The Fundamental Differential Form and Boundary Value Problems

A.S. Fokas[1], Maxim Zyskin
Basic Research Institute in the Mathematical Sciences
HP Laboratories Bristol
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Recently a new method for solving linear and integrable nonlinear PDE's in 2 dimensions has been introduced. This method is based on the construction of appropriate integral representations for both the solution and the spectral function. Here we present an alternative approach for constructing these integral representations. This approach is based on the introducing of what we call a fundamental differential form, and on a generalization of Green's formula. The new approach is illustrated for the Laplace and the modified Helmholtz equations in a convex polygon, for a general dispersive equation on the half-line, and for the heat equation on a domain involving a moving boundary.

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[1] Department of Mathematics, Imperial College, London, SW7 2BZ UK

The Fundamental Differential Form and Boundary Value Problems

A.S.Fokas [*], M. Zyskin [†]

Abstract

Recently a new method for solving linear and integrable nonlinear PDE's in 2 dimensions has been introduced. This method is based on the construction of appropriate integral representations for both the solution and the spectral function. Here we present an alternative approach for constructing these integral representations. This approach is based on the introducing of what we call a fundamental differential form, and on a generalization of Green's formula. The new approach is illustrated for the Laplace and the modified Helmholtz equations in a convex polygon, for a general dispersive equation on the half-line, and for the heat equation in a domain involving a moving boundary.

1 Introduction

A new approach for solving linear and integrable nonlinear PDE's was announced in [1] and further developed in [2]-[5]. For linear PDE's it involves 3 steps: 1. Given a PDE, construct two compatible linear eigenvalue equations [6], called a *Lax pair* [7]. 2. Given a domain, perform the *simultaneous spectral analysis* of the Lax pair. This yields an integral representation of the solution $q(x_1, x_2)$ in terms of a function $\hat{q}(k)$, called the *spectral function*, and an integral representation of $\hat{q}(k)$ in terms of q and of its derivatives on the boundary of the domain. 3. Given appropriate boundary conditions, analyse the *global relation* satisfied by the spectral function; this gives $\hat{q}(k)$ in terms of the given boundary conditions.

As an illustration of the above steps, consider the modified Helmholtz equation

$$q_{xx} + q_{yy} - 4\beta^2 q = 0, \tag{1.1}$$

[*]Department of Mathematics, Imperial College, London SW7 2BZ, UK.

[†]BRIMS, Hewlett Packard Labs, Filton Road, Stoke Gifford, Bristol, BS34 8QZ, UK.

where β is a constant.

A Lax pair of equation (1.1) is given by

$$\mu_x - i(k - \tfrac{\beta^2}{k})\mu = \tfrac{1}{2}(q_x - iq_y) + \tfrac{i\beta^2}{k}q,$$
$$\mu_y + (k + \tfrac{\beta^2}{k})\mu = \tfrac{1}{2}(iq_x + q_y) + \tfrac{\beta^2}{k}q. \tag{1.2}$$

Indeed, it is straightforward to verify that the compatibility of equations (1.2) yields equation (1.1).

Let q satisfy equation (1.1) in a convex polygon with corners $z_1, \ldots z_n$, $z_{n+1} = z_1$, $z = x + iy$, see Fig. 1.

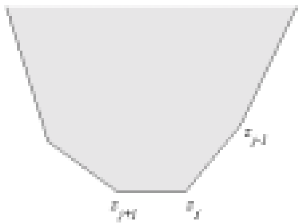


Figure 1.

It is shown in [3] that $q(x, y)$ can be expressed in the form

$$q(x,y) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{l_j} e^{ikz - \frac{i\beta^2}{k}\bar{z}} \, \rho_j(k) \, \frac{dk}{k}, \tag{1.3}$$

where l_j are the rays in the complex k plane

$$l_j = \{k \in \mathbb{C} : \; arg(k) = -arg(z_j - z_{j+1})\}, \; j = 1, \ldots n, \tag{1.4}$$

oriented from zero to infinity, and the spectral function $\{\rho_j(k)\}_{j=1}^{n}$ is defined by the line integrals

$$\rho_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz + \frac{i\beta^2}{k}\bar{z}} \left[\frac{1}{2}(q_x - iq_y)dz + \frac{i\beta^2}{k} q \, d\bar{z} \right]. \tag{1.5}$$

Let q be real and let q satisfy the boundary conditions

$$\alpha_j q_x + \beta_j q_y + \gamma_j q = f_j \tag{1.6}$$

on each side of the polygon, where $\alpha_j, \beta_j, \gamma_j$ are real constants and f_j is a real smooth function. Substituting equation (1.6) into equation (1.5), it follows that each ρ_j involves one unknown function, denoted by $\psi_j(k)$. The final step of the method involves determining $\{\psi_j(k)\}_{j=1}^{n}$ by analysing the global relation

$$\sum_{j=1}^{n} \rho_j(k) = 0, k \in \mathbb{C}. \tag{1.7}$$

A general procedure for using the *single* equation (1.7) to determine the *n unknown functions* $\psi_j(k)$ is discussed in [3] and further illustrated in [8].

The most difficult step of the above method is step (3). For evolution equations in a polygonal domain this step involves only algebraic manipulations [4]; for evolution equations in a time-dependent domain it involves the solution of a system of Volterra integral equations [5]; for elliptic equations in a polygonal domain it involves the solution of a matrix Riemann-Hilbert problem [3], [8].

The aim of this paper is to introduce an alternative approach for constructing integral representations for linear PDE's. This approach, instead of using the simultaneous spectral analysis of the Lax pair, makes use of a certain generalization of Green's formula, as explained below.

Definition 1.1. *Let $\mathfrak{p}(\partial_{x_1}, \partial_{x_2})$ be a polynomial differential operator with constant coefficients. We call a 1-form $\Omega(x_1, x_2, k_1, k_2)$ a fundamental differential form associated with the PDE*

$$\mathfrak{p}(\partial_{x_1}, \partial_{x_2})\ q(x_1, x_2) = 0, \tag{1.8}$$

iff the equation (1.8) can be represented in the form

$$d\Omega = \mathfrak{p}(ik_1, ik_2) q(x_1, x_2) e^{-i(k_1 x_1 + k_2 x_2)} dx_1 \wedge dx_2. \tag{1.9}$$

As an illustration of this definition we note that a fundamental differential 1-form associated with the modified Helmholtz equation (1.1) is

$$\Omega(x, y, k_1, k_2) = \Big\{[iq_x + q_y + (k_1 + ik_2)q]dx + [-q_x + iq_y + (-ik_1 + k_2)q]dy\Big\} e^{-i(k_1 x + k_2 y)}. \tag{1.10}$$

Indeed,

$$d\Omega = -[q_{xx} + q_{yy} + ({k_1}^2 + {k_2}^2)q]\ e^{-i(k_1 x + k_2 y)}\ dx \wedge dy,$$

which using equation (1.1) and $\mathfrak{p}(ik_1, ik_2) = -({k_1}^2 + {k_2}^2 + 4\beta^2)$, becomes an equation of the form (1.9).

Observation 1.1. *Let* Ω *be a fundamental differential form associated with the PDE*

$$\mathfrak{p}(\partial_{x_1}, \partial_{x_2})\ q(x_1, x_2) = 0.$$

Let $q(x_1, x_2)$ *satisfy this equation in the domain* $(x_1, x_2) \in M$*, where* $M \subset \mathbb{R}^2$ *is a compact domain in* $\mathbb{R}^2$ *with the boundary* ∂M*, given by a piecewise smooth curve. Then, if* $q(x_1, x_2)$ *exists, it can be represented in the form*

$$q(x_1, x_2) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} dk_1 dk_2 \int_{(x_1', x_2') \in \partial M} \frac{e^{i(k_1 x_1 + k_2 x_2)}\ \Omega(x_1', x_2', k_1, k_2)}{\mathfrak{p}(ik_1, ik_2)}\ . \tag{1.11}$$

Indeed, the Stokes formula together with the definition of Ω imply

$$\int_{\partial M} \Omega = \iint_M d\Omega = \iint_M \mathfrak{p}(ik_1, ik_2)\ q(x_1', x_2') e^{-i(k_1 x_1' + k_2 x_2')} dx_1' \wedge dx_2',$$

and (1.11) follows using

$$\frac{1}{2\pi} \int_{\mathbb{R}} dk\ e^{ik(x-x')} = \delta(x - x').$$

We note that equation (1.11) is formal; the regularisation of the right-hand side of this equation will be discussed in §3-6. For general considerations regarding this regularisation, see [12].

As an illustration of (1.11), consider the modified Helmholtz equation (1.1) in a convex polygon with corners $z_1, \dots z_n$, see Fig. 1. Equation (1.11) together with the fundamental form (1.10) imply

$$q(x_0, y_0) = -\frac{1}{(2\pi)^2} \sum_{j=1}^{n} \int_{\mathbb{R}^2} dk_1 dk_2 \int_{z_{j+1}}^{z_j} \frac{e^{i(k_1(x_0 - x) + k_2(y_0 - y))}}{{k_1}^2 + {k_2}^2 + 4\beta^2} \times \\ \{[iq_x + q_y + (k_1 + ik_2)q]dx + [-q_x + iq_y + (-ik_1 + k_2)q]dy\}\,. \tag{1.12}$$

It will be shown in §5 that using an appropriate contour integration, this equation yeilds equations (1.3)- (1.5).

This paper is organised as follows: In §2 we construct Ω for an arbitrary polynomial differential operator $\mathfrak{p}$ and discuss the question of non-uniquness of Ω. We also discuss the relation of equation (1.11) with the classical Green's formula. In §3 we apply (1.11) to the general dispersive evolution equation

$$\partial_t q + i \sum_{j=0}^{n} \alpha_j (-i\partial_x)^j q = 0, \tag{1.13}$$

where the coefficients α_j are real, $0 < x < \infty$, $0 < t < T$, and T is a finite positive constant. This will reproduce the integral representation obtained in [4]. In §4, using (1.11), we rederive the integral representation of the solution of the Laplace equation in an arbitrary convex polygon, presented in [8]. In §5 we derive equations (1.3)- (1.5). In §6 we discuss the heat equation in the domain $l(t) < x < \infty$, $0 < t < T$, where $l(t)$ is a smooth function with a monotonic first derivative. In §7 we compare the use of equation (1.11) with the spectral analysis of the Lax pair and discuss further these results.

We conclude this introduction by noting that the fundamental form Ω yields both the Lax pair as well as the global relation satisfied by the spectral function:

Observation 1.2. *Let Ω be a fundamental 1-form associated with the equation*

$$\mathfrak{p}(\partial_{x_1}, \partial_{x_2})\ q(x_1, x_2) = 0.$$

(a) If Ω is written in the form

$$\Omega(x_1, x_2, k_1, k_2) = \sum_{j=1}^{2} A_j(x_1, x_2, k_1, k_2) e^{-i(k_1x_1+k_2x_2)} dx_j, \tag{1.14}$$

then an associated Lax pair is given by

$$\begin{aligned} \mu_{x_1} - ik_1\mu &= A_1 \\ \mu_{x_2} - ik_2\mu &= A_2, \end{aligned} \tag{1.15}$$

where k_1 and k_2 satisfy the equation $\mathfrak{p}(ik_1, ik_2) = 0$.

(b) If $(x_1, x_2) \in M$, $M \subset \mathbb{R}^2$ is a compact domain, then the following global relation is valid

$$\int_{\partial M} \Omega(x_1, x_2, k_1, k_2) = 0, \tag{1.16}$$

provided $\mathfrak{p}(ik_1, ik_2) = 0$.

Indeed, if $\mathfrak{p}(ik_1, ik_2) = 0$, Ω is a closed 1-form and equation (1.16) follows. Also on a simply connected domain, a closed 1 form is exact, therefore if $\mathfrak{p}(ik_1, ik_2) = 0$, there exists a function $\nu(x_1, x_2, k_1, k_2)$, such that $\Omega = d\nu$. Taking $\nu = \mu e^{-i(k_1x_1+k_2x_2)}$, this equation implies equations (1.15).

As an illustration of equations (1.15) and (1.16) we note that if Ω is given by (1.10), then equations (1.15), (1.16) become equations (1.2) and (1.7), where we have parametrised the equation

$${k_1}^2 + {k_2}^2 + 4\beta^2 = 0$$

by

$$k_1 = k - \frac{\beta^2}{k}, k_2 = ik + i\frac{\beta^2}{k}.$$

2 The construction of the fundamental differential form

Proposition 2.1. *Let* $\mathfrak{p}(\partial_{x_1}, \partial_{x_2})$ *be defined by*

$$\mathfrak{p} \equiv \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \alpha_{n_1,n_2} \ \partial_{x_1}^{n_1} \partial_{x_2}^{n_2} \ , \tag{2.1}$$

where $\alpha_{n_1,n_2} \in \mathbb{C}$ *are constant coeffficeinets. A fundamental differential 1-form associated with the PDE*

$$\mathfrak{p}(\partial_{x_1}, \partial_{x_2}) \ q(x_1, x_2) = 0,$$

is

$$\begin{aligned}
\Omega = &\sum_{n_2=1}^{N_2} e^{-i(k_1x_1+k_2x_2)} \alpha_{0,n_2} \sum_{j=1}^{n_2} (ik_2)^{j-1} \partial_{x_2}^{n_2-j} q \ dx_1 - \\
&- \sum_{n_1=1}^{N_1} e^{-i(k_1x_1+k_2x_2)} \alpha_{n_1,0} \sum_{j=1}^{n_1} (ik_1)^{j-1} \partial_{x_1}^{n_1-j} q \ \ dx_2 + \\
&+ \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} e^{-i(k_1x_1+k_2x_2)} \alpha_{n_1,n_2} \left[\sum_{j=1}^{n_2} (ik_2)^{j-1} \partial_{x_1}^{n_1} \partial_{x_2}^{n_2-j} q dx_1 - (ik_2)^{n_2} \sum_{j=1}^{n_1} (ik_1)^{j-1} \partial_{x_1}^{n_1-j} q dx_2 \right].
\end{aligned} \tag{2.2}$$

Proof: We will verify that

$$d\Omega = \mathfrak{p}(ik_1, ik_2) \ q(x_1, x_2) e^{-i(k_1x_1+k_2x_2)} dx_1 \wedge dx_2, \tag{2.3}$$

where $\mathfrak{p}(ik_1, ik_2) = - \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \alpha_{n_1,n_2} \ (ik_1)^{n_1} (ik_2)^{n_2}$.

Indeed,

$$\begin{aligned}
&d \left(e^{-i(k_1x_1+k_2x_2)} \alpha_{0,n_2} \sum_{j=1}^{n_2} (ik_2)^{j-1} \partial_{x_2}^{n_2-j} q \ dx_1 \right) = \\
&-\alpha_{0,n_2} e^{-i(k_1x_1+k_2x_2)} \left(\sum_{j=1}^{n_2} (ik_2)^{j-1} \partial_{x_2}^{n_2-j+1} q - \sum_{j=1}^{n_2} (ik_2)^{j} \partial_{x_2}^{n_2-j} q \right) \ dx_1 \wedge dx_2 = \\
&= -\alpha_{0,n_2} e^{-i(k_1x_1+k_2x_2)} \left(\partial_{x_2}^{n_2} q - (ik_2)^{n_2} q \right) \ dx_1 \wedge dx_2,
\end{aligned}$$

where in the first equality, the first term comes from differentiating q, and the second comes from differentiating the exponent.

Similarly,

$$-d\left(e^{-i(k_1x_1+k_2x_2)}\alpha_{n_1,0}\sum_{j=1}^{n_1}(ik_1)^{j-1}\partial_{x_1}^{n_1-j}q\ \ dx_2\right)=$$
$$=-\alpha_{n_1,0}e^{-i(k_1x_1+k_2x_2)}\left(\partial_{x_1}^{n_1}q-(ik_1)^{n_1}q\right)\ dx_1\wedge dx_2,$$

and

$$d\left(e^{-i(k_1x_1+k_2x_2)}\alpha_{n_1,n_2}\sum_{j=1}^{n_2}(ik_2)^{j-1}\partial_{x_1}^{n_1}\partial_{x_2}^{n_2-j}qdx_1-(ik_2)^{n_2}\sum_{j=1}^{n_1}(ik_1)^{j-1}\partial_{x_1}^{n_1-j}qdx_2\right)=$$
$$=-e^{-i(k_1x_1+k_2x_2)}\alpha_{n_1,n_2}\left(\partial_{x_1}^{n_1}\partial_{x_2}^{n_2}q-(ik_1)^{n_1}(ik_2)^{n_2}q\right)\ dx_1\wedge dx_2.$$

Collecting these terms, we obtain

$$d\Omega=-e^{-i(k_1x_1+k_2x_2)}\sum_{0\le n_1\le N_1,0\le n_2\le N_2,(n_1,n_2)\neq(0,0)}\alpha_{n_1,n_2}\left(\partial_{x_1}^{n_1}\partial_{x_2}^{n_2}q-(ik_1)^{n_1}(ik_2)^{n_2}q\right)\ dx_1\wedge dx_2,$$

and using (2.1), equation (2.3) follows.

QED.

Example 2.1 A fundamental differential 1-form associated with equation (1.13) is

$$\Omega=\left(q(x,t)dx-\tilde{q}(x,t,k_1)dt\right)\ e^{-i(k_1x+k_2t)}, \tag{2.4}$$

where

$$\tilde{q}(x,t,k_1)=\sum_{j=1}^{n}\alpha_j\left[(-i\partial_x)^{j-1}+k_1(-i\partial_x)^{j-2}+{k_1}^2(-i\partial_x)^{j-3}+\ldots+{k_1}^{j-1}\right]q.$$

Example 2.2 A fundamental differential form associated with the equation

$$q_{xx}+q_{yy}+4\alpha q=0, \tag{2.5}$$

where α is a constant, is

$$\Omega=\left[(q_y+ik_2q)dx+(-q_x-ik_1q)dy\right]e^{-i(k_1x_1+k_2x_2)}. \tag{2.6}$$

Remark 2.1*(On the Nonuniqueness of Ω).* A fundamental form Ω is not unique. Indeed, $\Omega+f$ is also a fundamental form iff f is identically a closed 1-form. For example, the modified Helmholtz equation(1.1) is associated with the fundamental forms (1.10) and (2.6). These 1-forms differ by the exact form

$$f = \Big\{(iq_x + k_1 q)dx + (iq_y + k_2 q)dy\Big\}e^{-i(k_1x_1+k_2x_2)} = d\Big(iqe^{-ik_1x-ik_2y}\Big).$$

Remark 2.2. It is sometimes convenient to obtain an integral representation for $L\, q$, where L is some linear operator, instead of an integral representation for q. For example, let q satisfy the Laplace equation and let Ω be defined by

$$\Omega = \Big\{(iq_x + q_y)dx + (-q_x + iq_y)dy\Big\}e^{-i(k_1x_1+k_2x_2)} \tag{2.7}$$

Then

$$d\Omega = (q_x - iq_y)(ik_1 - k_2)e^{-i(k_1x_1+k_2x_2)}dx \wedge dy. \tag{2.8}$$

Thus, if $(x, y) \in M$,

$$q_x - iq_y = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} dk_1 dk_2 \int_{(x',y')\in\partial M} \frac{e^{i(k_1x+k_2y)}\ \Omega(x', y', k_1, k_2)}{ik_1 - k_2}. \tag{2.9}$$

This equation involves $ik_1 - k_2$, while the analogous equation for q involve ${k_1}^2 + {k_2}^2$, thus equation (2.9) is more convenient for performing a contour integration with respect to k_1 or k_2, see §4.

Remark 2.3*(Relationship with Green's formula).* We will illustrate the relationship between the classical Green's formula and the fundamental differential form introduced here using the example of the modified Helmholtz equation (1.1). Recall that using Green's formula, the solution of equation (1.1) can be represented in the form

$$q(x_0, y_0) = \int_{\partial M}\{-(q\partial_y G - G\partial_y q)dx + (q\partial_x G - G\partial_x q)dy\}, \tag{2.10}$$

where $G \equiv G(x, y; x_0, y_0)$ is the Green's function, defined by

$$\mathfrak{p}(\partial_x, \partial_y)G = \delta(x - x_0, y - y_0), \quad \mathfrak{p}(\partial_x, \partial_y) := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 4\beta^2.$$

Using the Fourier representation of the Green's function, i.e.

$$G(x, y; x_0, y_0) = \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} \frac{e^{-ik_1(x-x_0)-ik_2(y-y_0)}}{\mathfrak{p}(ik_1, ik_2)}\, dk_1 dk_2$$

in equation (2.10), we obtain the representation (1.11), where the fundamental 1-form Ω given by (2.6).

3 A general dispersive evolution equation

Proposition 3.1. *Let $q(x,t)$ satisfy equation (1.13) in the domain $0 < x < \infty$, $0 < t < T$, where T is a finite positive constant. If a solution $q(x,t)$ exists, is smooth up to the boundary of the domain, and has sufficient decay as $x \to \infty$, then $q(x,t)$ is given by*

$$q(x,t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{ikx-i\omega(k)t}\hat{q}_0(k)dk + \int_{\partial D_+} e^{ikx-i\omega(k)t}\hat{Q}(k)dk, \tag{3.1}$$

where

$$\omega(k) = \sum_{j=0}^{n}\alpha_j k^j, \tag{3.2.a}$$

∂D_+ is the boundary of the domain D_+

$$D_+ = \{k \in \mathbb{C} : Im\ \omega(k) > 0, Im\ k > 0\}, \tag{3.2.b}$$

$$\hat{q}_0(k) = \int_0^{+\infty} e^{-ikx}q(x,0)dx, \tag{3.2.c}$$

$$\hat{Q}(k) = \sum_{j=1}^{n}\alpha_j\left(\hat{Q}_{j-1}(k) + k\hat{Q}_{j-2}(k) + \ldots + k^{j-2}\hat{Q}_1(k) + k^{j-1}\hat{Q}_0(k)\right), \tag{3.2.d}$$

$$\hat{Q}_l(k) = \int_0^T e^{i\omega(k)t}(-i\partial_x)^l q(0,t)dt,\ l = 0,1,\ldots n-1,\ k \in \mathbb{C}. \tag{3.2.e}$$

Furthermore, the following relation is valid:

$$\hat{Q}(k) + \hat{q}_0(k) - e^{i\omega(k)T}\int_0^{+\infty} e^{-ikx}q(x,T)dx = 0,\ Im\ k \leq 0. \tag{3.3}$$

Proof: Equation (1.11) yields

$$q(x_0,t_0) = \frac{1}{i(2\pi)^2}\int_{\mathbb{R}^2} dk_1 dk_2 \int_{(x,t)\in\partial M} \frac{e^{i(k_1(x_0-x)+k_2(t_0-t))}\ \tilde{\Omega}(x,t,k_1,k_2)}{k_2 + \omega(k_1) - i0}, \tag{3.4}$$

where $\tilde{\Omega} = e^{i(k_1 x + k_2 t)}\Omega$, and Ω is defined by (2.4). The boundary ∂M consists of the union of $M_j, j = 1,2,3$, see Fig. 2, where

$$
\begin{array}{llll}
M_1 : \{ & t = T, & 0 < x < \infty & \} \\
M_2 : \{ & 0 < t < T, & x = 0 & \} \\
M_3 : \{ & t = 0, & 0 < x < \infty & \}.
\end{array}
$$

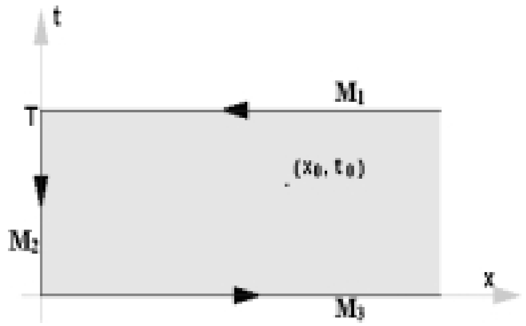


Figure 2.

Along M_1, since $t = T$ and $t_0 < T$, the coefficient of ik_2 in the exponential appearing in equation (3.4) is negative, thus we can perform the k_2 integration by integrating along a semicircle $|k_2| = R$, with $R \to \infty$, in the lower half complex k_2-plane ; this yields a zero contribution. Along M_3, since $t = 0$, the coefficient of ik_2 is positive, thus by integrating in the upper half complex k_2-plane, we obtain the first term in the equation (3.1) (we have renamed k_1 as k). Along M_2, since $x = 0$, the coefficient of k_1 is positive, thus we can integrate in the upper half complex k_1-plane. We first consider the particular equation $q_t - iq_{xx} = 0$, and then discuss the general case. For this particular equation, $\omega(k) = k^2$, thus equation (3.4) contains the term $k_2 + k_1^2 - i0$. Performing the integration over k_1 yields the integral

$$
\begin{aligned}
&-\frac{1}{2\pi} \int_{-\infty}^{0} dk_2 \int_0^T \frac{e^{i\sqrt{|k_2|}x_0 + ik_2(t_0 - t)}\ (iq_x - \sqrt{|k_2|}q)}{2\sqrt{|k_2|}}\ dt - \\
&-\frac{1}{2\pi} \int_0^{+\infty} dk_2 \int_0^T \frac{e^{-\sqrt{|k_2|}x_0 + ik_2(t_0 - t)}\ (iq_x - i\sqrt{k_2}q)}{2i\sqrt{k_2}}\ dt.
\end{aligned}
\tag{3.5}
$$

We introduce the change of variables

$$
k^2 + k_2 - i0 = 0,\ k_2 \in \mathbb{R},\ Imk \geq 0.
$$

Then the above integral becomes

$$
\frac{1}{2\pi} \int_{\partial I} dk e^{ikx_0 - ik^2 t_0}\ \hat{Q}(k),
$$

where ∂I denotes the boundary of the first quadrant in the complex k-plane. In the general case, one must first determine the zeroes of $k_2 + \omega(k_1) - i0$ in the upper half complex k_1-plane, then compute the k_1 integral, and then change the variables from the real k_2 to the complex k, defined by $\omega(k) + k_2 - i0 = 0$ The first step is cumbersome, particularly if there exist multiple roots. The step involving the change of variables is straightforward: the contour of integration over k is defined by

$$Im\ \omega(k) = \lim_{\varepsilon \to 0} \varepsilon,\ \ \varepsilon > 0, Im\ k \geq 0, \tag{3.6}$$

thus k is on the boundary of ∂D_+, where D_+ is defined by (3.2.b). Thus in general the integration along M_2 give rise to the second term in (3.1).

Equation (1.16) with Ω defined by (2.4) and $\partial M = \bigcup_{j=1}^{3} M_j$, becomes equation (3.3).

QED.

4 The Laplace Equation

Proposition 4.1. *Let $q(x,y)$ satisfy the Laplace equation in a convex closed polygon with corners $z_1, \ldots z_n$, $z_{n+1} = z_1$, $z = x + iy$. If $q(x,y)$ exists and it is smooth up to the boundary of the polygon, then $q(x,y)$ admits the integral representation*

$$q_x - iq_y = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{l_j} e^{ikz} \rho_j(k) dk, \tag{4.1}$$

where l_j are the rays in the complex k plane

$$l_j = \{k \in \mathbb{C} :\ arg(k) = -arg(z_j - z_{j+1})\},\ j = 1, \ldots n, \tag{4.2}$$

oriented from zero to infinity, and the spectral function $\{\rho_j(k)\}_{j=1}^{n}$ is defined by the line integrals

$$\rho_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}(q_x - iq_y)dz,\ k \in \mathbb{C}. \tag{4.2.b}$$

Furthermore, the spectral function satisfies the global relation

$$\sum_{j=1}^{n} \rho_j(k) = 0, k \in \mathbb{C}. \tag{4.3}$$

Proof: Equation (2.9) becomes

$$q_{x_0} - iq_{y_0} = \frac{i}{(2\pi)^2} \int_{\mathbb{R}^2} dk_1 dk_2 \sum_{j=1}^{n} \int_{z_{j+1}}^{z_j} \frac{e^{i(k_1(x_0-x)+k_2(y_0-y))}\,(q_x - iq_y)\,(dx + idy)}{ik_1 - k_2}. \quad (4.4)$$

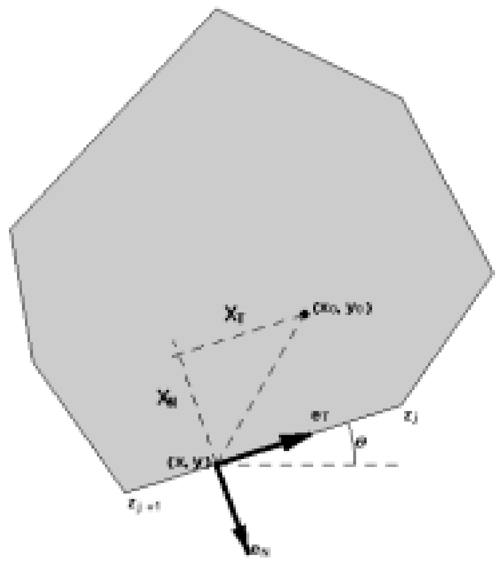


Figure 3.

Let us investigate the term in this equation corresponding to the integral along the side (z_{j+1}, z_j), see Fig. 3. For convinience of notation we drop the subscript j. Let e_T be the unit vector along this side, and e_N the unit vector perpendicular to this side and pointing outwards. Let k_T and k_N denote the components of the vector (k_1, k_2) along e_T and e_N,

$$k_T = k_1 \cos\Theta + k_2 \sin\Theta, \;\; k_N = k_1 \sin\Theta - k_2 \cos\Theta. \quad (4.5)$$

Similarly, let (X_T, X_N) denote the components of the vector $(x_0 - x, y_0 - y)$ along k_T and k_N. Then

$$k_1(x_0 - x) + k_2(y_0 - y) = k_T X_T + k_N X_N. \quad (4.6)$$

Since (x_0, y_0) is a point inside the convex polygon, and e_N is pointing outwards, it is easy to show that $X_N < 0$. Therefore, we can compute the above integral by integrating in the lower half complex k_N-plane. Using

$$ik_1 - k_2 = (k_N + ik_T)e^{i\Theta},$$

we find

$$\frac{1}{2\pi}\int_0^\infty dk_T \int_{z_{j+1}}^{z_j} e^{ik_T X_T + k_N X_N}(q_x - iq_y)\ (dx+idy)e^{-i\Theta}. \tag{4.7}$$

Introducing the change of variables $k_T = ke^{i\Theta}$, the above becomes

$$\frac{1}{2\pi i}\int_{l_j} e^{ikz_0}\rho_j(k)dk.$$

Equation (1.16) yields equation (4.3).

QED.

5 The modified Helmholtz equation

Proposition 5.1. *Let $q(x,y)$ satisfy equation (1.1) in a convex closed polygon with corners $z_1,\dots z_n$, $z_{n+1}=z_1$, $z=x+iy$. If $q(x,y)$ exists and it is smooth up to the boundary of the polygon, then $q(x,y)$ admits the integral representation (1.3), (1.5). Furthermore, the global relation (1.7) is valid.*

Proof: Equation (1.11) together with the fundamental form (1.10) yield

$$q(x_0,y_0) = -\frac{1}{(2\pi)^2}\sum_{j=1}^{n}\int_{\mathbb{R}^2} dk_1 dk_2 \int_{z_{j+1}}^{z_j} \frac{e^{i(k_1(x_0-x)+k_2(y_0-y))}}{{k_1}^2+{k_2}^2+4\beta^2}\times \\ [i(q_x-iq_y)(dx+idy)+(k_1+ik_2)q(dx-idy)]. \tag{5.1}$$

Using the notations of §4, see (4.5) -(4.6), we find that $X_N<0$, and

$${k_1}^2+{k_2}^2+4\beta^2 = {k_T}^2+{k_N}^2+4\beta^2.$$

Performing the integration over k_N, we obtain

$$-\frac{1}{2\pi^2}\int_{-\infty}^{+\infty} dk_T \int_{z_{j+1}}^{z_j} \frac{e^{ik_T X_T + \sqrt{{k_T}^2+4\beta^2}\ X_N}}{2\sqrt{{k_T}^2+4\beta^2}}\times \\ \Big(i(q_x-iq_y)\ (dx+idy) + i(k_T - \sqrt{{k_T}^2+4\beta^2})\ q\ e^{i\Theta}(dx-idy)\Big). \tag{5.2}$$

Making the change of variables

$$k_T = (\lambda e^{i\Theta}) - \frac{\beta^2}{(\lambda e^{i\Theta})},\ \ k_T\in\mathbb{R}, (\lambda e^{i\Theta})\in\mathbb{R}_+,$$

we find

$$\frac{1}{2\pi i}\int_{l_j}\frac{d\lambda}{\lambda}\int_{z_{j+1}}^{z_j} e^{i\lambda\Big((x_0-x)+i(y_0-y)\Big)-\frac{i\beta^2}{\lambda}\Big((x_0-x)-i(y_0-y)\Big)}\Big(\frac{1}{2}(q_x-iq_y)\,(dx+idy)+\frac{\beta^2}{\lambda}\,q\,(dx-idy)\Big), \tag{5.3}$$

where l_j are the rays defined in (1.4).

QED.

6 The heat equation in a time-dependent domain

Proposition 6.1. *Let $q(x,t)$ satisfy the heat equation in the following time dependent domain:*

$$q_t - q_{xx} = 0,\ \ l(t) < x < \infty,\ \ 0 < t < T, \tag{6.1}$$

where T is a positive constant, and $l(t)$ is a smooth function with a monotonic first derivative and with $l(0) = 0$. If a solution $q(x,t)$ exists, is sufficiently smooth up to the boundary of the domain, and has sufficient decay as $x \to \infty$, then $q(x,t)$ can be represented as follows:
(a) $\underline{l''(t) < 0}$:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}(k)dk + \frac{1}{2\pi}\int_{L(t)} e^{ikx-k^2t}\hat{Q}(k)dk, \tag{6.2.a}$$

where

$$\begin{aligned}\hat{q}(k) &= \int_0^{+\infty}\exp(-ikx)q(x,0)dx,\ \ Imk \le 0,\\ \hat{Q}(k) &= \int_0^{T}\exp(k^2t - ikl(t))\Big((l'(t)+ik)q(l(t),t) + q_x(l(t),t)\ \Big)dt, k \in \mathbb{C},\end{aligned} \tag{6.3.a}$$

and $L(t)$ is defined by:

$$L(t) = \{k = \alpha + i\beta, -\infty < \alpha < \infty,\ \beta = \frac{l'(t)}{2} + \sqrt{\alpha^2 + \frac{\left(l'(t)\right)^2}{4}},\ \ \alpha,\beta \in \mathbb{R}\ \},\ \tag{6.4.a}$$

see Figure 4.

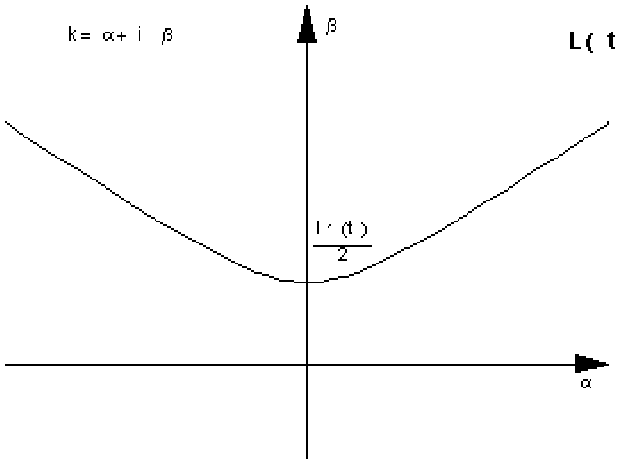


Figure 4.

(b) <u>$l''(t) > 0$:</u>

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}(k)dk + \frac{1}{2\pi}\int\int_D e^{ikx-k^2t}\rho(\alpha,\beta)d\alpha d\beta, \qquad (6.2.b)$$

where $k = \alpha + i\beta$*,*

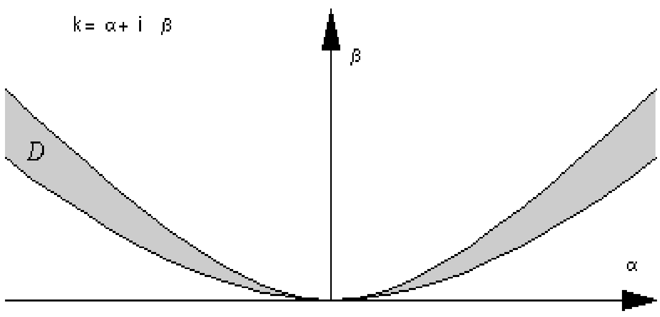
$$\hat{q}(k) = \int_0^{+\infty} \exp(-ikx)q(x,0)dx, Imk \leq 0,$$
$$\rho(\alpha,\beta) = \frac{\alpha^2 + 2i\alpha\beta + \beta^2}{\beta^2\ l''(\tau(\alpha,\beta))} e^{k^2\tau - ikl(\tau)}\Big((-\frac{\alpha^2}{\beta} + i\alpha)q + q_x\ \Big)\Big|_{(x,t)=(l(\tau(\alpha,\beta)),\tau(\alpha,\beta))}\ , \qquad (6.3.b)$$

$t = \tau(\alpha,\beta)$ *is the solution of the equation*

$$\frac{l'(t)}{2} + \sqrt{\alpha^2 + \frac{(l'(t))^2}{4}} = \beta,$$

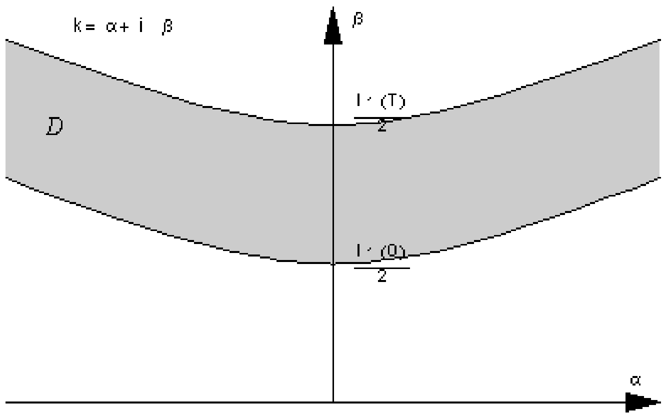
and the domain D *is given by*

$$D = \{(\alpha,\beta) : -\infty < \alpha < \infty, \frac{l'(0)}{2} + \sqrt{\alpha^2 + \frac{(l'(0))^2}{4}} < \beta < \frac{l'(T)}{2} + \sqrt{\alpha^2 + \frac{(l'(T))^2}{4}}\}. \qquad (6.4.b)$$



The case I'' > 0, I' (T) < 0

Figure 5 a



The case I'' > 0, I' (T) > 0

Figure 5b

Moreover, the following global relation holds in both cases:

$$\int_0^\infty q(x,0)\exp(-ikx)dx - \int_0^t e^{k^2\tau - ikl(\tau)}\ [q_x + (l' + ik)q](l(\tau),\tau)d\tau = \\ = e^{k^2 t}\int_{l(t)}^{+\infty} \exp(-ikx)q(x,t)dx, \tag{6.5.b}$$

where $k \in \mathbb{C}$, $\ Im\ k < 0$ *and* $0 \le t \le T$.

Proof: A fundamental 1-form asssociated with the heat equation is

$$\Omega = \Big(qdx + (q_x + ik_2 q)dt\Big)e^{-ik_1 t - ik_2 x}. \tag{6.6}$$

The integral representation (1.11) yields

$$q(x_0,t_0) = \frac{1}{i(2\pi)^2}\int_{\mathbb{R}^2} dk_1 dk_2 \int_{(x,t)\in\partial M} \frac{e^{ik_1(t_0-t)+ik_2(x_0-x)}\Big(qdx + (q_x + ik_2 q)dt\Big)}{k_1 - ik_2{}^2}. \tag{6.7}$$

The boundary ∂M consists of M_1, M_2, M_3, see Fig. 6, where

$$\begin{array}{llll} M_1 : \{ & t = T, & l(t) < x < \infty & \} \\ M_2 : \{ & 0 < t < T, & x = l(t) & \} \\ M_3 : \{ & t = 0, & 0 < x < \infty & \}. \end{array} \tag{6.8}$$

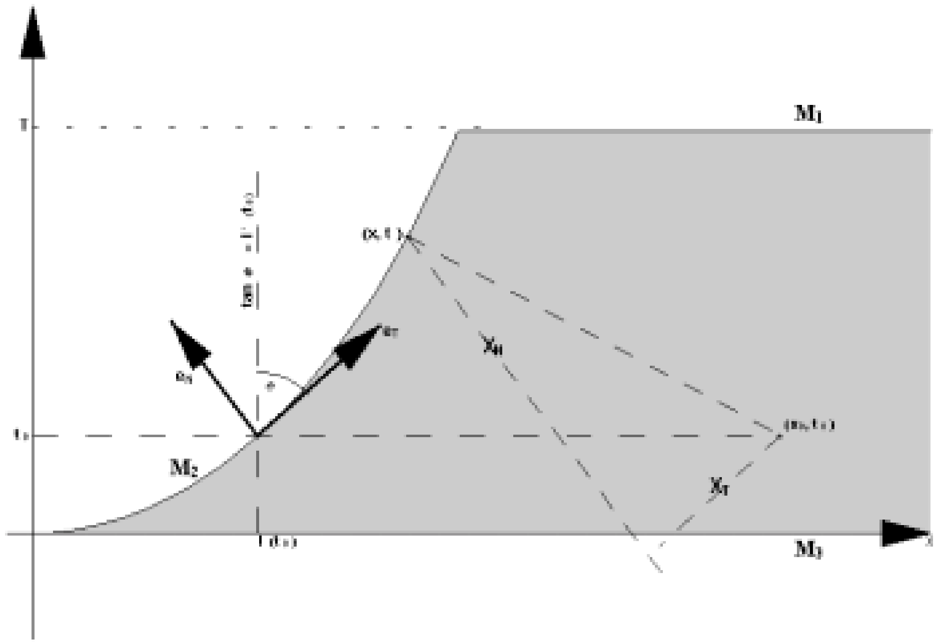


Figure 6

Along M_1, since $t = T$, and $t < T$, the coefficient of ik_1 in the exponential is negative, thus by integrating in the lower half complex k_1-plane, we obtain a zero contribution. The contribution of the M_3 integral yields the first term in equations. In order to compute the contribution of the M_2 integral, we distinguish two cases.

(a) $\underline{l''(t) < 0:}$

Let $e_T(t_0)$, and $e_N(t_0)$ be unit vectors along the tangent and the outward normal of the curve $(l(t), t)$ at $t = t_0$. Let k_T and k_N denote the components of the vector (k_T, k_N) along $e_T(t_0)$ and $e_N(t_0)$, that is

$$k_T = k_1 \cos\Theta_0 + k_2 \sin\Theta_0, \; k_N = k_1 \sin\Theta_0 - k_2 \cos\Theta_0, \quad \tan\Theta_0 = l'(t_0). \tag{6.9}$$

Similarly, (X_T, X_N) denote the components of the vector $(x_0 - x, t_0 - t)$ along $e_T(t_0)$ and $e_N(t_0)$,

$$X_T = (t_0 - t)\cos\Theta_0 + (x_0 - l(t))\sin\Theta_0, \; X_N = (t_0 - t)\sin\Theta_0 - (x_0 - l(t))\cos\Theta_0, \tag{6.10}$$

here $(x_0, t_0) \in M$ and $(t, l(t)) \in M_2 \subset \partial M$.

It can be shown that $X_N < 0$. Indeed, since $(x_0, t_0) \in M$, it follows that $x_0 > l(t_0)$. Using the mean value theorem,

$$l(t) - l(t_0) = l'(\tau)(t - t_0),$$

where τ is in between t and t_0. Thus

$$X_N = (t_0 - t)\sin\Theta_0 - (x_0 - l(t))\cos\Theta_0 < (t_0 - t)\sin\Theta_0 - (l(t_0) - l(t))\cos\Theta_0 = \\ = \cos\Theta_0 (t_0 - t)(l'(t_0) - l'(\tau)).$$

Since $l''(t) < 0$, it follows that

$$l'(\tau) < l'(t_0), t > t_0, \\ l'(\tau) > l'(t_0), t < t_0,$$

and therefore $X_N < 0$.

We can compute the integral corresponding to M_2 by integrating in the lower k_N-plane. This yields

$$-\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dk_T \int_0^T dt \frac{e^{ik_T X_T + i\tilde{k}_N X_N}\Big(ql' + (q_x + i(k_T \sin\Theta_0 - \tilde{k}_N \cos\Theta_0)q\Big)dt}{2\cos\Theta_0\Big((k_T \sin\Theta_0 - \tilde{k}_N \cos\Theta_0) - \frac{i}{2}\tan\Theta_0\Big)}, \tag{6.11}$$

where $\tilde{k}_N$ denotes the root of the quadratic equation

$$i(k_T \cos\Theta_0 + \tilde{k}_N \sin\Theta_0) + (k_T \sin\Theta_0 - \tilde{k}_N \cos\Theta_0)^2 = 0, \; k_T \in \mathbb{R}, \; Im\, \tilde{k}_N < 0. \tag{6.12}$$

Using the change of variables indicated below it follows that there exist only one such root.

Introduce the complex variable k by

$$k = k_T \sin\Theta_0 - \tilde{k}_N \cos\Theta_0. \tag{6.13}$$

Equation (6.12) yields

$$k_T \cos\Theta_0 + \tilde{k}_N \sin\Theta_0 = ik^2. \tag{6.14}$$

Using equations (6.13) and (6.14) we can express k_T and $\tilde{k}_N$ in terms of k:

$$\begin{pmatrix} k_T \\ k_N \end{pmatrix} = \begin{pmatrix} \cos\Theta_0 & \sin\Theta_0 \\ \sin\Theta_0 & -\cos\Theta_0 \end{pmatrix} \begin{pmatrix} ik^2 \\ k \end{pmatrix}$$

Let $k = \alpha + i\beta$, then

$$\begin{aligned} k_T &= -2\alpha\cos\Theta_0\,(\beta - \frac{\tan\Theta_0}{2}) + i\cos\Theta_0(\alpha^2 - (\beta - \frac{\tan\Theta_0}{2})^2 + \frac{\tan^2\Theta_0}{4}) \\ \tilde{k}_N &= -(2\alpha\beta\sin\Theta_0 + \alpha\cos\Theta_0) + i((\alpha^2-\beta^2)\sin\Theta_0 - \beta\cos\Theta_0). \end{aligned} \tag{6.15}$$

From the conditions

$$Im\ k_T = 0, Im k_N < 0, \tag{6.16}$$

it follows that

$$\alpha^2 = \beta(\beta - \tan\Theta_0),\ \ \beta > 0,\ \ \beta \geq \tan\Theta_0. \tag{6.17}$$

Thus

$$\begin{aligned} &\alpha = -sign(k_T)\sqrt{\beta(\beta - \tan\Theta_0)}, \\ &k_T = sign(k_T)\sqrt{\beta(\beta - \tan\Theta_0)}(\beta - \frac{\tan\Theta_0}{2}). \end{aligned} \tag{6.18}$$

Since $\beta > 0$ and $\beta > \tan\Theta_0$, the function $\sqrt{\beta(\beta - \tan\Theta_0)}(\beta - \frac{\tan\Theta_0}{2})$ is monotonic.

Using the change of variables from k_T to k defined by equation (6.13) in the integral (6.11), and noting that the Jackobian cancells with the denominator, we obtain the second term of the equation (6.2.a):

$$\frac{1}{(2\pi)} \int_{L(t_0)} dk \int_0^T dt\ e^{-k^2(t_0-t)+ik(x_0-x)}\ \Big((l'(t) + ik)q + q_x\Big), \tag{6.19}$$

where

$$L(t_0) = \{k = \alpha + i\beta : \beta = \frac{\tan\Theta_0}{2} + \sqrt{\alpha^2 + \frac{\tan^2\Theta_0}{4}}, \alpha \in \mathbb{R}\}. \tag{6.20}$$

For $k_1 = ik_2{}^2$, the form Ω defined by equation (6.6) is closed, and therefore for such k the integral of Ω over any closed path is zero; in particular, choosing the path consisting of the rays $\{x > 0, t = 0\}$, $\{x > l(\tilde{t}), t = \tilde{t}\}$, and the curve $\{(l(t), t), 0 < t < \tilde{t}\}$, we obtain the global relation (6.5.b).

(b) $\underline{l''(t) > 0:}$

The main difference with the case (a) is that now we expand along the unit vectors $e_T(t)$ and $e_N(t)$ at a point $(l(t), t)$ on the boundary, see Fig. 7.

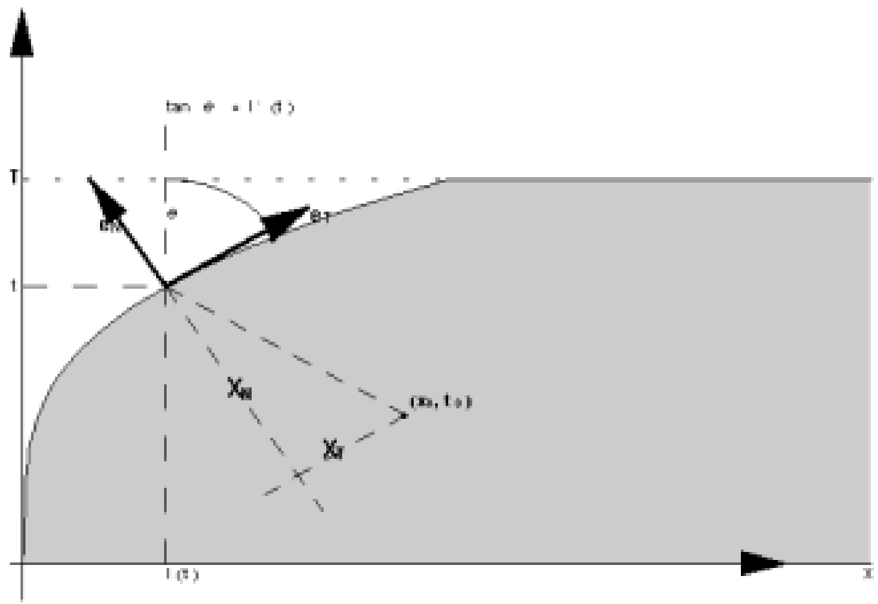


Figure 7.

In particular,

$$X_T = (t_0 - t)\cos\Theta(t) + (x_0 - l(t))\sin\Theta(t), \; X_N = (t_0 - t)\sin\Theta(t) - (x_0 - l(t))\cos\Theta(t), \tag{6.21}$$

where X_T and X_N denote the components of the vector $(x_0 - x, t_0 - t)$ along $e_T(t)$ and $e_N(t)$. As before, it can be shown that $X_N < 0$. Therefore, we can compute the integral by integrating in the lower half k_N plane. Similarly to the case (a), after the change of variables $k_T = ik^2\cos\Theta(t) + k\sin\Theta(t)$ we obtain

$$\frac{1}{(2\pi)}\int_0^T dt \quad \int_{L(t)} dk e^{-k^2(t_0-t)+ik(x_0-x)} \Big((l'(t) + ik)q + q_x\Big), \tag{6.22}$$

where

$$L(t) = \{k = \alpha + i\beta, \beta = \frac{l'(t)}{2} + \sqrt{\alpha^2 + \frac{(l'(t))^2}{4}}, \alpha \in \mathbb{R}\}. \quad (6.23)$$

The condition $l''(t) > 0$ implies that the function

$$\beta(\alpha, t) := \frac{l'(t)}{2} + \sqrt{\alpha^2 + \frac{(l'(t))^2}{4}}$$

is monotonically increasing with respect to t for all fixed α:

$$\beta' = \frac{l''}{2}\left(1 + \frac{\frac{l'(t)}{2}}{\sqrt{\alpha^2 + \frac{(l'(t))^2}{4}}}\right) > 0.$$

Hence, there exist an inverse function $t = \tau(\alpha, \beta)$. Changing the integration variables from (t, k) to (α, β) we get

$$q(x_0, t_0) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} d\alpha \int_{\frac{l'(0)}{2}+\sqrt{\alpha^2+\frac{(l'(0))^2}{4}}}^{\frac{l'(T)}{2}+\sqrt{\alpha^2+\frac{(l'(T))^2}{4}}} d\beta \; e^{-k^2 t_0 + ikx_0} \rho(\alpha, \beta),$$

where

$$\rho(\alpha, \beta) = \frac{\alpha^2 + 2i\alpha\beta + \beta^2}{\beta^2 \; l''(\tau(\alpha;\beta))} e^{k^2\tau - ikl(\tau)} \Big((-\frac{\alpha^2}{\beta} + i\alpha) q + q_x \Big)\Big|_{(x,t)=(l(\tau(\alpha,\beta)), \tau(\alpha,\beta))}. \quad (6.24)$$

QED.

7 Conclusions

This work is part of a larger program of study which investigates boundary value problems for linear and for integrable nonlinear PDE's. For linear PDE's , the method announced in [1] suggests that one first constructs appropriate integral representations for both the solution and the spectral function, and then one analyses the global relation satisfied by the spectral function. Untill now the relevant integral representations were constructed by performing the spectral analysis of the associated Lax pair. In this paper we introduced an alternative approach for constructing these representations, namely we analyse equation (1.11). The advantages and disadvantages of these two approaches appear to be the following:

The direct approach: 1. It can be immedeately generalised to more then 2 dimensions 2. Involves only one contour integration. However: 1. In some cases, one must first "regularise" the integral appearing in (1.11). 2. For equations containing derivatives of order higher then two, the integration is cumbersome, particularly for non-polygonal domains.

The simultaneous spectral analysis approach: 1. Can be suitably "nonlinearised", i.e. it can be extended to nonlinear integrable PDE's [2]. 2. It does not involve any contour integration. However: 1. It has not yet been extended to multidimensions. 2. It requires expertise with the Riemann-Hilbert and the d-bar problems.

The above discussion indicates that for linear PDE's containing only second order derivatives, the direct approach is advantageous, while for linear PDE's with higher order derivatives, the simultaneous spectral analysis approach is more efficient. For nonlinear PDE's, only the Lax pair approach can be used.

We conclude with some remarks:

1. For some linear second order equations in simple domains, the direct approach was used in [9], [10] to rederive some of the results of [1]. Here the direct approach is formulated for any linear constant coefficients PDE in 2 dimensions, and its application to complicated domains is illustrated.

2. The solution of a linear PDE in a given domain can be represented in several different ways. We believe that the "Ehrenpreis type representation" obtained in [1] - [5] and in this paper has distinctive advantages: (a) For evolution equations, it yields a representation with explicit x and t dependence. This makes it possible to obtain in a straightforward manner the long t behaviour of the solution [11]. (b) The integral representation expresses the solution in terms of the spectral data. Furthermore, these spectral data satisfy a global relation. Using these facts and exploiting *the analyticity properties of the integral representation*, it is possible in some problems to analyse the global relation using only algebraic manipulations. This is always possible for the case of evolution equations in a polygonal domain, and it is possible for some elliptic equations in simple polygonal domains satisfying a particular class of boundary conditions [8].

3. We emphasise that the derivation of appropriate integral representations for both the solution and its spectral function, is the "straightforward" part of the method of [1]. The challenging problem is the analysis of the global relation. For linear PDE's a general approach to this problem is developed in [3].

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