The Characteristic Polynomial of a Random Permutation Matrix

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We establish a central limit theorem for the logarithm of the characteristic polynomial of a random permutation matrix. With this result we can obtain a central limit theorem for the counting function for the eigenvalues lying in some interval on the unit circle.

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Abstract

We establish a central limit theorem for the logarithm of the characteristic polynomial of a random permutation matrix. With this result we can obtain a central limit theorem for the counting function for the eigenvalues lying in some interval on the unit circle.

1 Introduction and summary

The spectral properties of random matrices have received a lot of attention in recent years, and this is largely due to the striking (statistical) similarity between eigenvalue distributions of certain random matrices and the distribution of the zeros of the Riemann zeta function along the critical line. This similarity was first observed by Montgomery [6] (through conversations with Freeman Dyson in the early 1970's) and has since been supported by various numerical and analytic results (see, for example, [7, 8, 3]). Most of this work has focused on random matrices with continuous distributions, such as Haar measure on the unitary group (the CUE ensemble) or random hermitian matrices with complex iid Gaussian entries (the GUE ensemble). The discrete analogue of the unitary group is the symmetric group. The spectrum of a permutation matrix is completely determined by the cycle structure of the corresponding permutation, and the cycle structure of random permutations is very well understood. In both cases all of the eigenvalues lie on the unit circle. One might expect the spectrum of a random permutation matrix to look very different from that of a random unitary matrix, and indeed it does, but in many respects it is surprisingly similar. This was first observed in the PhD thesis of Wieand [10] (see also [11]) where it is shown that the order of the fluctuations for the number of eigenvalues in a (nice) subset of the circle is the same as in the unitary case[1]. The main result of this paper is that a similar statement is true for values of the characteristic polynomial.

The characteristic polynomial of a random unitary matrix was first studied by Keating and Snaith [4]. They propose a renormalized version of this random polynomial as a model for the Riemann zeta function along the critical line. They also prove a central theorem for the value distribution of the polynomial which is consistent with Selberg's central limit theorem for the value distribution of the Riemann zeta function along the critical line [9]. In order to state the theorem of Keating and Snaith, and the results of this paper, we need to introduce some notation.

For an n by n matrix U let

$$\chi_U(s) = \det(U - sI),$$

denote the characteristic polynomial and set

$$\zeta_U(s) = \det\left(I - \frac{1}{s}U\right) = \left(-\frac{1}{s}\right)^n \chi_U(s).$$

We denote the restriction of ζ_U to the unit circle by

$$Z_U(\theta) = \zeta(e^{i\theta}) \quad (0 \leq \theta < 2\pi).$$

Note that ζ_U and χ_U have the same zeros, and $\zeta_U(s) \to 1$ as $s \to \infty$. This allows us to choose a canonical branch of $\log \zeta_U(s)$ as follows. A branch of the log function can only be defined in a simply connected domain that does not include 0, so for each of the n zeros of Z (eigenvalues) we cut the complex plane from 0 to this value. We then choose the branch of $\log \zeta_U(s)$ of the rest of the plane which converges to zero as $s \to \infty$. We use this to define a function on the unit circle:

$$L_U(\theta) = \begin{cases} \log Z_U(\theta), & \text{where defined,} \\ 0, & \text{otherwise.} \end{cases}$$

Keating and Snaith [4] prove the following central limit theorem.

[1] It is also shown in [10] that the joint distributions for the numbers of eigenvalues in collections of intervals has essentially the same covariance structure as in the unitary case.

Theorem 1.1. *Fix $0 \le \theta < 2\pi$. Let U_n be a random n by n unitary matrix, chosen according to Haar measure on the unitary group. Then both the real and imaginary parts of*

$$\frac{L_{U_n}(\theta)}{\sqrt{\frac{1}{2}\log n}},$$

converge in distribution to a standard normal random variable.

In fact, although it is not proved, it is clear from the arguments in [4] that the joint distribution of the real and imaginary parts converges to a standard bivariate normal.

The main result of this paper is the following.

Theorem 1.2. *(i) Suppose $0 \le \theta < 2\pi$ and $\theta/2\pi$ is an irrational of finite type. Let P_n be a n by n permutation matrix, chosen uniformly at random. Then both the real and imaginary parts of*

$$\frac{L_{P_n}(\theta)}{\sqrt{\frac{\pi^2}{12}\log n}},$$

converge in distribution to a standard normal random variable.

(ii) Moreover, for any irrational $\theta/2\pi$,

$$\mathfrak{Im}\frac{L_{P_n}(\theta) - EL_{P_n}(\theta)}{\sqrt{\frac{\pi^2}{12}\log n}}$$

converges in distribution to a standard normal random variable.

The notion of type will be explained in Section 2. Here we will simply remark that the set of numbers which are not of finite type has Hausdorff dimension zero.

We can relate this result to a central limit theorem of Wieand, by considering the imaginary part of L_{P_n}. This is described in Section 4. Let X_n^l denote the number of eigenvalues of P_n which lie in the half open subinterval of the unit circle $(1, e^{il}]$. A particular case of the main result in [11], which we now state, follows from a straightforward modification of the proof of Theorem 1.2.

Theorem 1.3 (Wieand). *If $\frac{l}{2\pi}$ is irrational,*

$$\frac{X_n^l - EX_n^l}{\left(\frac{1}{3}\log n\right)^{1/2}}$$

converges in distribution to a standard normal random variable. If $\frac{l}{2\pi}$ is irrational of finite type, then

$$EX_n^l = \frac{nl}{2\pi} - \frac{1}{2}\log n + O(1).$$

The outline of the paper is as follows. In Section 2 we present some preliminary material on permutations and ergodic theory. In Section 3 prove the main result and, in Section 4, we relate this to the counting function X_n^l.

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2 Preliminaries

2.1 Permutations and the Feller coupling

Given a permutation π in the symmetric group S_n the corresponding *permutation matrix* is Π, where $\Pi_{i,j} = 1(\pi(i) = j)$. The permutation matrices form a subgroup of the unitary group. A *random n by n permutation matrix* is the matrix of a permutation chosen uniformly at random from S_n.

Let π be an n-cycle and Π its permutation matrix. The characteristic polynomial of Π is $\chi_\Pi(t) = t^n - 1$. Suppose $\pi \in S_n$ has cycle structure $1^{c_1}2^{c_2}\ldots$. Then $\chi_\Pi(t) = \prod_{j=1}^n (t^j - 1)^{c_j}$.

Let $C_j(n)$ denote the number of cycles of length j in a random permutation of length n. The Feller coupling, see for example [1], allows one to construct independent random variables Z_j on the same probability space as the $C_j(n)$ in such a way that Z_j is Poisson with mean $1/j$, and

$$E|\sum_{j=1}^n a_j(Z_j - C_j(n))| \leq \frac{2}{n+1}\sum_{j=1}^n |a_j|,$$

for any complex sequence a_j.

2.2 Uniformly distributed sequences and discrepancy

We need a number of results concerning uniformly distributed sequences which we collect together here. For a real number x we will write $\{x\}$ for the fractional part of x A real sequence $(x_n)_{n=1}^{\infty}$ is *uniformly distributed* if for all $0 \leq a < b \leq 1$ we have

$$\lim_{N\to\infty} \frac{|\{n \leq N : a < \{x_n\} < b\}|}{N} = b - a.$$

Lemma 2.1. *(x_n) is uniformly distributed if and only if for every properly Riemann integrable f defined on [0,1] we have $\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) \to \int_0^1 f(t)\, dt$*

Weyl's theorem states that if ϕ is irrational the sequence $x_n = \{n\phi\}$ is uniformly distributed.

Let $x = (x_n)_{n=1}^{\infty}$ be a real sequence. For $I \subset [0,1]$ write $A_N(I) = |\{n \leq N : \{x_n\} \in I\}|$. The *discrepancy* (at the N^{th} stage) is

$$D_N(x) = \sup_{0\leq a<b\leq 1} \left| \frac{A_N([a,b))}{N} - (b-a) \right|.$$

As one would expect, $D_N(x) \to 0$ as $N \to \infty$ precisely when x is uniformly distributed. For a proof of this (and other statements in this section) see Kuipers and Niederreiter [5]. We will find it useful to consider a slightly restricted notion of discrepancy:

$$D_N^*(x) = \sup_{0<a\leq 1} \left| \frac{A_N([0,a))}{N} - a \right|.$$

This is equivalent (in the sense that $D_N^* \leq D_N \leq 2D_N^*$) but has the advantage of having a more convenient expression: if we order $(\{x_n\})_{n=1}^{N}$ as $y_1 \leq \ldots \leq y_N$ then

$$D_N^*(x) = \max_{n\leq N} \max\left(\left| y_n - \frac{n}{N} \right|, \left| y_n - \frac{n-1}{N} \right| \right).$$

The discrepancy controls the rate of convergence in Lemma 2.1. The variation of a function f is defined to be $V(f) = \int_0^1 |df(t)|$, where df denotes the Lebesgue-Stieltjes measure of f on [0,1]. We have $V(f) = \int_0^1 |f'(t)|\, dt$ when f is differentiable.

Theorem 2.2. ***Koksma's inequality*** *Let f be a function on $[0,1]$ of bounded variation $V(f)$ and $x = (x_n)_{n=1}^{\infty}$ be a real sequence. Then*

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) - \int_0^1 f(t)\ dt\right| \le V(f)D_n^*(x).$$

We will be interested in the discrepancy of the sequence $x_n = \{n\phi\}$, where $\phi = \frac{\theta}{2\pi}$ is an irrational in [0,1]. This will depend upon the extent to which ϕ can be approximated by rationals.

For real α let $\|\alpha\|$ denote the distance from α to the nearest integer. The *type* of ϕ is the value of η which is the supremum of those γ for which lim inf $n^{\gamma}\|n\phi\| = 0$ as n runs through the natural numbers.

This implies that if m is less than the type of ϕ and $C > 0$ is a constant there are an infinite number of approximations by rationals satisfying

$$\left|\phi - \frac{p}{q}\right| \le \frac{C}{q^{m+1}}.$$

A simple application of the pigeonhole principle shows that any irrational has type at least 1. It is possible that the type of ϕ equals infinity; if this is not the case ϕ is of *finite type.* Roughly speaking, small type implies small discrepancy.

The set of Liouville numbers is defined by

$$\mathbb{L} = \{x \in \mathbb{R}\backslash\mathbb{Q} : \forall n \in \mathbb{N}, \exists q \in \mathbb{N} : \|qx\| < q^{-n}\}.$$

It is easy to see that if a number is not of finite type, then it is in $\mathbb{L}$. It is a classical result that $\mathbb{L}$ is a small set which has Hausdorff dimension 0, [2].

For numbers of finite type we have a strong control on their discrepancy.

Theorem 2.3. *Let ϕ have finite type η and let $x_n = \{n\phi\}$. Then for every $\epsilon > 0$ we have $D_N(x) = O\left(N^{-\frac{1}{\eta}+\epsilon}\right)$.*

2.3 Elementary analytic results

We conclude with two elementary results.

Lemma 2.4. *Let $(x_k)_{k=1}^{\infty}$ be a real sequence. Suppose $\frac{1}{n}\sum_{k=1}^{n} x_k \to L$. Then $\frac{1}{\log n}\sum_{k=1}^{n}\frac{x_k}{k} \to L$.*

Finally we record a summation by parts formula.

Lemma 2.5. *Let $(x_k)_{k=1}^n$ be a real sequence and $y(t)$ a function with a continuous derivative. Define $X(t) = \sum_{k\le t} x_k$. Then*

$$\sum_{k=1}^{n} x_k y(k) = X(n)y(n) - \int_1^n X(t)y'(t)dt. \tag{2.1}$$

3 Proof of the main result

In order to prove Theorem 1.2 we will consider the real and imaginary parts separately. First we observe that from the structure of random permutations,

$$L_{P_n}(\theta) = \sum_{j=1}^{n} C_j(n)\log(1-e^{-ij\theta}),$$

and hence

$$\mathfrak{Re} L_{P_n}(\theta) = \sum_{j=1}^{n} C_j(n)\log|1-e^{-ij\theta}|, \tag{3.1}$$

$$\mathfrak{Im} L_{P_n}(\theta) = \sum_{j=1}^{n} C_j(n)\arg(1-e^{-ij\theta}). \tag{3.2}$$

We now use the Feller coupling to determine a set of conditions required for the weighted sums of cycle length random variables to have a central limit theorem.

Lemma 3.1. *Let $(a_j)_1^\infty$ be real constants satisfying*

(1) $\frac{1}{n}\sum_{j=1}^n a_j \to 0$,

(2) $\frac{1}{\sqrt{\log n}}\sum_{j=1}^n \frac{a_j}{j} \to 0$,

(3) $\frac{1}{n}\sum_{j=1}^n a_j^k \to B_k$*, for all* $k>1$*, for some constants* B_k.

Let $A_n = \sum_{j=1}^n a_j C_j(n)$, where, as before, $C_j(n)$ is the number of j-cycles of a random permutation from S_n. Then

$$\frac{A_n}{\sqrt{B_2\log n}}$$

converges in distribution to a standard normal random variable. If conditions (1) and (3) are satisfied, then

$$\frac{A_n - EA_n}{\sqrt{B_2 \log n}}$$

converges in distribution to a standard normal random variable.

Proof: We will use the Feller coupling described in Section 2. Consider the sum $\tilde{A}_n = \sum_{j=1}^n a_j Z_j$ where the Z_j are independent Poisson random variables with mean $1/j$. By hypothesis and the fact that $\|.\|_1 \le \|.\|_2$,

$$E|\frac{A_n}{\sqrt{B_2 \log n}} - \frac{\tilde{A}_n}{\sqrt{B_2 \log n}}| \le \frac{2}{\sqrt{B_2 \log n}} \left(\frac{1}{n} \sum_{j=1}^n |a_j|^2 \right)^{1/2} \to 0.$$

The sequences A_n and $\tilde{A}_n$ are thus asymptotically equivalent and we will work with $\tilde{A}_n$ for the rest of the proof.

We consider the characteristic function

$$\chi_n(t) = Ee^{it\frac{\tilde{A}_n}{\sqrt{\log n}}}.$$

As

$$Ee^{itZ_j} = \exp\left(\frac{e^{it}-1}{j}\right),$$

and the Z_j are independent,

$$\chi_n(t) = \exp\left(\sum_{j=1}^n \frac{e^{\frac{a_j}{\sqrt{\log n}}it} - 1}{j}\right). \tag{3.3}$$

We also have Taylor expansions

$$\cos\frac{a_j}{\sqrt{\log n}}t - 1 = -\frac{a_j^2 t^2}{2\log n} + \frac{a_j^3 t^3}{6\log^{3/2} n}\sin\eta_j,$$

$$\sin\frac{a_j}{\sqrt{\log n}}t = \frac{a_j t}{\sqrt{\log n}} - \frac{a_j^3 t^3}{6\log^{3/2} n} + \frac{a_j^4 t^4}{24\log^2 n}\sin\omega_j,$$

(for some $0 < \eta_j, \omega_j < t$ when $t > 0$). Substituting these into (3.3) we obtain

$$\sum_{j=1}^{n} \frac{e^{\frac{a_j}{\sqrt{\log n}} it} - 1}{j} = \left(\frac{1}{\sqrt{\log n}} \sum_{j=1}^{n} \frac{a_j}{j} \right) it - \left(\frac{1}{2 \log n} \sum_{j=1}^{n} \frac{a_j^2}{j} \right) t^2 + I_n,$$

with

$$|I_n| < \frac{C_1(t)}{\log^{3/2} n} \sum_{j=1}^{n} \frac{|a_j|^3}{j} + \frac{C_2(t)}{\log^2 n} \sum_{j=1}^{n} \frac{|a_j|^4}{j},$$

where $C_1(t), C_2(t)$ are constants depending on t only.

Using condition (3) the error I_n defined above tends to 0, as by Lemma 2.1,

$$\frac{C_2(t)}{\log^2 n} \sum_{j=1}^{n} \frac{|a_j|^4}{j} = \frac{C_2(t)}{\log^2 n} \sum_{j=1}^{n} \frac{a_j^4}{j} \to 0,$$

and by Cauchy-Schwarz and Lemma 2.1,

$$\frac{C_1(t)}{\log^{3/2} n} \sum_{j=1}^{n} \frac{|a_j|^3}{j} \leq \frac{C_1(t)}{\log^{3/2} n} \left(\sum_{j=1}^{n} \frac{a_j^6}{j} \right)^{1/2} \left(\sum_{j=1}^{n} \frac{1}{j} \right)^{1/2} \to 0.$$

We deduce from conditions (1) and (2) that

$$\lim_{n\to\infty} \chi_n(t) = e^{-\frac{B_2}{2} t^2},$$

as required. The second part follows from the above: under conditions (1) and (3) we have

$$E \exp\left(it \frac{A_n - EA_n}{\sqrt{\log n}} \right) \sim E \exp\left(it \frac{\tilde{A}_n - E\tilde{A}_n}{\sqrt{\log n}} \right) \to e^{-\frac{B_2}{2} t^2},$$

as $n \to \infty$. □

Proof of Main Theorem:

1. The imaginary part.

We will use Lemma 3.1 with, by (3.2), the values of the a_j given by

$$r_j = \arg(1 - e^{-ij\theta}).$$

Define

$$r(t) = \arg(1 - e^{-2\pi i t}) = (\frac{1}{2} - t)\pi.$$

Then

$$\int_0^1 r^k(t)\ dt = \begin{cases} \frac{\pi^k}{2^k(k+1)}, & k \text{ even,} \\ 0, & k \text{ odd.} \end{cases}$$

This is a proper Riemann integral, so we can apply Lemma 2.1. Together with Lemma 2.4 we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{j=1}^{n} \frac{r_j^k}{j} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} r_j^k = \begin{cases} \frac{\pi^k}{2^k(k+1)}, & k \text{ even,} \\ 0, & k \text{ odd.} \end{cases}$$

Thus we have established conditions (1) and (3) of Lemma 3.1. This completes the proof of Theorem 1.2 (ii). To complete the proof of part (i) (in the imaginary case) we need only establish condition (2), namely that

$$\frac{1}{\sqrt{\log n}} \sum_{j=1}^{n} \frac{r_j}{j} \to 0.$$

As the function $r(t)$ is bounded we can use Koksma's inequality and our estimate on the discrepancy in Theorem 2.3 to obtain a rate of convergence for the coefficients of the imaginary part.

Lemma 3.2. *For ϕ of finite type $\frac{1}{n}\sum_{j=1}^{n} r_j$ decays polynomially as $n \to \infty$.*

To complete the proof of the first part of our main result we apply the summation by parts formula (2.1) with $x_k = r_k$, $y(t) = \frac{1}{t}$, to see that, for ϕ of finite type, $\sum_{j=1}^{n} \frac{r_j}{j}$ remains bounded as $n \to \infty$. Thus, in particular, $\frac{1}{\sqrt{\log n}} \sum_{j=1}^{n} \frac{r_j}{j} \to 0$ giving condition (2) of Lemma 3.1 as required.

2. The real part.

We follow the same approach as for the imaginary part. From (3.1) the values of a_j for the real part are

$$q_j = \log|1 - e^{-ij\theta}|.$$

Define

$$q(t) = \log|1 - e^{-2\pi i t}|.$$

This is unbounded as t tends to 0 or 1 and thus Lemma 2.1 cannot be applied directly. We will show how to adapt the method, but first we need to evaluate the q^k integrals. Term by term integration of the Fourier series gives

$$\begin{aligned}\int_0^1 q^k(t)dt &= \int_0^1 \left(\mathfrak{Re}\log\left(1-e^{-2\pi it}\right)\right)^k dt \\ &= (-1)^k \int_0^1 \left(\mathfrak{Re}\sum_{m=1}^{\infty}\frac{e^{-2\pi itm}}{m}\right)^k dt \\ &= (-1)^k \sum_{m_1}\cdots\sum_{m_k}\frac{1}{m_1\ldots m_k}\int_0^1 \cos 2\pi m_1 t\ldots\cos 2\pi m_k t dt.\end{aligned}$$

The interchange of integral and summation is a consequence of the dominated convergence theorem with dominating function $(\cos(2\pi t))^k$. The cases $k = 1, 2$ give:

$$\int_0^1 q(t)\ dt = 0,$$

$$\int_0^1 q^2(t)\ dt = \sum_{m=1}^{\infty}\frac{1}{m^2}\int_0^1 \cos^2 2\pi mt\ dt = \frac{\pi^2}{6}\cdot\frac{1}{2} = \frac{\pi^2}{12}.$$

In order to establish the conditions of Lemma 3.1 we require the following.

Lemma 3.3. *For ϕ of finite type $\frac{1}{n}\sum_{j=1}^{n} q_j$ decays polynomially to 0 as $n \to \infty$.*

Proof: Let $x_n = \{n\phi\}$ for $n \in \mathbb{N}$. As ϕ has finite type there are constants $C, m > 0$ such that $\|n\phi\| > \frac{C}{n^m}$, $\forall n \in \mathbb{N}$. Fix N and let $\delta = \frac{C}{N^m}$. Order $(\{x_n\})_{n=1}^N$ as $y_1 \leq \ldots \leq y_N$ and define $y_0 = \delta$, $y_{N+1} = 1-\delta$ (so $y_0 < y_i < y_{N+1}$ for each $1 \leq i \leq N$.) Recall that $q(t) = \log|1-e^{-2\pi it}|$ satisfies $\int_0^1 q(t)\ dt = 0$. It is also (infinitely) differentiable in the open interval (0,1) and satisfies $q(1-t) = q(t)$. (The second assertion is clear geometrically.) Consider the identity

$$\sum_{n=0}^{N}\int_{y_n}^{y_{n+1}}\left(t-\frac{n}{N}\right)q'(t)\ dt = \int_{\delta}^{1-\delta} tq'(t)\ dt\ -\ \frac{1}{N}\sum_{n=0}^{N} n\Big(q(y_{n+1})-q(y_n)\Big). \tag{3.4}$$

Integration by parts gives

$$\begin{aligned}\int_{\delta}^{1-\delta} tq'(t)dt &= (1-\delta)q(1-\delta) - \delta q(\delta) - \int_{\delta}^{1-\delta} q(t)dt \\ &= (1-2\delta)q(\delta) + 2\int_{0}^{\delta} q(t)\ dt.\end{aligned}$$

Using this in 3.4, we have

$$\sum_{n=0}^{N} \int_{y_n}^{y_{n+1}} \left(t - \frac{n}{N}\right) q'(t)\ dt = 2\left(\int_0^{\delta} q(t)\ dt\ -\ \delta q(\delta)\right)\ +\ \frac{1}{N}\sum_{n=1}^{N} q(y_n). \tag{3.5}$$

As in Koksma's inequality we can bound the modulus of the left hand side of (3.5) above by

$$D_N^* \int_{\delta}^{1-\delta} |q'(t)|\ dt,$$

to deduce that

$$\left|\frac{1}{N}\sum_{n=1}^{N} q(\{x_n\})\right| \leq D_N^* \int_{\delta}^{1-\delta} |q'(t)|\ dt\ +\ 2\left|\int_0^{\delta} q(t)\ dt\ -\ \delta q(\delta)\right|.$$

As $q(t) \sim \log t$ as $t \to 0$, it is not hard to see that this expression can be bounded by $\frac{C}{n^{\epsilon}}$ for some $C, \epsilon > 0$. □

Using the same method it can be shown that $\frac{1}{n}\sum_{j=1}^{n} q_j^k - \int_0^1 q^k(t)\ dt$ also decays polynomially to 0, for any $k > 1$. Thus we have conditions (1) and (3). The rate of decay is only important in the case $k = 1$, where it may be used to establish condition (2).

We may then complete the proof of the main theorem for the real case in the same way that we did for $r(t)$. □

Remark: A closer inspection of the proof of Theorem 1.2 (i) will reveal that the finite type condition is actually stronger than necessary. For the imaginary part we require $D_N(x) = o\left(\frac{1}{\sqrt{\log N}}\right)$. For the real part we require that there exists a function $\delta(N)$ such that $D_N(x)\log\delta(N)$ and $\delta(N)$ are both $o\left(\frac{1}{\sqrt{\log N}}\right)$.

4 The counting function

Let U be a unitary matrix and I a subinterval of the unit circle. The *counting function* $X_U(I)$ is the number of eigenvalues of U contained in I. The study of this function for various matrix groups is the subject of [10] and [11]. There is a close relationship between the counting function and the *imaginary part* of L_U and this allows us to relate the results we have obtained to the counting function, and to the results in [10] and [11].

The connection between the characteristic polynomial and the counting function can be summarised by the following lemma. This is well-known. We include a proof in the appendix for completeness.

Lemma 4.1. *Let U be a unitary matrix and I a subinterval of the unit circle whose endpoints e^{it_1}, e^{it_2} are not eigenvalues of U. Then*

$$X_U(I) = \frac{n(t_2 - t_1)}{2\pi} + \frac{1}{\pi}\mathfrak{Im}\Big(L_U(t_2) - L_U(t_1)\Big). \tag{4.1}$$

Set $X_n^l = X_{P_n}((1, e^{il}])$. By Lemma 4.1, $X_n^l = \frac{nl}{2\pi} + \frac{1}{\pi}F_n^l$, where

$$F_n^l = \mathfrak{Im}\Big(L_U(l) - \lim_{t\downarrow 0} L_U(t)\Big) = \sum_{j=1}^{n} C_j(n)\Big(r_j - \frac{\pi}{2}\Big),$$

where, as before, the $C_j(n)$ are the number of cycles of length j, and the $r_j = \arg(1 - e^{-ijl})$ are the imaginary coefficients for $L_U(l)$. We will apply a slightly modified version of Theorem 3.1 and take $a_j = r_j - 1/2$. Note that

$$\frac{1}{n}\sum_{j=1}^{n} r_j - \frac{\pi}{2} \to -\frac{\pi}{2},$$

$$\frac{1}{n}\sum_{j=1}^{n}(r_j - \frac{\pi}{2})^2 \to \frac{\pi^2}{12} + \frac{\pi^2}{4} = \frac{\pi^2}{3},$$

and the averages of the higher powers of the a_j also converge to constants. This implies that

$$\frac{X_n^l - EX_n^l}{\left(\frac{1}{3}\log n\right)^{1/2}}$$

converges to a standard normal variable. Finally, using our previous results about the behaviour of the r_j in the case where $l/2\pi$ is of finite type, we have

$$\frac{1}{\sqrt{\log n}}\sum_{j=1}^{n}\frac{r_j-\pi/2}{j}\sim\frac{1}{\sqrt{\log n}}\left(\sum_{j=1}^{n}\frac{r_j}{j}-\frac{\pi}{2}\log n\right)\to 0,$$

and hence

$$EX_n^l=\frac{nl}{2\pi}-\frac{1}{2}\log n+O(1).$$

A Appendix

Proof of Lemma 4.1

Recall that $\zeta_U(s)=\det\left(I-\frac{1}{s}U\right)$. If U is n by n unitary with eigenangles $\theta_1,\ldots,\theta_n$ then

$$\overline{(-s)^n\zeta(s)}=\prod_{k=1}^{n}\left(e^{-i\theta_k}-\bar{s}\right)=\overline{\det U}\;\zeta\left(\frac{1}{\bar{s}}\right)$$

Let α satisfy $\alpha^2=\frac{(-1)^n}{\det U}$ and define $\xi_U(s)=\alpha s^{n/2}\zeta(s)$. Then the functional equation is

$$\xi_U(\frac{1}{\bar{s}})=\overline{\xi_U(s)}. \tag{A.1}$$

Remarks: 1. When n is odd we have to make a choice of $s^{n/2}$ in the domain we use (which will never encircle zero.)
2. There is an intentional analogy with the Riemann zeta function, which we mention for the sake of interest. If one defines

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

then we have the functional equation $\xi(1-s)=\xi(s)$. This expresses the symmetry of ξ under reflection in the half line; the matrix equation expresses conjugate symmetry of ξ_U under inversion in the unit circle.

We use the following result of complex analysis:

Lemma A.1. *Let D be a domain, $f:D\to\mathbb{C}$ analytic and C a closed contour in D on which f is non-zero. Then $\frac{1}{2\pi i}\int_C\frac{f'(z)}{f(z)}\,dz$ is the number of zeros of f inside C.*

Let U be a unitary matrix and I a subinterval of the unit circle whose endpoints e^{it_1}, e^{it_2} are not eigenvalues of U. We apply this lemma to ξ_U, which has the same zeros as ζ_U on the unit circle. Our contour C is the boundary of the region $\{re^{it} \; : \; \frac{1}{2} \leq r \leq 2, \; e^{it} \in I\}$. C falls naturally into two pieces; that which is within the unit circle (C_{in}) and that which is without (C_{out}.) Inversion in the circle takes C_{in} to C_{out} with the opposite orientation, so applying the functional equation,

$$X_U(I) = \frac{1}{2\pi i}\int_C \frac{\xi_U'(s)}{\xi_U(s)}\, ds = \frac{1}{\pi}\mathfrak{Im}\int_{C_{out}} \frac{\xi_U'(s)}{\xi_U(s)}\, ds.$$

Our previous definition of $L_U(s)$ as a branch of $\log \zeta_U(s)$ is on a domain containing C_{out}. Within this we choose a neigbourhood of C_{out}, a branch of $\log s$ in this neighbourhood and so construct $\log \xi_U(s) = \log \alpha + \frac{n}{2}\log s + L_U(s)$. This is an indefinite integral of the integrand considered above, and hence

$$X_U(I) = \frac{n(t_2 - t_1)}{2\pi} + \frac{1}{\pi}\mathfrak{Im}\Big(L_U(t_2) - L_U(t_1)\Big). \tag{A.2}$$

We can think of the second term on the right hand side as the deviation of the number of eigenvalues in I from the number you would expect if each eigenvalue was uniformly distributed on the circle. □

Remark: Suppose we fix t_1 and let t_2 tend to an eigenangle t. Depending on the direction of approach the interval I will either increase to an interval that is open at e^{it} or decrease to an interval closed at e^{it}. We thus have a difference of the multiplicity of the eigenangle in the counting function; this corresponds to the difference of π between the clockwise and anticlockwise limits of the imaginary part of L_U approaching t.

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