that naive application of (22) yields an algorithm with $O(N^2)$ operations between two successive kicks. She, Aurell & Frisch (1992) observed that the minimizing position y is actually a monotonic non-decreasing function of x. This is indeed a simple consequence of the convexity of the parabolic term involved in (9). Hence, the determination of the y_{ℓ_k}'s can be performed using a binary-subdivision search which requires only $O(N \log_2 N)$ operations. This kind of algorithm is known under the name Fast Legendre Transform (FLT), since the minimum representation is equivalent to a Legendre transform (Hopf 1950). (An even faster algorithm requiring only $O(N)$ operations has been developed by Trussov (1996).)

We use an adaptation of the method of Noullez and Vergassola (1994; see also Vergassola *et al.* 1994) who developed an FLT algorithm using a binary-subdivision search combined with a reorganization of the search, permitting the use of very low in-core storage. We first determine the minimizing y_{ℓ_0} for the point $x_0 \equiv 0$. As the velocity-potential at the time t_{J_+} is periodic, it is easy to show that y_{ℓ_0} is within the interval $[-\pi, +\pi[$. The minimizing location corresponding to $x_N \equiv 2\pi$ is then given by periodicity and reads $y_{\ell_N} = y_{\ell_0} + 2\pi = y_{\ell_0+N}$. The search for all the other minimizing locations y_{ℓ_k} can then be restricted to indices ℓ_k such that $\ell_0 \le \ell_k \le \ell_0 + N$. We then compute $y_{\ell_{N/2}}$, corresponding to $x = \pi$. We can then further subdivide the x-interval by considering $k = N/4$ and $k = 3N/4$, for which the corresponding ℓ_k's satisfy $\ell_0 \le \ell_{N/4} \le \ell_{N/2} \le \ell_{3N/4} \le \ell_0 + N$. At the next stage, we compute $y_{\ell_{N/8}}$, $y_{\ell_{3N/8}}$, $y_{\ell_{5N/8}}$ and $y_{\ell_{7N/8}}$ for which we need ℓ_0, $\ell_{N/4}$, $\ell_{N/2}$ and $\ell_{3N/4}$ as search boundaries. We repeat this subdivision procedure $\log_2 N$ times to obtain the N values of y_{ℓ_k}.

The method just described is optimal for non-smooth solutions of the kind considered by Vergassola *et al.* (1994) who had initial conditions of Brownian type. For the case of smooth solutions considered here, a more accurate determination of the solution is required to obtain reliable results on space derivatives of the velocity. We now describe an improvement of the method allowing to calculate first- and second-order derivatives. We observe that in (22), when the discrete location y_ℓ is replaced by an arbitrary real number y, the minimum, for a given x_k is, in general, not achieved exactly on the grid at y_{ℓ_k}, but at a neighboring location $y(k)$ within less than one mesh. This location satisfies

$x_k = y(k) + (t - t_J)\,[u\,(y(k), t_{J_-}) + f_J(y(k))]$, obtained by requiring that the derivative of the r.h.s. of (22) vanish. For this, the velocity $u(y, t_{J_-})$ and the force $f_J(y(k))$) are Taylor expanded to second order to obtain the improved location $y(k)$. When required, the first and second space derivatives of the velocity $u(x,t)$ are calculated, not from the potential by finite differences, but by using exact expressions of these derivatives in terms of the Lagrangian map from the preceding kicking time, which holds for the unforced Burgers equation (E *et al.* 1997; Bec & Frisch 1999).

This FLT, implemented on a grid of $2^{17} \approx 10^5$ collocation points takes about 1 s of CPU on a 100 MFlops computer. Without use of the binary-subdivision search the CPU time would be several thousands times larger. Among the other advantages of FLT is that no viscosity is needed and that the solution can be calculated directly at the required output times without need to obtain it at many intermediate times (other than the kicking times).

3 Deterministic periodic kicking

From now on we shall consider exclusively the case where the kicking is periodic in both space and time. Specifically, we assume that the force in the Burgers equation is given by

$$f(x,t) = g(x) \sum_{j=-\infty}^{+\infty} \delta(t - jT), \tag{23}$$

$$g(x) \equiv -\frac{d}{dx} G(x), \tag{24}$$

where $G(x)$, the kicking potential, is a deterministic function of x which is periodic and sufficiently smooth (e.g. analytic) and where T is the kicking period. The initial potential $\psi_{\rm init}(x)$ is also assumed smooth and periodic. This implies that the initial velocity integrates to zero over the period. (The case where this assumption is relaxed will be considered briefly in the Conclusion in relation with the Aubry–Mather theory.)

The numerical experiments reported hereafter have been made with the kicking potential

$$G(x) = \frac{1}{3} \sin 3x + \cos x, \tag{25}$$

and a kicking period $T = 1$. Other experiments where done with (i) $G(x) = -\cos x$ and (ii) $G(x) = (1/2)\cos(2x) - \cos x$. The former potential produces a single shock and no preshock. As a consequence it displays no $-7/2$ law in the pdf of gradients. The latter potential gives essentially the same results as reported hereafter but has an additional symmetry which we avoided by the choice (25).

The number of collocation points chosen for our simulations is generally $N_x = 2^{17} \approx 1.31 \times 10^5$, with a few simulations done at $N_x = 2^{20}$ (for the study of the relaxation to the periodic regime presented below). Since our numerical method allows us to go directly to the desired output time (from the nearest kicking time) there is no need to specify a numerical time step. However, in order to perform temporal averages, e.g. when calculating pdf's or structure functions, without missing the most relevant events (which can be sharply localized in time) we need sufficiently frequent temporal sampling. We have taken for the total number of output times $N_t \approx 1000$ chosen such that the increment between successive output times is roughly the two-thirds power of the mesh (this is related to the structure of preshocks, see §4.1).

Figure 1 shows snapshots of the time-periodic solution at various instants. It is seen that shocks are always present (at least two) and that at each period two new shocks are born at

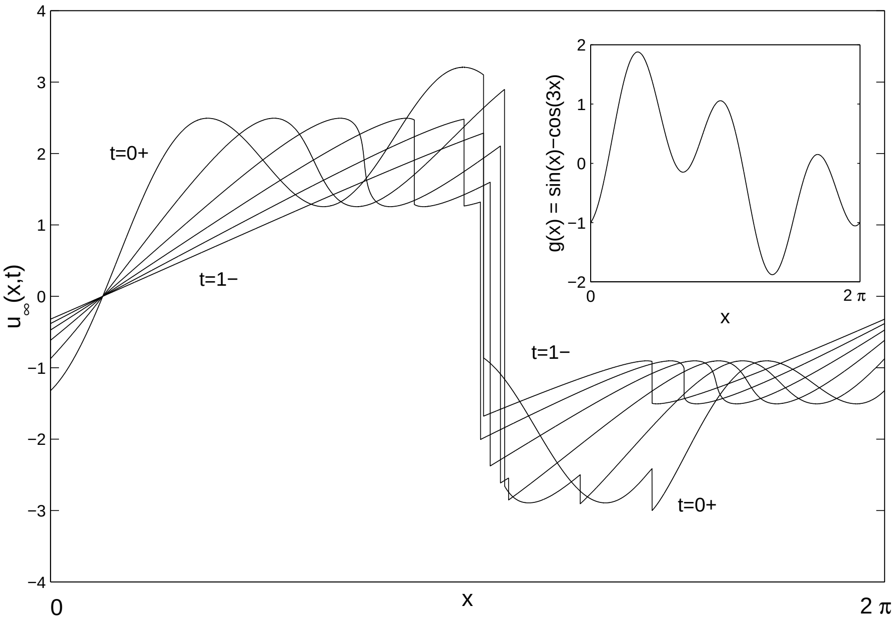

Figure 1: Snapshots of the velocity for the unique time-periodic solution corresponding to the kicking force $g(x)$ shown in the upper inset; the various graphs correspond to six output times equally spaced during one period. The origin of time is taken at a kick. Notice that during each period, two new shocks are born and two mergers occur.

$t_{\star 1} \approx 0.39$ and $t_{\star 2} \approx 0.67$. There is one main shock which remains near $x = \pi$ and which collides with the newborn shocks at $t_{c1} \approx 0.44$ and $t_{c2} \approx 0.86$. Figure 3 shows the evolution of the positions of shocks during one period.

We find that, for all initial conditions $u_0(x)$ used, the solution $u(x, t)$ relaxes exponentially in time to a unique function $u_\infty(x, t)$ of period 1 in time. Figure 2 shows the variation of $\int_0^{2\pi} |u(x, n_-) - u_\infty(x, 1_-)| \, dx/(2\pi)$ for three different initial conditions as a function of the discrete time n.

The phenomenon of exponential convergence to a unique space- and time-periodic solution is something quite general: whenever the kicking potential $G(x)$ is periodic and analytic and the initial velocity potential is periodic (so that the mean velocity $\langle u \rangle$ =0 at all times), there is exponential convergence to a unique piecewise analytic solution. This is proved rigorously in the Appendix for functions $G(x)$ which have a unique point of maximum with a nonvanishing second derivative (Morse generic functions). Here, we just explain the main ideas of the proof and give some additional properties of the unique solution.

One very elementary property of solutions is that, for any initial condition of zero mean value, the solution after at least one kick satisfies $|u(x, t)| \leq (1/2) + \max_x |dG(x)/dx|$. Indeed, at a time $t = n_-$ just before any kick we have $x = y + u(x, n_-)$ where y is the position just after the previous kick of the fluid particle which goes to x at time n_-. It follows from the spatial periodicity of the velocity potential that the location y which minimizes the action is within less than half a period from x. Thus, $|u(x, n_-)| \leq 1/2$. The additional $\max_x |dG(x)/dx|$ term comes from the maximum change in velocity from one kick. It follows that the solution is bounded. Note that if the spatial and temporal periods are L and T, respectively, the bound on the velocity becomes $L/(2T) + \max_x |dG(x)/dx|$.

The convergence at large times to a unique solution is related to properties of the two-dimensional conservative (area-preserving) dynamical system defined by the Euler–Lagrange

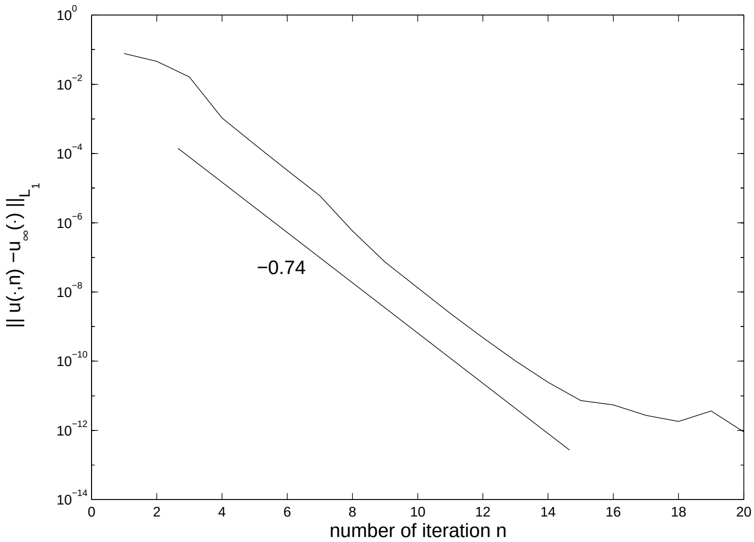

Figure 2: Exponential relaxation to a time-periodic solution for three different initial velocity data as labelled. The horizontal axis gives the time elapsed since $t = 0$.

Figure 3: Evolution of shock positions during one period. The beginnings of lines correspond to births of shocks (preshocks) at times $t_{\star 1}$ and $t_{\star 2}$; shock mergers take place at times t_{c1} and t_{c2}. The "main shock", which survives for all time, is shown with a thicker line.

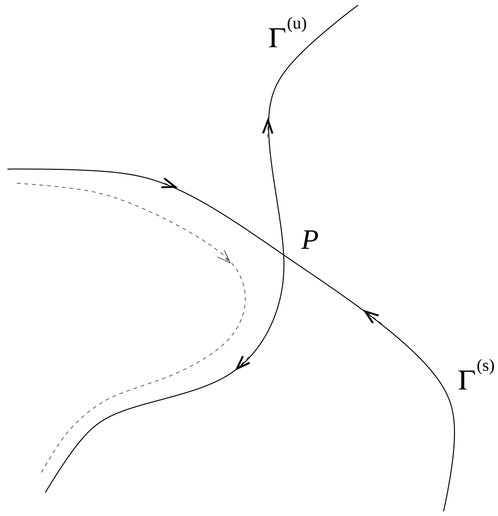

Figure 4: Sketch of a hyperbolic fixed point P with stable ($\Gamma^{(s)}$) and unstable ($\Gamma^{(u)}$) manifolds. The dashed line gives the orbit of successive iterates of a point near the stable manifold.

map (14)-(15) of §2.1. By construction, we have $u(x, 1_+) = \hat{u}(x) - dG(x)/dx$, where $\hat{u}(x)$ is the solution of the unforced Burgers equation at time $t = 1_-$ from the initial condition $u(x)$ at time $t = 0_+$. The map $u \mapsto \hat{u}(x) + g(x)$, where $g(x) \equiv -dG(x)/dx$, will be denoted B_g and is just the map which solves the kicked Burgers equation over a time of one. The problem is to show that the iterates $B_g^n u_0$ converge for $n \to \infty$ to a unique solution.

If it were not for the shocks it would suffice to consider the two-dimensional Euler–Lagrange map. Note that, for the case of periodic kicking, this map has an obvious fixed point P, namely $(x = x_c, v = 0)$, where x_c is the unique point maximizing the kicking potential. It is easily checked that this fixed point is an unstable (hyperbolic) saddle point of the Euler–Lagrange map with two eigenvalues $\lambda = 1 + c + \sqrt{c^2 + 2c}$ and $1/\lambda$, where $c = -\partial^2_{xx} G(x_c)/2$.

Like for any two-dimensional map with a hyperbolic fixed point, there are two curves globally invariant by the map which intersect at the fixed point: the stable manifold $\Gamma^{(s)}$, the set of points which converge to the fixed point under indefinite iteration of the map, and the unstable manifold $\Gamma^{(u)}$, the set of points which converge to the fixed point under indefinite iteration of the inverse map, as illustrated in figure 4 (see, e.g., Hénon 1983, Manneville 1990). It follows that any curve which intersects the stable manifold transversally (the tangents of the two curves are distinct) will, after repeated applications of the map, be pushed exponentially against the unstable manifold at a rate determined by the eigenvalue $1/\lambda$. In the language of Burgers dynamics, the curve in the (x, v) plane defined by an initial condition $u_0(x)$ will be mapped after time n into a curve very close to the unstable manifold. In fact, for the case studied numerically, $1/\lambda \approx 0.18$ is within one percent of the value measured from the exponential part of the graph shown in figure 2. Note that if the initial condition $u_0(x)$ contains the fixed point, the convergence rate becomes $(1/\lambda)^2$ (even higher powers of $1/\lambda$ are possible if the initial condition is tangent to the unstable manifold).

The fixed point P gives rise to a very simple global minimizer: $(y_j = x_c,\ v_j = 0)$ for all positive and negative j's. It follows indeed by inspection of (21) that any deviation from this minimizer can only increase the action; actually, it minimizes both the kinetic and the potential part of the action. Note that the corresponding fluid particle is at rest forever and will never be captured by a shock (it is actually the only particle with this

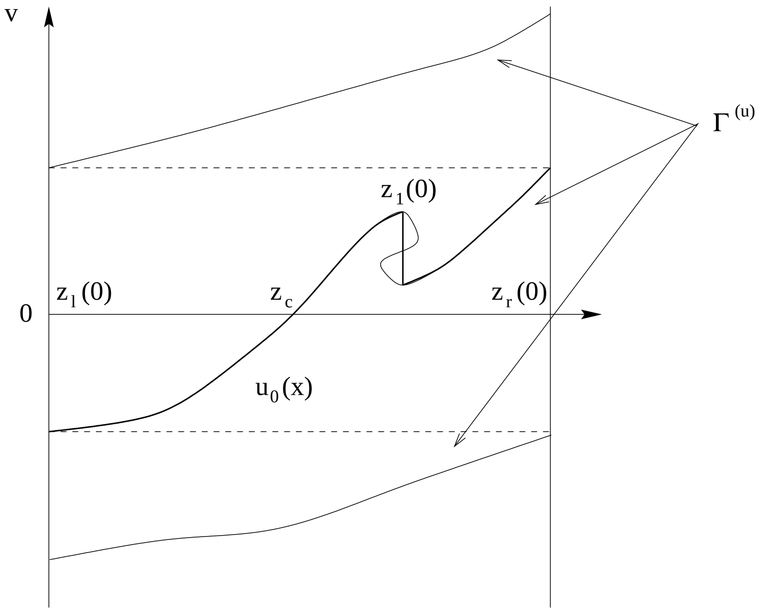

Figure 5: Unstable manifold $\Gamma^{(\mathrm{u})}$ on the (x, v)-cylinder (the x-coordinate is defined modulo 1) which passes through the fixed point $P = (x_c, 0)$. The bold line is the graph of $u_\infty(x, 1_-)$. The main shock is located at $x_l = x_r$. Another shock at x_1 corresponds to a local zig-zag of $\Gamma^{(\mathrm{u})}$ between A and B.

property). It is shown in the Appendix that any minimizer is attracted exponentially to such a global minimizer as $t \to -\infty$. Thus, any point (y_j, v_j) on a minimizer belongs to the *unstable manifold* $\Gamma^{(\mathrm{u})}$ and, hence, any regular part of the graph of the limiting solution $u_\infty(x)$ belongs to the unstable manifold $\Gamma^{(\mathrm{u})}$. This unstable manifold is analytic but can be quite complex. It can have several branches for a given x (see figure 5) and does not by itself define a single-valued function $u_\infty(x)$. The solution has shocks and is only piecewise analytic. Consideration of the minimizers is required to find the position of the shocks in the limiting solution: two points with the same x corresponding to a shock, such as A and B on figure 5 should have the same action.

Finally, we give the geometric construction of the main shock, the only shock which exists for an infinite time. Since λ is positive, locally, minimizers which start to the right of x_c approach the global minimizer from the right, and those which start to the left approach it from the left. Take the rightmost and leftmost points x_r and x_l on the periodicity circle such that the corresponding minimizers approach the global minimizer from the right and left respectively (see figure 6). These points are actually identical since there cannot be any gap between them that would have minimizers approaching the global minimizer neither from the right nor the left. The solution $u_\infty(x)$ has then its main shock at $x_l = x_r$.

4 Statistical properties for the periodically kicked case

We are here working with time- and space-periodic deterministic solutions of the kicked Burgers equation. We thus choose to define our statistical averages as averages over the two periods, here assumed to be both unity. Specifically, let $\mathcal{F}(u)$ be an observable (functional of the solution u) and let $T_{x,t}$ denote the space-time translation operator which shifts the

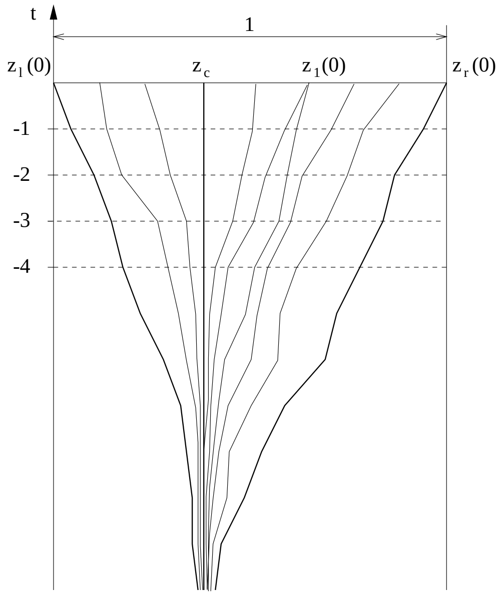

Figure 6: Minimizers (trajectories of fluid particles) on the (x,t)-cylinder. Time starts at $-\infty$. Shock locations at $t = 0_-$ are characterized by having two minimizers (an instance is at x_1). The main shock is at $x_l = x_r$. The fat line $x = x_c$ is the global minimizer.

solution u by a spatial amount x and a temporal amount t. We define

$$\langle \mathcal{F}(u) \rangle \equiv \int_0^1 \int_0^1 \mathcal{F}\left(T_{x,t}\, u\right)\, dx\, dt. \tag{26}$$

For example, with the observable $\mathcal{F}(u) \equiv [u(\Delta x, 0) - u(0,0)]^p$, we obtain the structure function of order p over a separation Δx:

$$S_p(\Delta x) \equiv \int_0^1 \int_0^1 [u(x + \Delta x, t) - u(x,t)]^p\, dx\, dt. \tag{27}$$

Such averages are easily calculated numerically. For example, pdf's are obtained from space-time histograms over all collocation points and a suitably large number of output times.

4.1 Pdf's of velocity derivatives and increments

For the periodic solution of §3 we calculate first and second space derivatives of the velocity; the corresponding pdf's are then determined as normalized space-time histograms after binning of derivative values (the bins are in geometric progression ; there are 100 bins per decade for the first derivative and 50 for the second). Figures 7 and 8 show the pdf's of the first and second space derivatives in log-log coordinates. Negative values are shown for the former and positive values for the latter. It is seen that clean power laws are obtained. More quantitative information about the values of the exponents of the power laws are obtained by measuring the "local scaling exponent", i.e. the logarithmic derivative of the pdf, calculated here using least-square fits on quarter decades. The results are shown as upper right insets on figures 7 and 8. It is seen that, over one decade, the local exponent for the pdf of the gradient is within less than five percent of the value $-7/2$ predicted by a simple theoretical

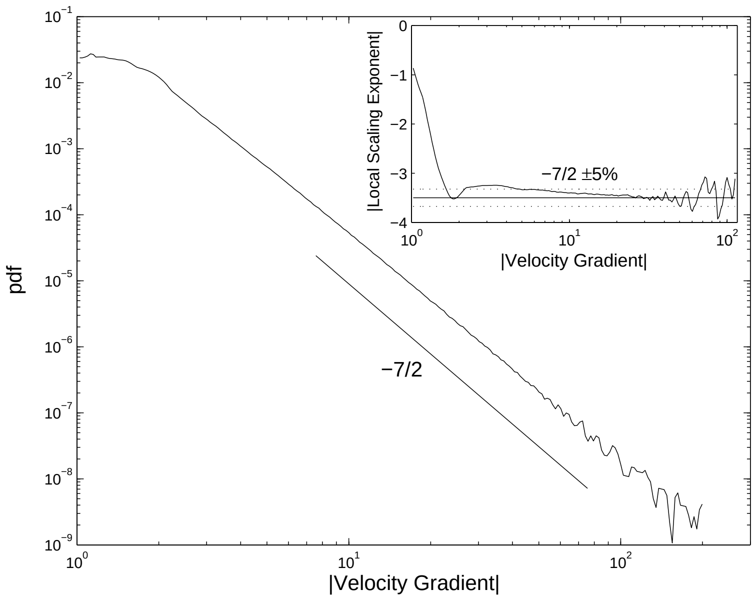

Figure 7: Pdf of the velocity gradient at negative values in log-log coordinates. Upper inset: local scaling exponent. A power law with exponent $-7/2$ is obtained at large arguments.

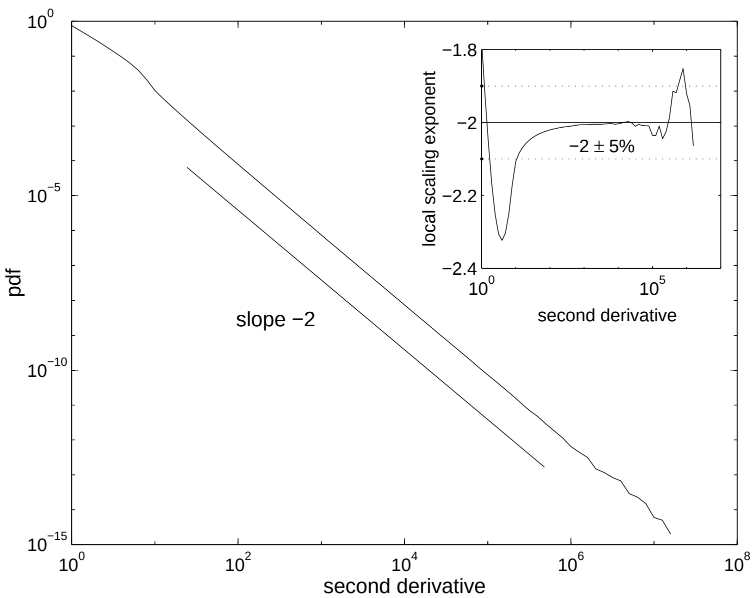

Figure 8: Same as figure 7 with the second space derivative of the velocity. The exponent is now -2.

arguments given hereafter; for the second derivative there are about four decades within five percent of the value -2.

The presence of a power-law pdf with $-7/2$ exponent is easily understood. It is just the signature of the preshocks which appear when new shocks are created during the unforced phase between two successive kicks. These preshocks are the only structures giving large *finite* negative gradients: shocks give infinite negative gradients unless a finite viscosity is introduced and the gradients in the immediate spatial neighborhood of a mature shock are not particularly large. The theory of the $-7/2$ law at large negative values ξ of the velocity gradient, developed by Bec & Frisch (1999) for the unforced case with *random* initial conditions is readily adapted to the deterministic case, provided we use space-time averages instead of ensemble averages. A simplified presentation, following in part E *et al.* (1997), is given hereafter for the case of a single preshock. The contributions of several preshocks to the pdf are just additive and it is proved in the Appendix that the periodic solution has finitely many preshocks.

We define a velocity in Lagrangian coordinates $u^{(\mathrm{L})}(a,t')$ with the origin of time just after a kick and $a=0$ at a (negative) minimum of $\partial_x u$. Without loss of generality, we assume $u^{(\mathrm{L})}(0,0)=0$ (otherwise we perform a Galilean transformation to bring it to zero). We then have, locally, $u^{(\mathrm{L})}(a,0) = -c_1 a + c_2 a^3 + \mathrm{h.o.t.}$, where c_1 and c_2 are positive constants and "h.o.t." stands for higher-order terms. No generality is lost by assuming $c_1 = 1$ (otherwise make a linear change on the a-coordinate). The fluid particle initially at a will be at time t' at $x = a + t'u^{(\mathrm{L})}(a,0) = a(1-t') + t'c_2a^3 + \mathrm{h.o.t.}$ This "Lagrangian map" becomes singular at $t' = t_\star = 1$, the instant of preshock (formation of a shock). We then have $a = (x/c_2)^{1/3} + \mathrm{h.o.t.}$ (The cubic root is here defined both for positive and negative values of its argument.) Since the Lagrangian velocity has not changed, the Eulerian velocity is given by $u(x,t_\star) = -(x/c_2)^{1/3} + \mathrm{h.o.t.}$, which has a cubic root structure and a gradient $-(x/c_2)^{-2/3}/(3c_2)$. Hence, the gradient takes large negative values for small x. Just before $t_\star$, at time $t' = 1-\tau$, we have, $x = \tau a + c_2 a^3 + \mathrm{h.o.t.}$ It follows that the cubic relation between a and x still holds, except in a region of Lagrangian width $\sim \tau^{1/2}$ and thus of Eulerian width $\sim \tau^{3/2}$, where the relation becomes linear to leading order. (It is because of this $\tau^{3/2}$ dependence that the time τ between successive outputs and the mesh $\delta x = 2\pi/N$ must be related by $\delta x \sim \tau^{3/2}$.)

The question is now: what is the fraction of space-time where the velocity gradient $\partial_x u < \xi$, where ξ is a large negative number? Because of the cubic root structure, x must be in a small interval of width $\sim |\xi|^{-3/2}$. The time must be sufficiently close to $t_\star$ for this interval still to be in the region of validity of the cubic relation, that is, within $\sim |x|^{2/3} \sim |\xi|^{-1}$. Hence, the relevant space-time fraction or, in other words, the cumulative probability to have $\partial_x u < \xi$ is $\sim |\xi|^{-5/2}$. This gives a pdf $\sim |\xi|^{-7/2}$ at large negative ξ's.

Actually, there is another contribution, also proportional to $|\xi|^{-7/2}$ stemming from a small time interval $\tau \sim |x|^{2/3} \sim |\xi|^{-1}$ just *after* $t_\star$ when small-amplitude shocks are present which have not yet completely destroyed the cubic root structure (Bec & Frisch 1999). For the case studied numerically, where the kicking potential is given by (25), there are two preshocks, each giving a contribution to the pdf of the gradient proportional to $|\xi|^{-7/2}$.

This argument is readily adapted to second space derivatives, yielding a pdf $\sim |\xi|^{-2}$ as observed in figure 8. (The same law holds also at large positive values since the second derivative near a preshock is an even function.)

We now turn to the pdf of (spatial) velocity increments over a separation Δx. We define

$$\Delta u(\Delta x; x, t) \equiv u(x + \Delta x, t) - u(x,t). \tag{28}$$

Its pdf's for various values of Δx are again calculated from space-time histograms. One

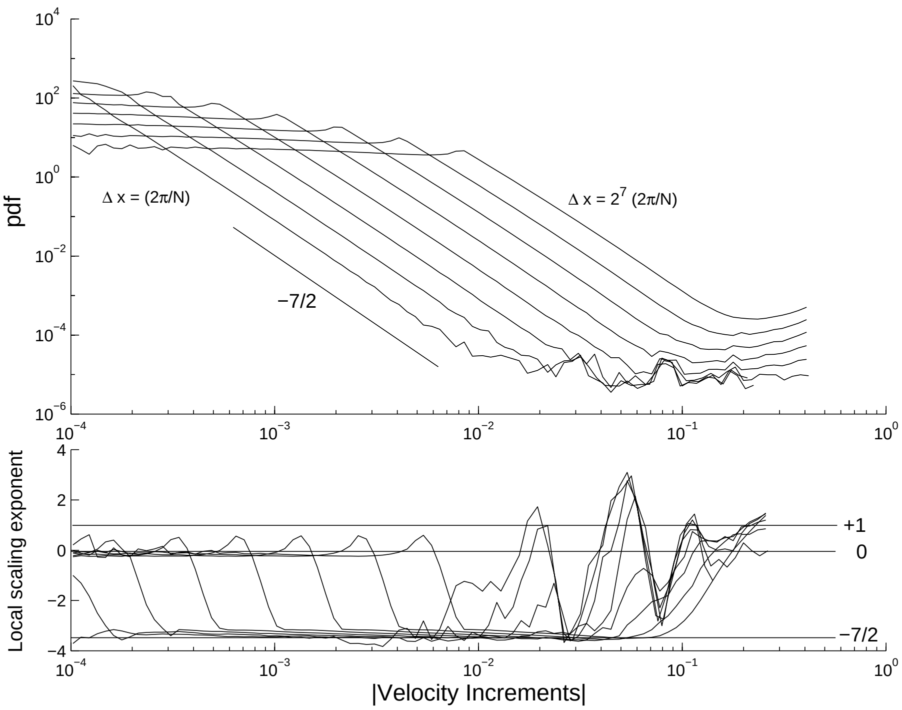

Figure 9: Upper part: Pdf of (negative) velocity increments in log-log coordinates for various values of the separation Δx in geometric progression from $2\pi/N$ to $2^7(2\pi/N)$. Lower part: the corresponding local scaling exponents.

hundred bins per decade are used. Δx is given the values $2\pi 2^p/N$, where $2\pi/N$ is the numerical mesh and p is varied from zero to seven. Figure 9 gives log-log plots of the pdf's of increments for the eight separations chosen. We limit ourselves to negative increments. The corresponding local scaling exponents are shown in figure 9. It is seen that, for moderately large increments, the pdf's have the same power-law behavior with exponent $-7/2$ as the pdf of the gradient. This universal behavior was also predicted by E *et al.* (1997) for white-in-time forced Burgers turbulence. Phenomenologically, this range is obtained simply by Taylor expanding the increment as $\Delta x\, \partial_x u$. At larger increments (in absolute value) the local scaling exponent rises quickly to positive values but does not saturate to the value $+1$ predicted by E *et al.* (1997) by the following argument, based on the consideration of nascent shocks and which applies also to the periodically kicked case: The probability (as fraction of space) to have a shock in an interval of length Δx is $\propto \Delta x$. Since the shock amplitude grows as $(t - t_\star)^{1/2}$, where $t_\star$ is the time of the preshock, the fraction of time for which the shock amplitude does not exceed a value $|\Delta u|$ is proportional to $(\Delta u)^2$. Hence, the cumulative probability to have a velocity increment (in absolute value) less than $|\Delta u|$ is proportional to $\Delta x(\Delta u)^2$ and the pdf is proportional to $\Delta x|\Delta u|$. By equating the contributions from the $-7/2$ and the $+1$ ranges, the transition between the two ranges is predicted to happen around an increment Δu_c which scales as $(\Delta x)^{1/3}$, in good agreement with our data. A clean $+1$ range is not seen and would require a resolution of well over one million collocation points.

Finally, the flat range seen in figure 9 for $|\Delta u| \ll \Delta x$, is a universal contribution $\propto (\Delta x)^{-1}$ from extremal points of the velocity (not predicted by E *et al.* (1997)). This range extends also to positive values of $\Delta u \ll \Delta x$ but there is no other universal range for positive increments; this is why pdf's are not shown for such increments.

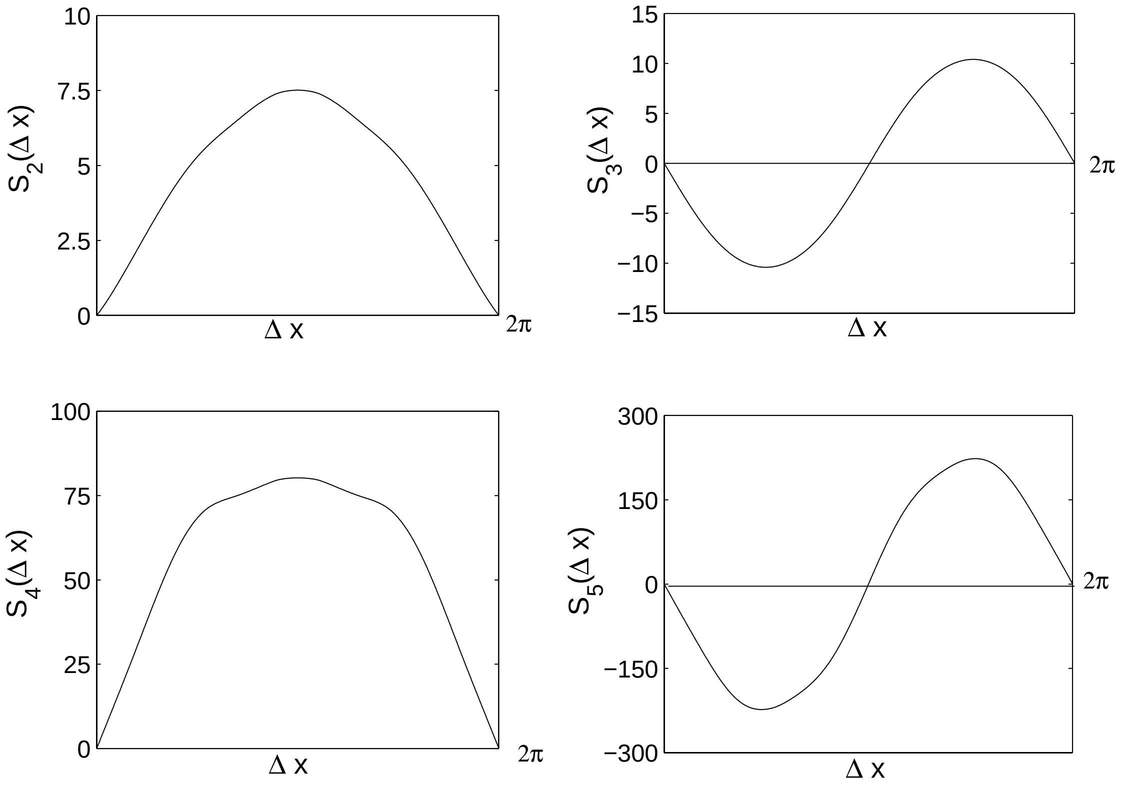

Figure 10: Structure functions $S_p(\Delta x)$ for $p = 2, 3, 4, 5$ as labelled. Note the linear behavior at small Δx.

4.2 Structure functions

We now study the structure functions for the limiting (unique) solution $u_\infty(x,t)$ of the periodically kicked Burgers equation. For numerical studies 2π-periodicity in space and 1-periodicity in time are assumed. Hence, the structure function of (integer) order p is given by

$$S_p(\Delta x) \equiv \frac{1}{2\pi} \int_0^1 dt \int_0^{2\pi} dx \, [u_\infty(x+\Delta x, t) - u_\infty(x,t)]^p \,. \tag{29}$$

The 2π-periodicity of $u_\infty(x,t)$ immediately implies that $S_p(\Delta x)$ is 2π-periodic in Δx and is an even/odd function for even/odd p.

Figure 10 shows the structure functions of order 2, 3, 4 and 5 (as labelled) in linear coordinates. It is seen that all these structure functions behave proportionally to Δx at small arguments (more precisely as $\Delta x \; (\mathrm{sign}\,(\Delta x))^{p+1}$). This is a well-known consequence of the presence of shocks (E *et al.* 1997). In the next two sections we shall show that all the structure functions except $S_3(\Delta x)$ are nonanalytic functions of Δx.

4.2.1 Analyticity of the third-order structure function

For notational convenience in this and the next sections we assume 1-periodicity in space and time. Space averages over the period are denoted $\langle\cdot\rangle_x$. Averages over both space and time are denoted $\langle\cdot\rangle$. We shall prove that, when the kicking potential $G(x)$ is analytic, the third-order structure function is also analytic.

This will be established as a consequence of the following relation for the unforced Burgers equation with space-periodic solution in the limit of vanishing viscosity. Let $u \equiv u(x,t)$ and $u' \equiv u(x+\Delta x, t)$, we have

$$\partial_t \left\langle u'u \right\rangle_x = \frac{1}{6} \partial_{\Delta x} \left\langle (u'-u)^3 \right\rangle_x . \tag{30}$$

It is here assumed that Δx is not an integer multiple of the spatial period and that no pair of shocks remains separated by exactly Δx for a finite amount of time (this holds for almost every Δx).

Proof. Let us denote by $X_i(t)$ ($i = 1, \ldots, N(t)$) the (Eulerian) ordered positions of shocks and by $[\![u]\!]_i \equiv u(X_i(t)_+, t) - u(X_i(t)_-, t)$ the (negative) velocity jump at the i-th shock. ($N(t)$ may change in time.) Except at shocks we can use the inviscid Burgers equation $\partial_t u + u \partial_x u = 0$. At shocks this has to be supplemented by the conditions

$$\dot{X}_i [\![u]\!]_i - [\![\frac{u^2}{2}]\!]_i = 0, \tag{31}$$

which follow from momentum conservation. We have

$$\left\langle u'u \right\rangle_x = \sum_{i=1}^{N} \int_{X_i(t)}^{X_{i+1}(t)} u'u \, dx, \tag{32}$$

where $X_{N+1}(t) \equiv X_1(t) + 1$. Differentiating with respect to t, we obtain

$$\begin{aligned}
\partial_t \left\langle u'u \right\rangle_x &= \left\langle u \partial_t u' + u' \partial_t u \right\rangle_x - \sum_{i=1}^{N} \dot{X}_i(t) [\![u]\!]_i \, u(X_i(t) + \Delta x, t) && (33)\\
&= -\sum_{i=1}^{N} \int_{X_i(t)}^{X_{i+1}(t)} (uu'\partial_x u' + u'u\partial_x u) \, dx - \sum_{i=1}^{N} \dot{X}_i(t) [\![u]\!]_i \, u(X_i(t) + \Delta x, t) && (34)\\
&= -\sum_{i=1}^{N} \int_{X_i(t)}^{X_{i+1}(t)} \left(\frac{1}{2} u \partial_{\Delta x} u'^2 - \frac{1}{2} u^2 \partial_{\Delta x} u' \right) dx \\
&\quad -\sum_{i=1}^{N} \dot{X}_i(t) [\![u]\!]_i \, u(X_i(t) + \Delta x, t) + \sum_{i=1}^{N} [\![\frac{u^2}{2}]\!]_i \, u(X_i(t) + \Delta x, t). && (35)
\end{aligned}$$

In going from (33) to (34) we used the inviscid decaying Burgers equation; from (34) to (35) we have performed an integration by parts and used $\partial_x u' = \partial_{\Delta x} u'$. From (31) follows that the last two terms in (35) cancel. Hence, we obtain

$$\partial_t \left\langle u'u \right\rangle_x = \frac{1}{2} \partial_{\Delta x} \left\langle -uu'^2 + u^2 u' \right\rangle_x = \frac{1}{6} \partial_{\Delta x} \left\langle (u'-u)^3 \right\rangle_x , \tag{36}$$

which completes the proof.

We now return to the case of the periodically kicked Burgers equation, with the unique solution $u_\infty(x,t)$. Using (30), integrated in time between two successive kicks, say at $t = 0$ and $t = 1$, we have

$$\begin{aligned}
\frac{1}{6} \partial_{\Delta x} S_3(\Delta x) &= \frac{1}{6} \partial_{\Delta x} \left\langle [u_\infty(x+\Delta x, t) - u_\infty(x,t)]^3 \right\rangle \\
&= \langle u_\infty(x+\Delta x, 1_-) u_\infty(x, 1_-) \rangle_x - \langle u_\infty(x+\Delta x, 0_+) u_\infty(x, 0_+) \rangle_x . && (37)
\end{aligned}$$

Next, we use

$$u_\infty(x, 0_+) = u_\infty(x, 0_-) + g(x) = u_\infty(x, 1_-) + g(x), \tag{38}$$

which follows from (5) and (23); here, $g(x) = -dG(x)/dx$ where $G(x)$ is the kicking potential. Substituting this in (37), we obtain

$$\frac{1}{6}\partial_{\Delta x} S_3(\Delta x) = \langle g(x)g(x+\Delta x)\rangle_x - \langle g(x)u_\infty(x+\Delta x, 0_+)\rangle_x - \langle g(x+\Delta x)u_\infty(x, 0_+)\rangle_x . \quad (39)$$

We now assume that the kicking potential and, hence, $g(x)$ are analytic functions and we find that all three terms on the r.h.s. of (39) are analytic functions of Δx. This follows indeed from the observation that the analyticity of $g(x)$ and the boundedness of $h(x)$ imply the analyticity in Δx of the integral $\int_0^1 g(x)h(x+\Delta x)\,dx$, which is basically a convolution integral. We have thus proved the analyticity of the third-order structure function in the separation Δx.

When the kicking potential $G(x)$ has only a finite number of Fourier harmonics a stronger result holds: the third-order structure function has exactly the same harmonics as the kicking potential. This follows because the r.h.s. of (39) is a convolution integral. For the case of the kicking potential given by (25), which has the harmonics of wavenumber 1 and 3, we thus have

$$S_3(\Delta x) = \lambda \sin(\Delta x) + \mu \sin(3\Delta x). \quad (40)$$

(The presence of only sine functions is due to the odd character of the third-order structure function.) We have indeed checked that the structure function $S_3(\Delta x)$ calculated numerically at the beginning of §4.2 has a global fit of this form with $\lambda \approx -10.9953$ and $\mu \approx -1.1463$ with an error of less than 10^{-5}.

We finally observe that the analyticity result for the third-order structure function is quite general and has been proved also for the case of white-noise forcing (E & Vanden Eijnden 1999b).

4.2.2 Nonanalyticity of the structure functions of order $p \neq 3$

We now concentrate on integer values of $p > 1$. Indeed, for non-integer values, the structure function is not defined, unless we take the absolute value of the velocity increment which results trivially in nonanalyticity and, for $p = 1$, the structure function vanishes. We intend to show that

$$S_p(\Delta x) = \begin{cases} A_p|\Delta x| + B_p(\Delta x)^2 + o\left((\Delta x)^2\right), & \text{for even } p; \\ A_p\Delta x + B_p\Delta x|\Delta x| + o\left((\Delta x)^2\right), & \text{for odd } p, \end{cases} \quad (41)$$

where the constant A_p never vanishes and the constant B_p vanishes for $p = 3$ and never vanishes for $p > 3$ (the expressions of these constants will be given below). This will then imply (i) that all structure functions are proportional to the first power of the separation (a well-known result; see, e.g. E *et al.* 1997) and (ii) that all structure functions of order $p \neq 3$ are nonanalytic functions of Δx. Actually, we shall establish (41) only for $\Delta x > 0$; the extension to $\Delta x < 0$ follows then from the even/odd character of structure functions of even/odd orders.

The idea of the proof is to observe that the only possible sources of nonanalyticity are singularities of the solution in the space-time domain, namely, preshocks, shocks and shock mergers. The contributions from the analytic regions to $S_p(\Delta x)$ is clearly $O\left((\Delta x)^p\right)$ and must therefore be retained in (41) only for $p = 2$. Let us now concentrate on the contributions from singularities.

It is easily shown that preshocks contribute at most terms $O\left((\Delta x)^{(p+5)/3}\right)$ which are only higher order corrections to (41). This follows from the scaling properties of the pdf of increments as discussed in §4.1 and in Bec & Frisch (1999) which is itself a consequence of the cubic root structure of preshocks.

As to the contribution of (mature) shocks, we obtain it by first calculating the contribution, denoted $S_p^{\rm shock}(\Delta x)$, coming from the neighborhood of individual shocks, ignoring shock mergers; then we determine the correction due to mergers, denoted $\Delta S_p^{\rm merge}(\Delta x)$.

Let $X_j(t)$ denote the positions of the various shocks (their number may change in time). Let $u_j^+(x,t)$ and $u_j^-(x,t)$ denote the velocity in the immediate right and left neighborhood of the j-th shock. Let $C_j(t) \equiv u_j^+(X_j(t),t) - u_j^-(X_j(t),t)$ denote the (negative) jump at $X_j(t)$. (The jump $C_j(t)$ is taken equal to zero when a shock has not yet been born or has disappeared by merger.) Since we take $\Delta x > 0$ the requirement that x and $x + \Delta x$ straddle the j-th shock limits the domain of x-integration to the interval $]X_j(t) - \Delta x, X_j(t)[$. Hence, we have

$$S_p^{\rm shock}(\Delta x) = \int_0^1 dt \sum_j \int_{X_j(t)-\Delta x}^{X_j(t)} dx \left[u_j^+(x+\Delta x,t) - u_j^-(x,t)\right]^p. \tag{42}$$

Since $u_j^+(x,t)$ and $u_j^-(x,t)$ are smooth functions of x, we can Taylor expand them near $X_j(t)$. For our interest only the first two terms are relevant. We thus obtain

$$\begin{aligned} S_p^{\rm shock}(\Delta x) = &\int_0^1 dt \sum_j \int_{X_j(t)-\Delta x}^{X_j(t)} dx \Big\{ u_j^+(X_j(t),t) + \left[\partial_x u_j^+(X_j(t),t)\right](x + \Delta x - X_j(t)) \\ &- u_j^-(X_j(t),t) - \left[\partial_x u_j^-(X_j(t),t)\right](x - X_j(t)) \Big\}^p + o\left((\Delta x)^2\right). \end{aligned} \tag{43}$$

We then use the following relation which governs the evolution of shock jumps:

$$\frac{d}{dt} C_j(t) = -\left[\partial_x u_j^+(X_j(t),t) + \partial_x u_j^-(X_j(t),t)\right] \frac{C_j(t)}{2}. \tag{44}$$

(This relation is obtained by using the inviscid Burgers equation on both sides of the shock and the fact that the shock velocity is half the sum of the right and left velocities.) Using (44) in (43), performing all the space integrals and keeping only terms up to $O\left((\Delta x)^2\right)$, we obtain

$$S_p^{\rm shock}(\Delta x) = \int_0^1 dt \sum_j \left[C_j^p(t)\Delta x - \frac{p}{p-1}\frac{d}{dt}\left(C_j^{p-1}(t)\right)(\Delta x)^2\right] + o\left((\Delta x)^2\right). \tag{45}$$

We turn to the contributions of the finite set $\{M_k\}$ of shock mergers taking place in the periodic space-time domain. (We only consider mergers of two shocks since events with more than two shocks merging are not generic; furthermore, it may be checked that they do not change our conclusions.) Associated with each event M_k, we define: $\tilde{t}_k$, the time of merger, $X_{kl}(t)$ and $X_{kr}(t)$, the positions of the left and the right shocks about to merge and $C_{kl}(t)$ and $C_{kr}(t)$, the respective jumps across these merging shocks. Let θ_k be the first instant of time t when the distance $X_{kr}(t) - X_{kl}(t)$ becomes less than Δx. It is clear that, for small Δx, we have $\tilde{t}_k - \theta_k = O(\Delta x)$ (see figure 11). Furthermore, we have

$$X_{kr}(t) - X_{kl}(t) = \Delta x + (t - \theta_k)\frac{C_{kl}(\tilde{t}_k) + C_{kr}(\tilde{t}_k)}{2} + o(\Delta x). \tag{46}$$

After θ_k, an interval $]x, x + \Delta x[$ may either straddle a single of the two merging shocks or both. The calculation above did not take into account the possibility of straddling two shocks.

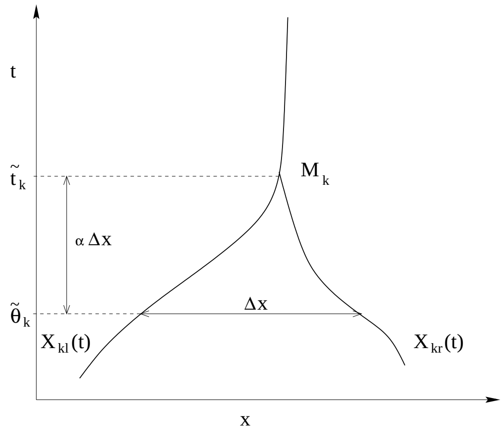

Figure 11: Merger of two shocks $X_{kl}(t)$ and $X_{kr}(t)$ on the (x,t)-cylinder. The merger (event M_k) takes place at $t = \tilde{t}_k$. At the earlier time $t = \theta_k$ the shocks are within a distance Δx.

For $\theta_k < t < \tilde{t}_k$, this happens when $X_{kr}(t) - \Delta x < x < X_{kl}(t)$. We therefore must introduce a correction to (45). To leading order we obtain

$$\Delta S_p^{\mathrm{merge}}(\Delta x) = -(\Delta x)^2 \sum_k \frac{(C_{kl}(\tilde{t}_k) + C_{kr}(\tilde{t}_k))^p - C_{kl}^p(\tilde{t}_k) - C_{kr}^p(\tilde{t}_k)}{C_{kl}(\tilde{t}_k) + C_{kr}(\tilde{t}_k)} + o\left((\Delta x)^2\right). \quad (47)$$

We now assemble the various contributions. The second term in the integral on the r.h.s. of (45) can be integrated explicitly as a sum of terms coming from the birth and death of shocks. Preshocks do not contribute because they have vanishing jumps. Shock mergers give three contributions: two from the incoming shocks with jumps $C_{kl}(\tilde{t}_k)$ and $C_{kr}(\tilde{t}_k)$ and one from the merged shock with jump $C_{kl}(\tilde{t}_k) + C_{kr}(\tilde{t}_k)$. We finally obtain the following expressions for the coefficients in the expansion (41) of the structure functions in terms of the shock jumps $C_j(t)$:

$$A_p = \sum_j \int_0^1 C_j^p(t)\,dt, \quad \text{for all } p > 1 \quad (48)$$

$$B_p = -\sum_k \left\{ \frac{p}{p-1} \left[C_{kl}^{p-1}(\tilde{t}_k) + C_{kr}^{p-1}(\tilde{t}_k) - \left(C_{kl}(\tilde{t}_k) + C_{kr}(\tilde{t}_k)\right)^{p-1} \right] + \frac{(C_{kl}(\tilde{t}_k) + C_{kr}(\tilde{t}_k))^p - C_{kl}^p(\tilde{t}_k) - C_{kr}^p(\tilde{t}_k)}{C_{kl}(\tilde{t}_k) + C_{kr}(\tilde{t}_k)} \right\}, \quad \text{for all } p > 2. \quad (49)$$

For $p = 2$, we must add to the expression given by the r.h.s. of (49) the contribution from the analytic regions which give also terms $O\left((\Delta x)^2\right)$. It is readily seen that B_p given by (49) vanishes for $p = 3$ and only for that value.

It must be stressed that, for structure function of order $p > 3$, the $\propto (\Delta x)^2$ corrections to the leading $\propto \Delta x$ terms come entirely from mergers. For cases which have a single shock with no mergers and no preshocks (an instance is $G(x) = -\cos x$) this correction is absent.

5 Concluding remarks

We have shown that Burgers turbulence in the inviscid limit with periodic large-scale kicking is characterized by universal properties originally conjectured to hold for the case of random forcing. In particular, there is a power-law tail with exponent $-7/2$ in the pdf of negative velocity gradients. This law was proposed by E *et al.* (1997) for the case of random forcing which is smooth in space and white noise in time. The validity of the $-7/2$ law for the latter case is still an open question. It is clear, that the $-7/2$ law is unescapable as soon as preshocks are present and well separated. Some hypothetical fractal clustering of preshocks could invalidate the $-7/2$ law for the white-in-time case. Careful numerical experimentation using a sequence of random forces which approach white-in-time forcing should be able to shed some light on this issue. This can in principle be done using kicked Burgers turbulence in which the instants of kicking are taken closer and closer and the successive spatial impulses are taken random and independent. Obtaining sufficiently clean power-law scaling to distinguish, e.g. between an exponent -3 and $-7/2$ in the random case may require extremely large computational resources. It is probably simpler to just investigate the statistical distribution of preshocks, a question we leave for future work.

Let us briefly now address the question of the effect of a finite small viscosity. Basically, this will broaden the shocks giving them a hyperbolic tangent structure of width $\propto \nu$. From this it is easily inferred that the maximum negative gradient is $|\xi|_{\max} = O(\nu^{-1})$ and that the shoulders of such viscous shocks contribute a term $\propto \nu|\xi|^{-1}$ to the pdf of (negative) gradients (Gotoh & Kraichnan 1998). This term will dominate over the inviscid contribution $\propto |\xi|^{-7/2}$ beyond a crossover value of the gradient $|\xi|_c \propto \nu^{-2/5}$. A small viscosity will also regularize preshocks, giving them a finite velocity gradient $|\xi|_{\max\,\mathrm{pr}} \propto \nu^{-1/2}$ (Crighton & Scott 1979). This gives only subdominant contributions for all ξ's.

The Burgers equation constitutes a dissipative dynamical system because of the presence of shocks which introduce an essentially irreversible element into the dynamics. Many features of our periodically kicked Burgers problem are actually in exact correspondence with those of a conservative (Hamiltonian) dynamical system, namely the equilibrium positions of a one-dimensional chain of (classical) atoms connected by elastic springs in the presence of a space-periodic external potential (Frenkel–Kantorova model). This problem has been investigated by Aubry (1983) and Mather (1982). The potential energy which has to be minimized to obtain the ground state has the following form:

$$H(\{y_j\}) = \sum_j \frac{1}{2}(y_{j+1} - y_j - a)^2 - \varepsilon G(x), \tag{50}$$

where the y_j's are the positions of the atoms, a is the unstretched length of the springs and $\varepsilon > 0$ measures the depth of the periodic external potential $\varepsilon G(x)$. Nontrivial properties of the ground states reflect the competition between the tendency of the atoms to sit at the minimum of the potential $-\varepsilon G(x)$ and to be within a distance a from each other. It is easily checked that the action (in the sense of §2) for the kicked Burgers equation with periodic forcing is exactly given by the Aubry–Mather Hamiltonian (50) if we take a forcing potential $\varepsilon G(x)$ and a mean velocity $\langle u \rangle = a$. The velocity in the Burgers equation is now the analog of the distance between adjacent atoms. Note that j is a space index in the chain model, whereas it is a time index in the Burgers equation. In this paper we have assumed a vanishing mean velocity ($a = 0$). In this case there is no competition between the aforementioned two tendencies, and the global minimizer $y_j = x_c$ corresponds to a trivial ground state minimizing both parts of the Hamiltonian. More delicate effects arising in the case $\langle u \rangle \neq 0$ and in more than one dimension will be discussed in forthcoming work. We note that the connection between Aubry–Mather theory and Burgers equation for the case of time-periodic potentials with continuous time was discussed for the first time in Jauslin, Kreiss & Moser (1997) and in E *et al.* (1998). The theory has been recently developed further in E (1999) and in Sobolevski (1999).

Finally, we discuss some possible extensions of the ideas of the present paper to Navier–Stokes turbulence. Obviously, the method of forcing by kicks applied at discrete instants of time can also be used for Navier–Stokes. One of the things which made this method particularly valuable for the Burgers case is the existence of an efficient numerical algorithm to solve the purely decaying Burgers equation in the inviscid limit. At the moment there is nothing comparable for Navier–Stokes. Note that periodic kicking will not result in a unique time-periodic solution for Navier–Stokes since periodicity-breaking bifurcations leading to chaos will unavoidably occur.

The results concerning analyticity of structure functions are likely to be the same for Burgers turbulence and isotropic 3-D Navier–Stokes turbulence (in the limit of vanishing viscosity). Indeed, in the latter case all structure functions of order $p \neq 3$ are generally believed to have scaling properties at small separations with nontrivial (and certainly noninteger) exponents. As to the third-order structure function, our proof of analyticity for analytic-in-space forcing can be extended if one assumes that Kolmogorov's four-fifths law is valid (Kolmogorov 1941; see also Chapter 6 of Frisch 1995).

Burgers turbulence presents an algebraic tail for the pdf of the velocity gradient but nothing similar is known for Navier–Stokes turbulence. This may be telling us something about the possible singularities associated to the Navier–Stokes and Euler equations. A remarkable feature for the Burgers equation is that only preshocks but not mature shocks contribute to the power-law tail. There is indeed a basic difference between the two types of singularities. For analytic forcing, near a mature shock, the solution is not only piecewise analytic (i.e. on each side of the shock) but uniformly so: the radius of convergence of the Taylor series remains finite as one approaches a mature shock. In contrast, when approaching a cubic root preshock singularity, the radius of convergence goes to zero and gradients become very large. It is the algebraic behavior of their strength which causes the power-law tail. If the Navier–Stokes equation, in the limit of vanishing viscosity, were to develop any singularity of this kind (accompanied by algebraically large gradients) it should also display a power-law-tail pdf. Further increases in the quality of experimental and numerical turbulence or convection data are needed to find if such singularities are really ruled out.

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APPENDIX

Statement of the results for periodic kicking

Here, we formulate and prove formally the statements presented in §3. The kicking is assumed 1-periodic in both space and time. The force in the Burgers equation is given by

$$f(x,t) = -\frac{d}{dx}G(x) \sum_{j=-\infty}^{+\infty} \delta(t-j), \tag{A.1}$$

where $G(x)$ is a deterministic function of x which is 1-periodic and three times continuously differentiable ($G \in C^3$). For some of the statements below it is assumed that G is analytic.

We shall also assume that the kicking potential $G(x)$ is generic in the Morse sense. This implies that $G(x)$, considered on the circle $0 \le x < 1$, has a unique point of maximum x_c and that $G(x)$ is non-degenerate at x_c, i.e. $\partial_{xx}G(x_c) < 0$. Without loss of generality we can assume that $G(x_c) = 0$. We denote by $c = -\partial_{xx}G(x_c)/2 > 0$. The initial potential $\psi_0(x)$ is also assumed 1-periodic. This implies that $\int_0^1 u_0(x)dx = 0$, where $u_0(x) = u(x, 0_+)$ is the initial velocity.

We now solve the unforced Burgers equation between the times $t = 0$ and $t = 1$ and get $\hat{u}(x) \equiv u(x, 1_-)$. Then, we determine $u(x, 1_+) = \hat{u}(x) + g(x)$, where $g(x) = -\partial_x G(x)$. Denote by B_g the transformation from $u(x, 0_+)$ to $u(x, 1_+) : B_g u = \hat{u} + g$. Clearly, B_g transforms $L_0^1 = \{u(x) \in L^1[0, 1], \int_0^1 u(x)dx = 0\}$ into itself. Then, the following statements hold.

S1 The functional transformation B_g has a unique fixed point u_∞: $B_g u_\infty = u_\infty$, $\max_x |u_\infty(x)| \le 1/2 + \max_x |g(x)|$. The fixed point u_∞ is a function of bounded variation, it is continuous everywhere except at the set of shock points, which is at most countably infinite.

S2 Let x be a point of continuity for u_∞. Then, for all u_0 such that $\int_0^1 u_0(x)dx = 0$,

$$B_g^n u_0(x) \to u_\infty(x) \quad \text{as} \quad n \to \infty. \tag{A.2}$$

S3 The unique global minimizer γ_c corresponds to a particle with zero velocity sitting at the point x_c of maximum kicking potential.

S4 There exists a unique entropy weak solution $u_\infty(x, t)$ to the kicked Burgers equation in the semi-infinite domain $]-\infty, T]$ with zero mean velocity. This solution is 1-periodic in time and it is generated by $u_\infty(x)$: $u_\infty(x, k+) = u_\infty(x)$ for all integer k. The solution $u_\infty(x, t)$ satisfies the estimate: $\max_{x,t} |u_\infty(x, t)| \le 1/2 + \max_x |g(x)|$.

S5 For arbitrary t there exists a unique main shock.

S6 Convergence in (A.2) is exponentially fast in n. If u_∞ is differentiable at x, then there exists a constant $C(x)$ which does not depend on u_0 such that

$$|B_g^n u_0(x) - u_\infty(x)| \le C(x)\lambda^{-n}, \tag{A.3}$$

where $\lambda = 1 + c + \sqrt{c^2 + 2c} > 1$. If x_c is a point of continuity for u_0 and $u_0(x_c) \ne 0$ then there exists a constant $c(u_0) > 0$ such that for all x the following estimate holds:

$$|B_g^n u_0(x) - u_\infty(x)| \ge c(u_0)\lambda^{-n}. \tag{A.4}$$

S7 If the kicking potential $G(x)$ is analytic, then for any t the solution $u_\infty(x, t)$ is a piecewise analytic function of x. The number of pieces is finite and is equal to the total number of shocks at time t. The number of preshock events between time 0 and 1 is also finite.

Proof

Uniqueness of the solution

Denote by $\mathcal{T}$ the Euler–Lagrange diffeomorphism of the phase-space cylinder $\mathcal{C} \equiv \{0 \le y < 1, -\infty < v < \infty\}$ generated by the system of equations (15), i.e. $\mathcal{T}(y, v) = (y', v')$, where

$$\begin{aligned} v' &= v + g(y) \\ y' &= y + v + g(y) \pmod 1. \end{aligned} \tag{A.5}$$

Since $G \in C^3$, the diffeomorphism $\mathcal{T} \in C^2$. The global minimizer in the sense of §2.1 is a sequence $\{y_j = x_c\}$ for all j. Here, it is trivial and it corresponds to a fixed point $P = (x_c, 0)$ for the diffeomorphism $\mathcal{T}$, i.e. the corresponding trajectory of $\mathcal{T}$ is a stationary trajectory $(y_j, v_j) = (x_c, 0) = P$. Let $\{y'_j,\ j \le J,\ y'_J = x\}$ be an arbitrary minimizer. Then $y'_j \to x_c$ as $j \to -\infty$. Indeed, if it were not the case, then a sequence $\{y''_J = x,\ y''_j = x_c,\ j_0 \le j \le J-1,\ y''_j = y'_j,\ j \le j_0 - 1\}$ would have smaller action than $\{y'_j\}$ for any j_0 which is sufficiently negative. (The action is given by (21).) Convergence $y'_j \to x_c$ implies that the corresponding trajectory of $\mathcal{T}$, $\{(y'_j, v'_j = \pm\rho(y'_j, y'_{j-1}))\}$ converges to P as $j \to -\infty$. Here, ρ denotes the distance between two points on the circle. Easy calculation shows that P is a saddle point for the diffeomorphism $\mathcal{T}$ with the eigenvalues $\lambda_1 = \lambda = 1 + c + \sqrt{c^2 + 2c} > 1$ and $0 < \lambda_2 = 1 + c - \sqrt{c^2 + 2c} < 1$. It follows that there exist two C^2-smooth curves $\Gamma^{(\mathrm{s})}$ and $\Gamma^{(\mathrm{u})}$ which are stable and unstable manifolds for the point P. Both curves pass through P and consist of all points (y, v) whose trajectories approach P as j tends to $+\infty$ and $-\infty$, respectively (see figure 4). In fact, convergence is exponentially fast and the rate is given by $1/\lambda$. Since $(y'_j, v'_j) \to P$ as $j \to -\infty$, a point (y'_J, v'_J) belongs to $\Gamma^{(\mathrm{u})}$. Let us associate a minimizing curve $\gamma_{x,J}$ on the space-time cylinder $M = \{0 \le y < 1,\ -\infty < t < +\infty\}$ with an arbitrary minimizer $\{y'_j, j \le J\}$. To construct $\gamma_{x,J}$ we just connect all the pairs of points $(y'_j, j), (y'_{j-1}, j-1)$ for $j \le J$ by straight segments of minimal length. Denote by $y'_{x,J}(t)$ a piecewise linear function such that $\gamma_{x,J} = \{(y'_{x,J}(t), t),\ -\infty < t < +\infty\}$. Clearly, $y'_{x,J}(j) = y'_j$. Denote also by γ the minimizing curve $\{(x_c, t),\ -\infty < t < +\infty\}$ corresponding to a global minimizer $\{y_j = x_c\}$.

We have shown above that any minimizing curve $\gamma_{x,J}$ is asymptotic to γ as $t \to -\infty$. By a standard argument this implies that any two minimizing curves do not intersect each other, except if they start from the same point x. It follows that for all but at most countably many x and all J there exists a unique minimizing curve $\gamma_{x,J}$. Denote by S the exceptional set of x's where a minimizer is not unique. Obviously, for fixed x and different J's minimizing curves $\gamma_{x,J}$ are connected by a time shift. It follows that the set S does not depend on J. Hence, for all x outside of S one can define a function $v(x) \equiv dy'_{x,J}(t)/dt\ |_{t=J_-} = \pm\rho(y'_J, y'_{J-1})$. It is easy to see that for any solution $u(x,t)$, $t \le T$ to the kicked Burgers equation in the semi-infinite domain $]-\infty, T]$ one has: $u(x, k_-) = v(x)$ for all integer $k \le T$ and x outside of S. This implies that the solution to the kicked Burgers equation in the semi-infinite domain $]-\infty, t]$ is unique and is generated by $v(x)$. It also follows that $u_\infty(x) = v(x) + g(x)$ is a unique fixed point for B_g. The set S is a set of shocks at integer moments of time. It follows from the closeness of the set of minimizers that $v(x)$ is continuous outside of S. The non-intersecting property implies that for arbitrary $x \in S$ there exist $v(x_-) = \lim_{y \to x_-} v(y)$, $v(x_+) = \lim_{y \to x_+} v(y)$, and that $v(x_-) > v(x_+)$. It also follows that $v(x)$ is a function of bounded variation. Clearly, $v(x_-)$ and $v(x_+)$ are the velocities of two minimizing curves which start at (x, J) for any integer J. Notice that there can be more than two minimizing curves starting at a shock point x; their velocities are between $v(x_+)$ and $v(x_-)$. This happens, e.g., at shock mergers. Since $|v(x)| = \rho(y'_J, y'_{J-1})$, we have: $\max_x |v(x)| \le 1/2$, $\max_x |u_\infty(x)| \le 1/2 + \max_x |g(x)|$. We have seen above that the points of continuity of u_∞, and hence of v, are exactly the points of uniqueness of minimizing curves. Since any limiting point of a sequence $B_g^n u_0(x) - g(x)$ gives a velocity of a minimizing curve starting at (x, J), we have $\lim_{n\to\infty}(B_g^n u_0(x) - g(x)) = v(x)$, or $B_g^n u_0(x) \to u_\infty(x)$ as $n \to \infty$ for any x outside of S. Suppose now that there exists another global minimizer $\{\bar{y}_j\}$. The same argument as above shows that necessarily $\lim_{j\to\infty} \bar{y}_j = \lim_{j\to-\infty} \bar{y}_j = x_c$. If $\bar{y}_i \ne x_c$ for some i, then one can construct a sequence with smaller action by taking $\tilde{y}_j = x_c$, $|j| \le j_0$, $\tilde{y}_j = \bar{y}_j$, $|j| > j_0$ for sufficiently large j_0. Such construction contradicts to global minimality of $\{\bar{y}_j\}$ and proves uniqueness of the global minimizer. Statements 1 - 4 are thus proved.

Uniqueness of the main shock

Till now we have not used the hyperbolicity properties of the fixed point P. We have seen above that if v is a velocity of a minimizer which starts at (x, J), then the point (x, v) belongs to $\Gamma^{(\mathrm{u})}$. Denote by s the natural parameter of length along $\Gamma^{(\mathrm{u})}$, i.e.

$$\Gamma^{(\mathrm{u})} = (x(s), v(s)),\ \ s \in \mathbb{R},\ \ (dx/ds)^2 + (dv/ds)^2 = 1,\ \ (x(0), v(0)) = (x_c, 0). \tag{A.6}$$

The orientation of s is fixed by the condition $x(s) \uparrow x(0)$ as $s \uparrow 0$. Define $(x_j(s), y_j(s)) \equiv \mathcal{T}^j(x(s), v(s)),\ j \le 0$ and the C^2 function

$$A(s) \equiv \sum_{j \le 0} \left[\frac{\rho^2(y_j, y_{j-1})}{2} - G(y_j) \right]. \tag{A.7}$$

The series above converges since $G(x_c) = 0$. It is easy to see that a point $(x(s), v(s))$ corresponds to a minimizer if and only if $A(s) = \min_{\tilde{s}: x(\tilde{s}) = x(s)} A(\tilde{s})$. Denote by $\bar{A}(x) \equiv \min_{\tilde{s}: x(\tilde{s}) = x} A(\tilde{s})$. Notice that for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $A(s) > \delta(\epsilon)$ for all $|s| > \epsilon$. Fix ϵ small enough, so that $x(s)$ is a monotone function for $|s| \le \epsilon$. Now, choose ϵ_0 so small that $\max_{|s| \le \epsilon_0} A(s) < \delta(\epsilon)$. Then for all $|s| \le \epsilon_0$, a point $(x(s), v(s))$ corresponds to the unique minimizer at $x(s)$. Hence, we have shown that there are no shocks inside some neighborhood of x_c.

We now construct the main shock. Fix an arbitrary time t. Consider the situation on the space-time cylinder M. All minimizing curves approach the global minimizing curve γ either from the right or from the left (see figure 6). Denote by $B_r(t)$ and $B_l(t)$ the sets of points x on the circle such that there exists a minimizing curve starting at (x, t) which approaches γ from the right and from the left, respectively. Since minimizing curves do not intersect, $B_r(t)$ and $B_l(t)$ are closed intervals, $B_r(t) \bigcup B_l(t) = S^1 = [0, 1[$ and $B_r(t) \bigcap B_l(t)$ consists of just two points. One of them is x_c. Denote the other one $x_{\mathrm{msh}}(t)$. It follows immediately from the construction that $x_{\mathrm{msh}}(t)$ is a shock point; moreover it is the main shock. To prove uniqueness consider any other shock at time $t = J$ at point x. Then, $(x, v(x_-))$ and $(x, v(x_+))$ either belong both to the negative-s part of $\Gamma^{(\mathrm{u})}$ or both to its positive-s part; that is, there exists s_1, s_2 such that $(x, v(x_-)) = (x(s_1), v(s_1))$ and $(x, v(x_+)) = (x(s_2), v(s_2))$ with $s_1 s_2 > 0$. For j sufficiently negative, both $(x_j(s_1), y_j(s_1)) = \mathcal{T}^j(x(s_1), v(s_1))$ and $(x_j(s_2), y_j(s_2)) = \mathcal{T}^j(x(s_2), v(s_2))$ belong to an ϵ_0-neighborhood of $(x(0), v(0))$ where there is no shock. Hence a prehistory of an original shock is not longer than $|j|$. Statement 5 is thus proved.

Exponential convergence to the unique solution

Consider a small neighborhood U of the point P. It is well known (Hartman 1960; Belitskii 1973) that if U is small enough then, inside U, $\mathcal{T}$ is C^1-smoothly conjugate to a linear transformation. This means that there exists a local C^1-smooth change of variables such that in the new coordinates (X, V) the map $\mathcal{T}$ becomes $\mathcal{T}_\lambda : (X, V) \mapsto (\lambda X, \lambda^{-1} V)$ and P is the origin in the coordinates X, V. Denote $u_n(x) \equiv B_g^n u_0$ and recall that $\int_0^1 u_0(x) dx = 0$. For arbitrary x consider a point $(x, v) = (x, u_n(x))$ on the cylinder $\mathcal{C}$ and its backward trajectory $(x(-j), v(-j)) = \mathcal{T}^{-j}(x, v),\ 0 \le j \le n$. It is easy to see that there exists a neighborhood $U_1 \subset U$ such that if two points of this backward trajectory belong to U_1, then all points in between belong to U. Notice, that there exists n_1 which depends only on U_1 such that, uniformly in n, at most n_1 points of the backward trajectory $(x(-j), v(-j)),\ 0 \le j \le n$, are outside of U_1. This implies that for some j_1, $0 \le j_1 \le n_1$, and j_2, $n - n_1 \le j_2 \le n$, we have $(x(-j_1), v(-j_1)) \in U_1, (x(-j_2), v(-j_2)) \in U_1$. Hence, $(x(-j), v(-j)) \in U$ for all $j_1 \le j \le j_2$ and $j_2 - j_1 \ge n - n_1$. Then, there exists a constant $C_1 > 0$ such that the distance between $(x(-j_1), v(-j_1))$ and a piece of local unstable manifold inside U is less than $C_1\lambda^{-n}$. Denote by $]s^{(1)}, s^{(2)}[$, $s^{(1)} < 0$, $s^{(2)} > 0$ an interval of the parameter which corresponds to this piece of local unstable manifold. Then the Euclidian distance $\mathrm{dist}\,((x(-j_1), v(-j_1)), (x(s), v(s))) \le C_1\lambda^{-n}$ for some $s_n \in [s^{(1)}, s^{(2)}]$. Denote by $\bar{s}^{(1)}$ and $\bar{s}^{(2)}$ the values of the parameter s corresponding to $\mathcal{T}^{n_1}(x(s^{(1)}), v(s^{(1)}))$ and $\mathcal{T}^{n_1}(x(s^{(2)}), v(s^{(2)}))$, respectively. Then, there exists a constant $\tilde{C} > 0$ such that $\mathrm{dist}\,((x, v), (x(\bar{s_n}), v(\bar{s}_n))) \le \tilde{C}\lambda^{-n}$ for some $\bar{s_n} \in [\bar{s}^{(1)}, \bar{s}^{(2)}]$. It might happen that $\bar{s_n}$ does not correspond to a minimizer. However, $A(\bar{s}_n) \to \bar{A}(x)$ as $n \to \infty$. More precisely, one can show that there exists a constant $C_2 > 0$ such that

$$A(\bar{s}_n) - \bar{A}(x(\bar{s}_n)) \le C_2 n\lambda^{-n}, \quad |\bar{A}(x(\bar{s}_n)) - \bar{A}(x)| \le C_2\lambda^{-n} \tag{A.8}$$

Suppose now that u_∞ is differentiable at x (recall that it is differentiable almost everywhere). Denote by s_x the value of parameter s corresponding to the unique minimizer at x. Then, $dx(s)/ds\,|_{s=s_x} \ne 0$. Then, there exist ϵ, $\delta > 0$ such that $|dx(s)/ds| \ge \delta$, $|\frac{dy}{dx}(s)| \le \delta^{-1}$ for all $s \in]s_x - \epsilon, s_x + \epsilon[$. Denote by $\nu \equiv \min_{s \in [\bar{s}^{(1)}, s_x - \epsilon] \bigcup [s_x + \epsilon, \bar{s}^{(2)}]} \mathrm{dist}\,\{(x, \bar{A}(x)), (x(s), A(s))\} > 0$. Clearly, there exists N which depends only on ν such that $\mathrm{dist}\,\{(x, \bar{A}(x)), (x(\bar{s}_n), A(\bar{s}_n))\} < \nu$ for all $n > N$. Hence, $\bar{s}_n \in]s_x - \epsilon, s_x + \epsilon[$ for all $n > N$. This and the estimates for the derivatives immediately imply that there exists a constant $C(x) > 0$ such that $|B_g^n u_0(x) - u_\infty(x)| \le C(x)\lambda^{-n}$.

To prove an estimate from below, notice that, for any backward trajectory $(x(-j), v(-j))$, $0 \le j \le n$, the last point $(x(-n), v(-n))$ cannot be too close to P if $u_0(x_c) \ne 0$. Indeed, the initial potential ψ_0 has non-zero slope at x_c since we assumed $u_0(x_c) \ne 0$. Hence, it is possible to make the action smaller by moving further from x_c. It is easy to show that there exist $\epsilon(u_0) > 0$ such that $\mathrm{dist}\,\{(x(-n), v(-n)), (x_c, 0)\} \ge \epsilon(u_0)$. It follows that a point (x, v) cannot be too close to $(x(s), v(s))$, $s \in [\bar{s}^{(1)}, \bar{s}^{(2)}]$, i.e. there exists a constant $c(u_0) > 0$ such that $|B_g^n u_0(x) - u_\infty(x)| \ge c(u_0)\lambda^{-n}$. Statement 6 is thus proved.

Analyticity and finiteness of the number of shocks

To prove Statement 7 we first notice that $\Gamma^{(\mathrm{u})}$ is analytic, provided G is analytic (see Moser 1956). Denote by $\{x_i\}$ the (at most countable) set of shock points other than the main shock at an integer time J. As we have already seen above, for every x_i there exists an open set $]s_1(i), s_2(i)[$ of the parameter s, where the parameter values $s_1(i), s_2(i)$ correspond to the points $(x_i, v(x_{i-}))$ and $(x_i, v(x_{i+}))$, respectively. Since x_i is not the main shock, $s_1(i)s_2(i) > 0$. The non-intersecting property of minimizing curves implies that different intervals $]s_1(i), s_2(i)[$ do not intersect. Also, $]s_1(i), s_2(i)[\subset [s_{2\mathrm{msh}}, s_{1\mathrm{msh}}]$, where $s_{1\mathrm{msh}}$ and $s_{2\mathrm{msh}}$ are the parameter values corresponding to $(x_{\mathrm{msh}}, v(x_{\mathrm{msh}-}))$ and $(x_{\mathrm{msh}}, v(x_{\mathrm{msh}+}))$, respectively. It is easy to see that all $s \in [s_{2\mathrm{msh}}, s_{1\mathrm{msh}}] - \bigcup_i (s_1(i), s_2(i))$ correspond to minimizers. Clearly, for all i there exists $s(i) \in (s_1(i), s_2(i))$ such that $dx(s)/ds\,|_{s=s(i)} = 0$. Suppose that there be infinitely many shocks. Then there exists an accumulation point s_∞ for the sequence $\{s(i)\}$. It follows that all derivatives of $x(s)$ vanish at s_∞. The analyticity of $x(s)$ then implies that $x(s)$ is a constant function. This contradiction proves that the number of shocks is finite. The same argument works for all times t, since for all t the Lagrangian map transforms $\Gamma^{(\mathrm{u})}$ into an analytic curve. Denote by $\Gamma_t^{(\mathrm{u})} = (x_t(s), v_t(s))$ for $0 \le t \le 1$ the image of $\Gamma^{(\mathrm{u})}$ under the Lagrangian map at time t, where s is a natural parameter along $\Gamma_t^{(\mathrm{u})}$. Suppose that the number of non-main shocks at time t is K, so that the total number of shocks is $K+1$. As above, denote by $]s_1(i), s_2(i)[$ the intervals of the parameter s generated by the i-th shock, and by $s_{2\mathrm{msh}}$ and $s_{1\mathrm{msh}}$ the values of the parameter corresponding to the main shock. The intervals $]s_1(i), s_2(i)[$, $1 \le i \le K$, divide $[s_{2\mathrm{msh}}, s_{1\mathrm{msh}}]$ into $K+1$ closed intervals I_i, $1 \le i \le K+1$. It is easy to see that each of those intervals corresponds to an analytic piece of $u_\infty(x,t)$, i.e. $u_\infty(x(s),t) = v_t(s)$, $s \in I_i$. Finally, we show that the number of preshock events between time 0 and 1 is finite. Suppose the number of preshocks were infinite. Denote by t_i, s_i the time of the i-th preshock and the corresponding value of the parameter s. Denote by (t^*, s^*) an arbitrary point of accumulation for a sequence (t_i, s_i). It is easy to see that all derivatives of $x_{t^*}(s)$ vanish at point s^*. This implies that $x_{t^*}(s)$ is a constant function. Again, we get a contradiction, which finishes the proof of Statement 7.

References

[1] Aubry, S. 1983. The twist map, the extended Frenkel–Kontorova model and the devil's staircase, *Physica* D **7**, 240–258.

[2] Barabási, A.-L. & Stanley, H.E. 1995. *Fractal Concepts in Surface Growth*, Cambridge University Press, Cambridge.

[3] Bec, J. & Frisch U. 1999. Pdf's of derivatives and increments for decaying Burgers turbulence, cond-mat/9906047, *submitted to Phys. Rev.* E.

[4] Belitskii, G.R. 1973. Functional equations and conjugacy of local diffeomorphisms of a finite smoothness class, *Funct. Analys. Applic.* **7**, 268–277. (Translated from Russian: Funktsional'nyi Analiz i Ego Prilozheniya **7** (1973), 17–28).

[5] Boldyrev, S.-A. 1997. Velocity-difference probability density functions for Burgers turbulence, *Phys. Rev.* E **55**, 6907–6910.

[6] Bouchaud, J.-P., Mézard, M. & Parisi, G. 1995. Scaling and intermittency in Burgers turbulence, *Phys. Rev.* E **52**, 3656–3674.

[7] Cole, J.D. 1951. On a quasi-linear parabolic equation occurring in aerodynamics, *Quart. Appl. Math.* **9**, 225–236.

[8] Crighton, D.G. & Scott, J.F. 1979. Asymptotic solutions of model equations in nonlinear acoustics, *Phil. Trans. Roy. Soc. London* **292**, 101–134.

[9] E, W. 1999. Aubry–Mather theory and periodic solutions for the forced Burgers equation, *Comm. Pure Appl. Math.* **52**, 0811–0828.

[10] E, W., Khanin, K., Mazel, A. & Sinai, Ya. 1997. Probability distribution functions for the random forced Burgers equation, *Phys. Rev. Lett.* **78**, 1904–1907.

[11] E, W., Khanin, K., Mazel, A. & Sinai, Ya. 1998. Invariant measures for Burgers equation with stochastic forcing, *submitted to Ann. Math.*

[12] E, W. & Vanden Eijnden, E. 1999a. Asymptotic theory for the probability density functions in Burgers turbulence, *Phys. Rev. Lett.* **83**, 2572-2575. (chao-dyn/9901006)

[13] E, W. & Vanden Eijnden, E. 1999b. Statistical theory for the stochastic Burgers equation in the inviscid limit, *Comm. Pure Appl. Math.*, in press.

[14] Frisch, U. 1995. *Turbulence: the Legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge.

[15] Gotoh, T. 1999. Probability density functions in steady-state Burgers turbulence, *Phys. Fluids.* **11**, 2143–2148.

[16] Gotoh, T. & Kraichnan, R.H. 1998. Steady-state Burgers turbulence with large-scale forcing, *Phys. Fluids.* **10**, 2859–2866.

[17] Hartman, P. 1960. On local homeomorphisms of Euclidean spaces, *Bol. Soc. Mat. Mexicana* (2) **5**, 220–241.

[18] Hénon, M. 1983, Numerical exploration of Hamiltonian systems, in *Chaotic behaviour of deterministic systems, Les Houches 1981* G. Iooss, R. Helleman & R. Stora, eds., pp. 53–170, North Holland.

[19] Hopf, E. 1950. The partial differential equation $u_t + uu_x = u_{xx}$, *Comm. Pure Appl. Mech.* **3**, 201–230.

[20] Jauslin, H.R., Kreiss, H.O. & Moser, J. 1997. On the forced Burgers equation with periodic boundary conditions, preprint.

[21] Kardar, M., Parisi, G. & Zhang, Y.-C. 1986. Dynamical scaling of growing interfaces, *Phys. Rev. Lett.* **56**, 889–892.

[22] Kolmogorov, A.N. 1941. Dissipation of energy in locally isotropic turbulence, *Dokl. Akad. Nauk SSSR* **32**, 16–18 (reprinted in *Proc. R. Soc. Lond.* A **434**, 15–17 (1991)).

[23] Kraichnan, R.H. 1999. Note on forced Burgers turbulence, chao-dyn/9901023.

[24] Lax, P.-D. 1957. Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.* **10**, 537–566.

[25] Manneville, P. 1990, *Dissipative Structures and Weak Turbulence*, Perspectives in Physics, Academic Press.

[26] Mather, J.N. 1982. Existence of quasi-periodic orbits for twist homeomorphisma of the annulus, *Topology* **21**, 457–467.

[27] Moser, J. 1956. The analytic invariants of an area-preserving mapping near a hyperbolic fixed point, *Comm. Pure Appl. Math.* **9**, 673–692.

[28] Noullez, A. & Vergassola, M. 1994. A fast algorithm for discrete Legendre transforms, *J. Sci. Comp.* **9**, 259–281.

[29] Oleinik, O. 1957. Discontinuous solutions of nonlinear differential equations, *Uspekhi Mat. Nauk* **12**, no 3, 3–73. (*Russ. Math. Survey.*, Amer. Math. Transl. Series 2 **26**, 95–172).

[30] Polyakov, A.M. 1995. Turbulence without pressure, *Phys. Rev.* E **52**, 6183–6188.

[31] She, Z.S., Aurell, E. & Frisch, U. 1992. The inviscid Burgers equation with initial data of Brownian type, *Commun. Math. Phys.* **148**, 623–641.

[32] Sobolevski, A. N. 1999 On periodic solutions of a Hamilton–Jacobi equation with periodic forcing, *Mat. Sbornik* (to appear, in Russian). Also chao-dyn/9906035 in English.

[33] Trussov, A. 1996. Linear time algorithms for discrete Legendre transform and for the numerical simulation of Burgers' equation, *Computational Seismology* **29**, 45–53.

[34] Vergassola, M., Dubrulle, B., Frisch, U. & Noullez, A. 1994. Burgers' equation, Devil's staircases and the mass distribution for large-scale structures, *Astron. Astrophys.* **289**, 325–356.

FIGURE CAPTIONS

Figure 1: Snapshots of the velocity for the unique time-periodic solution corresponding to the kicking force $g(x)$ shown in the upper inset; the various graphs correspond to six output times equally spaced during one period. The origin of time is taken at a kick. Notice that during each period, two new shocks are born and two mergers occur.

Figure 2: Exponential relaxation to a time-periodic solution for three different initial velocity data as labelled. The horizontal axis gives the time elapsed since $t = 0$.

Figure 3: Evolution of shock positions during one period. The beginnings of lines correspond to births of shocks (preshocks) at times $t_{\star 1}$ and $t_{\star 2}$; shock mergers take place at times t_{c1} and t_{c2}. The "main shock", which survives for all time, is shown with a thicker line.

Figure 4: Sketch of a hyperbolic fixed point P with stable ($\Gamma^{(\mathrm{s})}$) and unstable ($\Gamma^{(\mathrm{u})}$) manifolds. The dashed line gives the orbit of successive iterates of a point near the stable manifold.

Figure 5: Unstable manifold $\Gamma^{(\mathrm{u})}$ on the (x, v)-cylinder (the x-coordinate is defined modulo 1) which passes through the fixed point $P = (x_c, 0)$. The bold line is the graph of $u_\infty(x, 1_-)$. The main shock is located at $x_l = x_r$. Another shock at x_1 corresponds to a local zig-zag of $\Gamma^{(\mathrm{u})}$ between A and B.

Figure 6: Minimizers (trajectories of fluid particles) on the (x, t)-cylinder. Time starts at $-\infty$. Shock locations at $t = 0_-$ are characterized by having two minimizers (an instance is at x_1). The main shock is at $x_l = x_r$. The fat line $x = x_c$ is the global minimizer.

Figure 7: Pdf of the velocity gradient at negative values in log-log coordinates. Upper inset: local scaling exponent. A power law with exponent $-7/2$ is obtained at large arguments.

Figure 8: Same as figure 7 with the second space derivative of the velocity. The exponent is now -2.

Figure 9: Upper part: Pdf of (negative) velocity increments in log-log coordinates for various values of the separation Δx in geometric progression from $2\pi/N$ to $2^7(2\pi/N)$. Lower part: the corresponding local scaling exponents.

Figure 10: Structure functions $S_p(\Delta x)$ for $p = 2, 3, 4, 5$ as labelled. Note the linear behavior at small Δx.

Figure 11: Merger of two shocks $X_{kl}(t)$ and $X_{kr}(t)$ on the (x, t)-cylinder. The merger (event M_k) takes place at $t = \tilde{t}_k$. At the earlier time $t = \theta_k$ the shocks are within a distance Δx.