Renormalization of Hierarchical Sequences

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Renormalization of hierarchical sequences

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Abstract

We generate a hierarchical sequence of random variables from a map taking k copies of a Banach space to itself. With a simple assumption on the map and an integrability condition for the initial random variables we prove that the renormalized sequence converges almost surely to a constant. This result can be applied to the renormalization of electrical networks and homogenization of Laplace operators on fractals.

1 Introduction

Let $\{X_i^{(0)}; i \in \mathbb{N}\}$ be a sequence of independent and identically distributed random variables taking values in a Banach space, B. Consider a map $f : B^k \to B$, a function of k Banach space valued variables. We can use this map to generate a sequence of random variables by setting

$$X_i^{(n+1)} = f(X_{ik^n+1}^{(n)}, \ldots, X_{(i+1)k^n}^{(n)}), \;\; i \in \mathbb{N}, \;\; \forall n \geq 0.$$

The question of interest to us is a simple version of renormalization group theory concerning the convergence of the hierarchical sequence given by $\{X_1^{(n)}; n \in \mathbb{N}\}$.

The renormalization group is a widely used technique of mathematical physics which is often difficult to make mathematically rigorous. Hierarchical models provide a setting where exact computations are possible and they are also interesting models in their own right. In this paper we will establish an abstract renormalization theorem, giving a condition on the map f which guarantees almost sure convergence, under a finite mean assumption on the random variables $X^{(0)}$. Note that the usual strong law of large numbers can

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be embedded into this problem by setting $f(x_1, \ldots, x_k) = (x_1 + \cdots + x_k)/k$. Our condition on f will be satisfied for concave homogeneous functions with a non-trivial fixed point. The hierarchical sequence generated by a concave homogeneous function occurs in the renormalization of electrical networks, where the effective resistance is a concave function of the individual resistors in the network.

Previous work on this problem can be found in [14] and [15] in the case of real valued random variables. In [14] the random variables were assumed to be bounded and it was proved that for a hierarchical sequence generated by a positive homogeneous concave function with a fixed point, there was convergence in probability to a constant. In [15] another assumption was made, that the function was bounded by a normalized sum of random variables, which allowed strong law techniques, in particular the convergence of backward martingales, to be used. With this assumption it is possible to prove almost sure convergence of the sequence. However in many cases it is not easy to determine if this condition holds and it may well not be true. We will extend the ideas of [15], to weaken the condition on the function, and to consider Banach space valued random variables. We note that this type of renormalization problem has been studied for a spin glass model on a Bethe lattice in [3]. In that setting the renormalization map does not satisfy our assumptions and a non-constant limit is observed.

A motivating example for this work is the homogenization problem for finitely ramified fractals. The homogenization problem is to determine the effective macroscopic properties of a medium which has microscopic irregularities. A finitely ramified fractal is a fractal subset of $\mathbb{R}^d$ with the property that any connected subset can be disconnected by the removal of only a finite number of points. The Sierpinski gasket provides a basic example. We will consider graphs whose large scale structure is that of a finitely ramified fractal and with random resistors on each edge. The question we consider is when the rescaled effective conductivity across the graph converges to a non-trivial limit.

The construction of a Laplace operator on a finitely ramified fractal already involves a deterministic renormalization problem. The Laplacian is obtained from a compatible sequence of electrical networks which approximate the fractal. For the networks to be compatible they must be constructed from the scaled fixed point of a renormalization map from one level to the next. The scaling is called the conductance scale factor. For general finitely ramified self-similar sets there is a deterministic fixed point problem which needs to be solved in order to deduce the existence of an operator. As the electrical network can be viewed as a Dirichlet form, the fixed point problem can be regarded as an eigenvalue problem on the cone of Dirichlet forms. Conditions for the existence and uniqueness of a fixed point have been found in [13].

We will consider a random version of this fixed point problem. We take an infinite fractal lattice, consisting of translates of a basic cell, and place random resistors on each such basic cell, chosen as random elements of the cone of Dirichlet forms on the cell. We rescale the lattice by the natural length scale factor and consider then effective conductance across the unit cell scaled by the conductance scale factor. Our interest is in when this sequence of effective conductances converges to a non-degenerate limit. This problem was first considered in [8] in which it was shown that for nested fractals, under some assumptions corresponding to the uniqueness of the fixed point and boundedness of the conductors, that there was L^1 convergence of the conductance. In [6] the case of the Vicsek set was considered. This is a fractal for which there is not a unique fixed point for the renormalized conductivity map. However the Vicsek set is sufficiently simple that the renormalization map is effectively linear and the strong law of large numbers can be used to give the almost sure convergence of the effective conductance under a first moment condition on the conductors. Here we will prove that, only assuming a finite mean, the effective conductance converges almost surely under rescaling by the conductance scale factor. However, to ensure positivity of the limit in general, we require our conductors to be bounded away from 0.

The paper will begin with some preliminary results giving the necessary background for the convergence of reversed submartingales in Banach spaces. We will then state our main result in Section 3 and use it to extend the previous work on hierarchical sequences of random variables. The final section will discuss some examples where these ideas can be applied.

2 Preliminaries

In order to prove our results we introduce a suitable setting for understanding convergence of hierarchical sequences. The main result we will need is the reversed submartingale theorem for Banach spaces. We will also record a Banach space analogue of the dominated convergence theorem for conditional expectations and a version of the Hahn-Banach theorem.

Let B be a σ-complete Banach lattice with the property that the norm is continuous in the order topology. (For the properties of Banach lattices that we use here we refer the reader to [1]). A sequence of strongly measurable random variables $\{X_n\}_{n\geq 0}$ with respect to a decreasing filtration $\{\mathcal{F}_n\}_{n\geq 0}$ is a reversed submartingale if for each $n \leq 0$, $E|X_n| \in B$ and

$$X_{n+1} \leq E(X_n|\mathcal{F}_{n+1}).$$

We will denote by P the positive cone in the Banach space B.

Theorem 2.1 *Let $\{X_n\}$ be a reversed submartingale in P with respect to a decreasing filtration $\mathcal{F}_n$. Then X_n converges almost surely (in the metric topology) to a random variable X_∞.*

Proof: We prove this using a martingale convergence theorem for Banach spaces and a Doob decomposition. Observe that we can write the submartingale in terms of a martingale M_n null at 0, and a decreasing process A_n as $X_n = M_n + A_n$ where

$$\begin{aligned} M_n &= \sum_{i=1}^{n} E(X_{i-1}|\mathcal{F}_i) - X_{i-1}, \\ A_n &= X_0 + \sum_{i=1}^{n} E(X_i - X_{i-1}|\mathcal{F}_i). \end{aligned}$$

The martingale part M_n is a reversed martingale and hence converges (in the metric topology) by the usual reversed martingale convergence theorem for Banach spaces, [2] Theorem 4.

For the decreasing part we have

$$0 \leq E(A_n) = E(X_n) \leq E(X_0) = X_0.$$

We want to prove that the sequence A_n will converge. In order to do this we will show that $0 \leq A_n \leq X_0$. The upper bound is obvious. For the lower bound we assume that the sequence A_n leaves the positive cone. Thus there exists an element $\xi = A_{p+1}$ where $p = \sup\{n : A_n \in P\}$, the first point at which the sequence was not in the cone. As the sequence A_n is decreasing we see that $a_n = P(A_n \leq \xi) \to 1$ as $n \to \infty$. Now

$$0 \leq E(A_n) \leq a_n\xi + (1 - a_n)X_0 \to \xi,$$

as $n \to \infty$. Thus we have a contradiction and the sequence must remain in the positive cone.

As we have a monotone decreasing sequence which is order bounded below it converges in the order topology and hence in the norm. Thus $X_n \to X_\infty$ almost surely as required. ∎

Corollary 2.2 *A reversed submartingale in B which is bounded below converges almost surely.*

We also require a dominated convergence theorem for conditional expectations.

Lemma 2.3 *For a sequence $Y_n \to Y$ almost surely (in the metric topology) in B, with $\|Y_n\| \leq Z$ for all n, where $E(Z) < \infty$ and a filtration $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, then*

$$E(Y_n|\mathcal{F}_n) \to E(Y|\mathcal{F}_\infty).$$

Proof: We follow the proof given in [4] for the real valued case. Set $W_N = \sup\{\|Y_n - Y_m\| : n, m \geq N\}$. Note that $W_N \leq 2Z$ and hence $EW_N < \infty$. Then

$$\limsup_{n\to\infty} E(\|Y_n - Y\||\mathcal{F}_n) \leq \lim_{n\to\infty} E(W_N|\mathcal{F}_n) = E(W_N|\mathcal{F}_\infty).$$

The last equality follows from [4] (5.5). Hence, by the triangle inequality and a Banach space version of Jensen's inequality, we have

$$\begin{aligned}
\|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)\| &\leq \|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_n)\| + \\
&\qquad \|E(Y|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)\| \\
&\leq E(\|Y_n - Y\||\mathcal{F}_n) + \|E(Y|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)\| \\
&\to 0.
\end{aligned}$$

The fact that the second term vanishes is a consequence of a martingale convergence theorem given in [2]. ∎

We will be studying concave functions on the k-fold products of the Banach space and the next lemma gives us control on such functions.

Lemma 2.4 *Let $f : P^k \to P$ be a concave homogeneous function for which there is a fixed point $x^0 \in P\backslash\{0\}$, that is $f(x^0, \ldots, x^0) = x^0$. Then there exists $w \in \mathbb{R}^k_+$ such that $\sum_{i=1}^k w_i = 1$ and*

$$f(x_1, \ldots, x_k) \leq \sum_{i=1}^k w_i x_i.$$

Proof: We use the geometric version of the Hahn-Banach Theorem as given in Edwards p117. Consider the Banach space $E = B^{k+1}$ and let

$$A = \{\mathbf{x} \in (P^\circ)^{k+1} : f(x_1, \ldots, x_k) > x_{k+1}\}.$$

It is easy to see by the concavity of f that the set A is convex and open. Let $\mathbf{x}^0$ be the point containing $k+1$ copies of the fixed point and let $M = \{\alpha \mathbf{x}^0 : \alpha \in \mathbb{R}\}$. Observe that, by the homogeneity of f, $M \cap A = \emptyset$. Applying the Hahn-Banach Theorem, there exists a hyperplane H such that $M \subset H$ and $H \cap A = \emptyset$. Since the hyperplane H in E contains the origin it can be written as $\sum_{i=1}^{k+1} u_i x_i = 0$. As the point $\mathbf{x}^0$ lies in H and is non-zero we see that $\sum_{i=1}^{k+1} u_i = 0$ and we can write the hyperplane as

$$x_{k+1} = \sum_{i=1}^{k} w_i x_i,$$

where $\sum_{i=1}^{k} w_i = 1$. Now as $H \subset A^c$ we have

$$f(x_1, \ldots, x_k) \leq x_{k+1} = \sum_{i=1}^{k} w_i x_i.$$

To see that the $w_i \geq 0$ we observe that, if not, we get a contradiction to the positivity of f. ∎

3 Convergence under Renormalization

The results of this section extend those of [14] and [15] in the real valued case. The proof of the main result follows a similar line of argument as [15]. As in Section 2, B is a σ-complete Banach lattice with the property that the norm is continuous in the order topology.

Theorem 3.1 *Let $f : B^k \to B$ be a function bounded below with the property that there exists $w \in \mathbb{R}_+^k$ such that $w_1 + \cdots + w_k = 1$ and*

$$f(x_1, \ldots, x_k) \leq w_1 x_1 + \cdots + w_k x_k$$

for all $x \in B^k$. We also assume that f is symmetric in any two of its arguments. Let $X_1, X_2, \ldots$ be a sequence of iid integrable random variables *in B. For each j, set* $X_j^{(0)} = X_j$*, and for* $n \geq 0$*,*

$$\begin{aligned} X_1^{(n+1)} &= f(X_1^{(n)}, \dots, X_k^{(n)}) \\ X_2^{(n+1)} &= f(X_{k+1}^{(n)}, \dots, X_{2k}^{(n)}) \\ &\vdots \\ X_k^{(n+1)} &= f(X_{k^2-k+1}^{(n)}, \dots, X_{k^2}^{(n)}). \end{aligned}$$

Then the sequence $X_1^{(n)}$ *converges almost surely in* B^k*.*

Proof: Without loss of generality we can assume that $w_1 > 0$. Set

$$\mathcal{T}_m = \sigma(X_{m+1}, X_{m+2}, \dots)$$

and

$$\mathcal{F}_n = \sigma(Z_n) \bigvee \mathcal{T}_{k^n},$$

where

$$Z_n := w_1 X_1^{(n)} + \cdots + w_k X_k^n.$$

Observe that Z_n is independent of $\mathcal{T}_{k^{n+1}}$. Also, $\mathcal{F}_n$ is decreasing and Z_n is $\mathcal{F}_n$-measurable. We will now show that Z_n is a reversed submartingale with respect to $\mathcal{F}_n$.

$$\begin{aligned} E(Z_n|\mathcal{F}_{n+1}) &= E(Z_n|Z_{n+1}) \\ &= E[w_1(w_1 X_1^{(n)} + \cdots + w_k X_k^{(n)}) + w_2(w_1 X_{k+1}^{(n)} + \cdots \\ &\quad + w_k X_{2k}^{(n)}) + \cdots + w_k(w_1 X_{k^2-k+1}^{(n)} + \cdots + w_k X_{k^2}^{(n)})|Z_{n+1}] \\ &\geq E[w_1 X_1^{(n+1)} + \cdots + w_k X_k^{(n+1)}|Z_{n+1}] \\ &= Z_{n+1} \end{aligned}$$

It then follows from Corollary 2.2 that Z_n converges almost surely to a random variable Z.

Without loss of generality we can assume that f is symmetric in the first two arguments. As observed in [15] a variant of the Hewitt-Savage law can be invoked to show that Z is almost surely a constant. Indeed, while Z is not necessarily a symmetric function of the $X_i^{(0)}$ it is invariant under special permutations of the indices. In particular for each n it does not change its value under the permutation which transposes 1 with $k^n + 1$, 2 with $k^n + 2$, $\dots$,k^n with $2k^n$. Note that here we are using the fact that f is symmetric in its first two arguments. For the rest of the proof we shall assume without loss of generality that $Z = 0$.

In Banach lattices metric convergence is equivalent to relative uniform star-convergence. A sequence x_n converges relatively uniformly to x if and only if, for some u and $\lambda_n \downarrow 0$, $|x_n - x| \leq \lambda_n u$. Relatively uniform star convergence is defined to be relative uniform convergence of some subsequence of every subsequence.

We now show that $X_1^{(n)}$ converges to 0 in probability. Consider a particular subsequence n'. Since Z_n star-converges relatively uniformly there exists a further subsequence n'', and some u and $\lambda_{n''} \downarrow 0$, such that

$$|Z_{n''}| \leq \lambda_{n''} u.$$

By the definition of Z_n and the fact that we can choose $w_1 > 0$, we have

$$P(Z_{n''} > \lambda_{n''} u) \geq P(X_1^{(n'')} > \lambda_{n''} u / w_1) P(X_1^{(n'')} \geq 0)^{k-1}.$$

Similarly

$$P(Z_{n''} < -\lambda_{n''} u) \geq P(X_1^{(n'')} < -\lambda_{n''} u / w_1) P(X_1^{(n'')} \leq 0)^{k-1}.$$

and hence

$$P(|X_1^{(n'')}| > \lambda_{n''} u / w_1) \to 0.$$

As a sequence of real numbers converges to a limit if and only if every subsequence contains a further subsequence which converges to the same limit, we have

$$P(|X_1^{(n)}| > \lambda_n u / w_1) \to 0.$$

This implies that $\|X_1^{(n)}\| \to 0$ in probability.

Following [15] we truncate our random variables by setting, for any random variable Y,

$$Y^* = \frac{Y}{\max\{\|Y\|, 1\}}.$$

Let $V_n = w_1^{-1} \sum_{i=2}^k w_i X_i^{(n)}$ and observe that since $Z_n \to 0$ almost surely, $X_1^{(n)} + V_n \to 0$ almost surely. It follows that $(X_1^{(n)})^* + V_n^* \to 0$ almost surely. Let $\mathcal{G}_n = \sigma(X_1, \ldots, X_{k^n})$. Then

$$\begin{aligned} (X_1^{(n)})^* &= E((X_1^{(n)})^* | \mathcal{G}_n) \\ &= E((X_1^{(n)})^* + V_n^* | \mathcal{G}_n) - E(V_n^*) \end{aligned}$$

By Lemma 2.3 we have $(X_1^{(n)})^* + EV_n^* \to 0$ almost surely. It remains to show that $EV_n^* \to 0$. Since $\|X_1^{(n)}\| \to 0$ in probability, $\|(X_1^{(n)})^*\| \to 0$ in probability and hence $EV_n^* \to 0$ as required. ∎

Remark: The symmetry condition is only required for the constancy of the limit. An alternative condition is that there exists $0 < c < 1$ such that

$$\|f(x_1, \dots, x_k) - f(\tilde{x}_1, x_2, \dots, x_k)\| \leq c\|x_1 - \tilde{x}_1\|.$$

Indeed, this ensures that Z is in the tail σ-field.

Combining the above Theorem with Lemma 2.4 we have

Corollary 3.2 *Let $f: P^k \to P$ be a concave homogeneous function, symmetric in any two of its arguments with a non-zero fixed point. Then $X^{(n)}$ converges almost surely to a constant in P.*

3.1 On the positivity of the limit

We have established convergence for our hierarchical sequence but there is still the possibility that the limit is 0. We give some conditions which will ensure strict positivity.

Let $p_n = P(X_1^{(n)} > 0)$. Following [12] we define

$$\mathcal{U} = \{A \subset \{1, \dots, k\} : x_j > 0, \forall j \in A \Rightarrow f(x_1, \dots, x_k) > 0,\ \forall x_i \text{ with } i \in A^c\},$$

and

$$g(p) = \sum_{A \in \mathcal{U}} p^{|A|}(1-p)^{k-|A|}.$$

Lemma 3.3 *If $f : P^k \to P$ is concave, then $p_{n+1} = g(p_n)$ for all n.*

Proof: It suffices to show that

$$x_j > 0, \forall j \in A \Rightarrow f(x_1, \dots, x_k) > 0,\ \forall x_i \text{ with } i \in A^c$$

if and only if

$$\exists x_j > 0, j \in A \text{ such that } f(x_1, \dots, x_k) > 0,\ \forall x_i \text{ with } i \in A^c.$$

We will use the positivity and concavity of the function f. Fix A and label its elements $i_1, \dots, i_m$. Now define $f_A : P^{|A|} \to P$ by

$$f_A(x_1, \dots, x_m) = \inf\{f(y_1, \dots, y_k) : y_{i_j} = x_j, j = 1, \dots, m\}.$$

Note that f_A is concave.

We will show that if f_A is positive at any point in the interior of $K = P^{|A|}$, then f_A is positive throughout the interior. Suppose otherwise: Suppose $f_A(x) > 0$ for some $x \in K^\circ$ and $f_A(y) = 0$ for some $y \in K^\circ$. Let $l(t) = x + t(y-x)$ for $t \geq 0$. There exists $\epsilon > 0$ such that $z = l(1+\epsilon) \in K^\circ$. Note that

$$y = \frac{\epsilon}{1+\epsilon}x + \frac{1}{1+\epsilon}z,$$

and hence by concavity,

$$0 = f_A(y) \geq \frac{\epsilon}{1+\epsilon}f_A(x) + \frac{1}{1+\epsilon}f_A(z).$$

Thus $f_A(z) < 0$ contradicting the positivity. ∎

We state some properties of g. Firstly $g(0) = 0, g(1) = 1$, g is an increasing function and $g'(0)$ is the number of $A \in \mathcal{U}$ with exactly one element. It is shown in [12] that g has a unique repulsive fixed point p_g in $[0,1]$. Also

$$\lim_{n\to\infty} p_n = \begin{cases} 0, & p_0 < p_g, \\ 1, & p_0 > p_g. \end{cases}$$

We summarise the above in a Lemma.

Lemma 3.4 *If $f : P^k \to P$ is concave, then $X^{(n)}$ to0 if $p_0 < p_g$.*

It is more difficult to establish strict positivity.

Lemma 3.5 *In the context of Corollary 3.2, a sufficient condition for a strictly positive limit is that there exists a neighbourhood U of the origin such that $x_i \notin U$ for all i implies that $f(x) \notin U$ and $P(X^{(0)} \notin \epsilon U) = 1$ for some $\epsilon > 0$.*

Proof: The conditions ensure that $P(X^{(n)} \notin \epsilon U) = 1$ for all n and hence the limit is strictly positive. ∎

3.2 Conjugacy

An alternative approach to this problem which seems to have some potential is to use the idea of conjugacy. We begin with an example. Let $P = \mathbb{R}_+^2$

and $f(x_1, x_2) = x_1x_2/(x_1 + x_2)$. If we define $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\phi(x) = 1/x$, then

$$f(\phi(x_1), \phi(x_2)) = \phi(g(x_1, x_2)), \tag{3.1}$$

where $g(x_1, x_2) = (x_1 + x_2)/2$. Thus

$$X^{(n)} = \phi\left(2^{-n}\sum_{i=1}^{2^n} X_i^{(0)}\right),$$

and $X^{(n)}$ converges almost surely by the usual strong law of large numbers provided $\phi(E(X^{(0)})) < \infty$. Note that the electrical network generating this map is a pair of resistors in parallel and hence the map ϕ examines the effective conductivity instead of the effective resistance across the network.

This extends the usual notion of conjugacy to our setting: We will say that a non-linear function $f : P^k \to P$ is conjugate to a linear function $g : P^k \to P$ if there exists a one-to-one function ϕ satisfying the extension of (3.1) to k variables. (Note that this idea also works in the case where ϕ is not one-to-one; this is the natural analogue of semi-conjugacy.) As in the usual dynamical systems context conjugacy is a powerful tool for reducing non-linear problems to linear ones even if it is not possible to determine the conjugating function explicitly; there are abstract theorems giving sufficient conditions for the existence of conjugating functions (see for example [7]).

In the case of a homogeneous function $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ with a fixed point at $(1, 1)$, we can apply the standard results in [7] directly. More generally (as far as we know) there are no abstract theorems on this form of conjugacy.

4 Examples

1. Electrical networks

We begin with a class of simple hierarchical models motivated by the diamond lattice discussed in [15]. The diamond lattice is the hierarchical graph in which each edge is replaced with a parallel pair of sequentially connected resistors. We can treat a larger class of models, in particular networks with asymmetric arrangements of conductors. For example, the case where each edge is replaced by three sets of sequentially connected resistors, one with two in series, one with three in series and the third consisting of 6 in series. For this network the map $f : \mathbb{R}_+^{11} \to \mathbb{R}_+$ is

$$f(x_1, \ldots, x_{11}) = \frac{1}{1/x_1 + 1/x_2} + \frac{1}{1/x_3 + 1/x_4 + 1/x_5} + \frac{1}{1/x_6 + \cdots + 1/x_{11}}.$$

This determines the effective conductivity across the unit edge with the addition of the new lattice. Using the relationship between harmonic and arithmetic means we have

$$f(x_1, x_2, \ldots, x_{11}) \le \frac{1}{4}(x_1 + x_2) + \frac{1}{9}(x_3 + x_4 + x_5) + \frac{1}{36}(x_6 + \cdots + x_{11}).$$

Thus the function f is bounded below by 0 and above by a suitable hyperplane, it is also symmetric in any pair of the variables taken from a set of the resistors in series, and hence we can apply our Theorem 3.1 to deduce that if the conductors are chosen randomly according to any distribution with $EX < \infty$, then the effective conductance $X^{(n)}$, will converge almost surely. Thus

$$X^{(n)} \to X, \ \text{ as } \ n \to \infty.$$

Using Lemma 3.4 the limit will be 0 if $P(X^{(0)} > 0) < p_c$, where p_c is the probability that there is percolation across the graph, that there is a connection in the limit between the initial points, by edges with non-zero conductivity. In this one-dimensional setting we can conclude that the limit will be strictly positive if $P(X^{(0)} > 0) > p_c$.

In general random resistor networks lead to concave functions. To see this we recall the definitions of effective conductance. Let $G = (V, E)$ be a graph and assign conductors $c(e)$ to each edge $e \in E$. We will write an edge $e = (x, y)$ where x and y are the vertices at each end of the edge. The energy of a function f defined on the graph is then

$$\mathcal{E}_c(f, f) = \frac{1}{2} \sum_{e \in E} (f(x) - f(y))^2 c(e). \tag{4.1}$$

The effective conductance $C(x, y)$ between two vertices x, y in the graph is defined to be

$$C_c(x, y) = \inf\{\mathcal{E}_c(f, f) : f(x) = 0, f(y) = 1\}. \tag{4.2}$$

This is easily seen to be a concave function of the underlying resistors c.

2. Fractals

The homogenization problem for fractals has been treated in [8] and [6]. We consider a fractal graph, such as the Sierpinski gasket graph in which we take unit equilateral triangles and extend them to infinity using the structure of the Sierpinski gasket. The problem is to assign a random resistor to each edge of the graph and then rescale the graph according to the appropriate length scale factor. As we iterate this process, does the effective conductivity, renormalized by the appropriate factor, converge? The previous work on this problem made either strong assumptions about the random variables or looked at sufficiently simple fractals.

We will consider the class of nested fractals first defined by [9]. For $l > 1$, an *l-similitude* is a map $\psi : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\psi(x) = l^{-1}U(x) + x_0, \tag{4.3}$$

where U is a unitary, linear map and $x_0 \in \mathbb{R}^d$. Let $\psi = \{\psi_1, \ldots, \psi_m\}$ be a finite family of maps where ψ_i is an l-similitude. For $B \subset \mathbb{R}^d$, define

$$\Psi(B) = \bigcup_{i=1}^{m} \psi_i(B),$$

and let

$$\Psi_n(B) = \Psi \circ \ldots \circ \Psi(B).$$

The map Ψ on the set of compact subsets of $\mathbb{R}^d$ has a unique fixed point F, which is a self-similar set satisfying $F = \Psi(F)$.

As each ψ_i is a contraction, it has a unique fixed point. Let F_0' be the set of fixed points of the mappings ψ_i, $1 \le i \le m$. A point $x \in F_0'$ is called an *essential fixed point* if there exist $i, j \in \{1, \ldots, m\}$, $i \ne j$ and $y \in F_0'$ such that $\psi_i(x) = \psi_j(y)$. We write F_0 for the set of essential fixed points. Now define

$$\psi_{i_1,\ldots,i_n}(B) = \psi_{i_1} \circ \ldots \circ \psi_{i_n}(B), \quad B \subset \mathbb{R}^D.$$

The set $F_{i_1,\ldots,i_n} = \psi_{i_1,\ldots,i_n}(F_0)$ is called an *n-cell.* The lattice of fixed points F_n is defined by

$$F_n = \Psi_n(F_0), \tag{4.4}$$

and the set F can be recovered from the essential fixed points by setting

$$F = cl(\cup_{n=0}^{\infty} F_n).$$

We can now define a nested fractal as follows.

Definition 4.1 The set F is a nested fractal if $\{\psi_1, \ldots, \psi_m\}$ satisfy:
(A1) *(Connectivity)* For any 1-cells C and C', there is a sequence $\{C_i : i = 0, \ldots, n\}$ of 1-cells such that $C_0 = C, C_n = C'$ and $C_{i-1} \cap C_i \ne \emptyset, \quad i = 1, \ldots, n$.
(A2) *(Symmetry)* If $x, y \in F_0$, then reflection in the hyperplane $H_{xy} = \{z :$

$|z-x| = |z-y|\}$ maps F_n to itself.
(A3) *(Nesting)* If $\{i_1, \ldots, i_n\}, \{j_1, \ldots, j_n\}$ are distinct sequences, then

$$\psi_{i_1,\ldots,i_n}(F) \bigcap \psi_{j_1,\ldots,j_n}(F) = \psi_{i_1,\ldots,i_n}(F_0) \bigcap \psi_{j_1,\ldots,j_n}(F_0).$$

(A4) *(Open set condition)* There is a non-empty, bounded, open set V such that the $\psi_i(V)$ are disjoint and $\cup_{i=1}^m \psi_i(V) \subset V$.

Let ψ_1 denote the similitude which has the origin as a fixed point. We can construct the pre-fractal, an infinite graph, by setting $G^{(n)} = \psi_1^{-n}(G_0)$, where G_0 is the complete graph on the essential fixed points F_0. The pre-fractal graph is then

$$G^{(\infty)} = \bigcup_{n=0}^{\infty} G^{(n)}.$$

Let $\mathcal{T}_n$ denote the set of translations of G_0 in $G^{(n)}$ and define $\mathcal{T} = \cup_n \mathcal{T}_n$. Then the infinite fractal lattice can be written, in terms of translations of the complete graph on F_0, as

$$G^{(\infty)} = \bigcup_{\tau \in \mathcal{T}} \tau(G_0).$$

We now recall the fixed point problem for nested fractals which we set up as an eigenvalue problem for a nonlinear map on discrete Dirichlet forms. We let c_0 be a set of conductors on G_0, which we can view as an $|E_0| \times |E_0|$ matrix. Define the quadratic form $\mathcal{E}^{(0)}$ as in (4.1) by

$$\mathcal{E}^{(0)}(f,g) = \frac{1}{2} \sum_{x,y \in G_0} (f(x) - f(y))(g(x) - g(y))c(x,y),$$

with domain $\mathcal{F}_0 = \{f | f : F_0 \to \mathbb{R}\}$.

The renormalization map we are interested in can be defined as the composition of two maps. Let positive real numbers r_i be given and set

$$\Psi(\mathcal{E}_1, \ldots, \mathcal{E}_N)(f,g) = \mathcal{E}_{(1)}(f,g) = \sum_{i=1}^{N} r_i^{-1} \mathcal{E}_i(f \circ \psi_i, g \circ \psi_i),$$

for $f, g \in \mathcal{F}_1 := \{f | f : F_1 \to \mathbb{R}\}$. Define the renormalized Dirichlet form by taking its trace on F_0,

$$\Phi(\mathcal{E}_{(1)})(f,f) = \tilde{\mathcal{E}}_0(f,f) = \inf\{\mathcal{E}_{(1)}(g,g) | g = f \text{ on } F_0\}.$$

From (4.2) this is just computing the effective conductance between vertices of F_0 induced by the weighted electrical network F_1. The operation we have described is a nonlinear map on conductors and we can write this as a map on Dirichlet forms as $\Lambda = \Phi \circ \Psi$. For a given set of r, the existence problem for the corresponding Laplace operator on the fractal is solved, if we can find a set of conductors c and a constant λ such that

$$\Lambda(\mathcal{E}_0, \ldots, \mathcal{E}_0)(f,f) = \tilde{\mathcal{E}}_0(f,f) = \lambda \mathcal{E}_0(f,f), \quad \forall f \in \mathcal{F}_0.$$

From this point on we will only consider the case $r_i = 1$ for all i. This will ensure that our function is symmetric in at least one pair of variables.

Let $\mathbb{D}$ denote the set of Dirichlet forms on F_0 and let $\mathbb{P}$ denote the set of positive semidefinite forms on F_0. The space $\mathbb{B} := \mathbb{D} - \mathbb{D}$ with norm $\|\mathcal{E}\|^2 = \sup\{\mathcal{E}(f,f) | f \in \mathcal{F}_0, < f, f >= 1\}$ is a Banach space. Let $\mathbb{P}^\circ$ be interior of $\mathbb{P}$, the set of irreducible positive semidefinite forms. A Dirichlet form is said to be irreducible if the only functions for which $\mathcal{E}(f,f) = 0$ are the constant functions. Thus we have $\mathbb{D} \subset \mathbb{P}$ and the set of irreducible forms in $\mathbb{D}$ is $\mathbb{D} \cap \mathbb{P}^\circ$. The sets $\mathbb{D}, \mathbb{P}$ are cones and we refer to [10] for more details.

We will work with Hilbert's projective metric on the cone $\mathbb{P}$ and will be interested in fixed points which lie in $\mathbb{D} \cap \mathbb{P}^\circ$. In order to define the projective metric we must work with comparable forms. Two elements $\mathcal{E}, \mathcal{E}'$ of our cone are comparable if there exist real constants $a, b > 0$ such that $a\mathcal{E} \le \mathcal{E}' \le b\mathcal{E}$. This is an equivalence relation and the cone can be divided into parts based on this relation. If we take $\mathcal{E}, \mathcal{E}'$ to be two non zero comparable elements of our cone, then we can define

$$h(\mathcal{E}, \mathcal{E}') = \ln \frac{M(\mathcal{E}/\mathcal{E}')}{m(\mathcal{E}/\mathcal{E}')},$$

where

$$\begin{aligned} M(\mathcal{E}/\mathcal{E}') &= \inf\{b > 0 : \mathcal{E} \le b\mathcal{E}'\}, \\ m(\mathcal{E}/\mathcal{E}') &= \sup\{a > 0 : a\mathcal{E}' \le \mathcal{E}\}. \end{aligned}$$

If $\mathcal{E}, \mathcal{E}'$ are not comparable, then $h(\mathcal{E}, \mathcal{E}') = \infty$. The function h is called Hilbert's projective metric, even though it is not a metric. It is not difficult to see that if we restrict h to a part of the cone which intersects with the surface of the unit ball in the Banach space $\mathbb{B}$, then it is a metric. The projective metric, h and the norm $\|.\|$ are locally equivalent on the surface of the unit ball in $\mathbb{B}$.

We now consider the renormalization map Λ, which we can regard as $\Lambda : \mathbb{D}^N \to \mathbb{D}$. By using the fact that the unit ball in $\mathbb{B}$ is compact and

Brouwer's fixed point Theorem, the eigenvalue problem can be solved, to show there exists $\alpha \in \mathbb{R}$ and $\mathcal{E} \in \mathbb{D}$ such that $\Lambda(\mathcal{E}, \ldots, \mathcal{E}) = \alpha\mathcal{E}$. However this solution does not necessarily correspond to an irreducible Dirichlet form. From [10], the map is non-expansive on $\mathbb{P}^\circ$, in Hilbert's projective metric and we can show, using this metric, that there exists a fixed point $\hat{\mathcal{E}} \in \mathbb{D} \cap \mathbb{P}^\circ$ such that

$$\Lambda(\hat{\mathcal{E}}, \ldots, \hat{\mathcal{E}}) = \lambda\hat{\mathcal{E}}.$$

The existence was first demonstrated in [9], and the uniqueness of $\mathcal{E}$ for nested fractals with conductors invariant under the full symmetry group was first established by [13] and via an alternative approach in [11].

Let $X_i^{(0)}$ be a sequence of $\mathbb{D} \cap \mathbb{P}^\circ$ valued random variables. We can place these on the infinite nested fractal lattice $G^{(\infty)}$ to define a random Dirichlet form

$$\mathcal{E}_0^{(\infty)}(f,g) = \sum_{\tau \in \mathcal{T}} X_\tau^{(0)}(f \circ \tau, g \circ \tau),$$

for $f, g \in \{f | f : G^{(\infty)} \to \mathbb{R}\}$. Our map Λ can be extended to act on this random form. Hence we can construct the sequence $X^{(n)}$ as

$$X^{(n+1)} = \lambda^{-n}\Lambda(X_1^{(n)}, \ldots, X_N^{(n)}),$$

with associated random form

$$\mathcal{E}_n^{(\infty)}(f,g) = \sum_{\tau \in \mathcal{T}} X_\tau^{(n)}(f \circ \tau, g \circ \tau),$$

where the $X_\tau^{(n)}$ are independent copies of $X_1^{(n)}$. Note that $X_1^{(n)}$ is the Dirichlet form on the unit cell induced by rescaling the lattice by l^{-n}.

Theorem 4.2 *If $Eh(\hat{\mathcal{E}}, X^{(0)}) < \infty$, then the rescaled effective conductances across $G^{(n)}$ converge,*

$$X_1^{(n)} \to X^{(\infty)}, \ \ a.s. \ \ as \ \ n \to \infty,$$

where $X^{(\infty)} = \gamma\hat{\mathcal{E}}$, a scaled copy of the fixed point. If there exists $\epsilon > 0$ such that $P(X^{(0)} > \epsilon\hat{\mathcal{E}}) = 1$, then $\gamma > 0$. Also the sequence $\mathcal{E}_n^{(\infty)}$ of random Dirichlet forms on $G^{(\infty)}$ converges to $\hat{\mathcal{E}}^{(\infty)}$, the Dirichlet form in which each unit cell has a copy of $\gamma\hat{\mathcal{E}}$.

Proof: As the norm on $\mathbb{B}$ is monotone we see that $\mathcal{E} \leq \mathcal{E}'$ implies $\|\mathcal{E}\| \leq \|\mathcal{E}'\|$ and hence convergence in the order topology gives convergence in norm.

The map Λ is derived from an electrical network and it is straightforward to show that it is positive and concave, [10] Theorem 2.2, and symmetry follows from our symmetry assumption on the operator. By assumption that the Hilbert distance from the fixed point is finite, the $X_i^{(0)}$ are integrable in the Banach space. As there is also a fixed point $\hat{\mathcal{E}}$, we can apply Theorem 3.1 to deduce the convergence of the hierarchical sequence.

All we need to prove now is that the limit $X^{(\infty)}$ is a multiple of the fixed point. As the limit is constant it must satisfy the fixed point equation and hence either be zero or a positive multiple of the fixed point.

As the map Λ is monotone we know that if $P(X^{(0)} > \epsilon\hat{\mathcal{E}}) = 1$, then $P(X^{(n)} > \epsilon\hat{\mathcal{E}}) = 1$ for all n and hence for the limit (recall Lemma 3.5). ∎

Remark: This theorem extends the first result obtained in [8] on the L^1 convergence of the effective conductance. We do not need the elements of the cone $\mathcal{D}$ to be bounded away from ∞ and we obtain an almost sure limit. For strict positivity of the limit in general we require the same bounded below condition. In the case of the Vicsek set, a tree like fractal, [6] proved that the limit is strictly positive provided $P(X^{(0)} \in \mathcal{D} \cap \mathbb{P}^\circ) = 1$.

We note that if the fixed point of the map Λ is not unique, we still have convergence to a limit. In this case the set of fixed points forms an arc and our sequence will converge (up to a scale factor) to a point on this arc. For the Vicsek set, this limit was explicitly identified in [6].

3. Hierarchical fiber models.

In [15] a hierarchical fiber model is analysed and we can use our set up to investigate higher dimensional analogues. Here we present a simple example. Assume that a fiber has forces applied to it in a direction which is not parallel to the direction of the fiber. The force vector will have two components and the fiber will have a certain break point in each component. If we have two fibers, with break points (x_1, y_1) and (x_2, y_2) respectively, then the load will be distributed between the two fibers. If a fiber breaks, then the other inherits the full force. Thus the break point of a pair of fibers in each component will be

$$(2(x_1 \wedge x_2) \vee x_1 \vee x_2, 2(y_1 \wedge y_2) \vee y_1 \vee y_2).$$

The fixed point for this model is the point $x = y = 1$ and thus we can define our function and then check that

$$\begin{aligned} f((x_1, y_1), (x_2, y_2)) &= ((x_1 \wedge x_2) \vee x_1/2 \vee x_2/2, (y_1 \wedge y_2) \vee y_1/2 \vee y_2/2) \\ &\leq \frac{1}{2}\left((x_1, y_1) + (x_2, y_2)\right). \end{aligned}$$

As the function is clearly symmetric we can apply our Theorem 3.1 to see that, provided $E(|X^{(0)}|, |Y^{(0)}|) < \infty$, the hierarchical sequence defined by this map converges. On the question of strict positivity of the limit we note that the function $g(p)$ of subsection 3.1 has only fixed points at 0 and 1, and hence does not give any information. If we assume that the variables are bounded away from 0, then applying Lemma 3.5, the limit is strictly positive.

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