Single-Track Circuit Codes

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Abstract

Single-track circuit codes are circuit codes with codewords of length n such that all the n tracks which correspond to the n distinct coordinates of the codewords are cyclic shifts of the first track. These codes simul taneously generalise single-track Gray codes and ordinary circuit codes. They are useful in angular quantisation applications in which error detecting and/or correcting capabilities are needed. A parameter, k, called the spread of the code, measures the strength of this error control capability. We consider the existence of single-track circuit codes for small lengths $n \leq 17$ and spreads $k \leq 6$, constructing some optimal and many good examples. We then give a general construction method for single-track circuit codes which makes use of ordinary circuit codes. We use this construction to construct examples of codes with 360 and 1000 codewords which are of practical importance. We also use the construction to prove a general result on the existence of single-track circuit codes for general spreads.

Keywords

Gray code; snake in the box code; circuit code; single-track; digital encoding; absolute angle measurement; quantisation; error correction

I. Introduction

A length n Gray code is simply a cyclic list of distinct binary n-tuples, called the codewords, with the property that any two adjacent codewords differ in exactly one component. A common use of Gray codes is in reducing quantisation errors in various types of analogue-to-digital conversion systems [11], [12]. They have also found applications in many other areas of coding and computing science — see the introduction to [18] for a list of references.

Spread k circuit codes are a generalisation of Gray codes: they can be thought of as being Gray codes having additional error-detecting capability. For $k \geq 1$, a spread k code is defined to be a Gray code in which two words of the code either lie at most $k-1$ positions apart in the list of codewords or differ in at least k components. Thus a spread 1 code is just a Gray code. Spread 2 codes are more commonly known as snake-in-the-box codes. Circuit codes have a long history (see [1] and the references cited there), and many optimal codes and general constructions for families of codes are known: these results are summarised in Section III below.

As an example of the use of circuit codes in analogue-to-digital conversion, a length n, spread k circuit code C can be used to record the absolute angular positions of a rotating wheel by encoding (e.g. optically) the codewords of C in sectors on n concentrically arranged tracks. Then n reading heads, mounted radially across the tracks suffice to recover the codewords. The number of codewords in C determines the accuracy with which angles can be resolved. Quantisation errors are minimised by using a Gray encoding while errors resulting from equipment malfunction can be dealt with using the spread capability of the code: any error of weight $r < k$ either results in an angle precisely r sectors away from the correct sector or leads to a word W that does not lie in C (so that the error is detectable). In the latter case, if $2r < k$, then the word W' in C that is closest to W in Hamming distance is in turn at most distance r from the correct codeword. The resulting angular error is at most r sectors. In this way, errors of weight up to $\lfloor \frac{k-1}{2} \rfloor$ can be 'partially corrected'.

As resolution and error-correcting capability increase, so must the code length and number of concentric tracks n. The end result is that when high resolution and/or error-tolerant codes are needed, encoders with large physical dimensions must be used. This poses problems for the design of small-scale or high-speed devices. Single-track Gray codes were proposed in [10] and further explored in [9], [18] as a way of overcoming these problems. If a length n single-track Gray code is used in the quantisation application above, then the bits of any codeword can be obtained solely from a single track, the n reading heads being spaced around that single track at some fixed relative positions. Thus the physical dimensions of an encoder can be much reduced

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over those of one using a traditional Gray code. One of the main contributions of [10], [9] was to show that for most resolutions of practical interest, the use of single-track Gray codes does not entail a significant increase in the code length n. Thus practical single-track encoders can be realised using almost the same number of reading heads as is needed for their multi-track counterparts. For example, a length 9 single-track code with 360 codewords was reported in [9], and no Gray code with 360 codewords can have length less than this. So a one-degree resolution code can be realised using just a single track and the minimum of 9 reading heads.

Single-track Gray codes reduce quantisation errors but do not provide any means for error-correction or error-detection. It is therefore very natural to ask for single-track versions of spread k circuit codes. If it is possible to attain a particular resolution using a spread k *single-track* circuit code without significantly increasing the length n over that of a spread k standard circuit code, then once again we would be able to realise spread k single-track encoders using almost the same number of reading heads as for the usual spread k multi-track encoders. There already exist bounds limiting the number of codewords in a spread k circuit code (see Section III for a summary of these) and of course, a spread k single-track circuit code is also a single-track Gray code, so the necessary conditions of [10] apply to the parameters of such a code. Thus the possible number of codewords in a spread k single-track circuit code is already limited.

In this paper we address both the important practical question of finding spread k single-track circuit codes of a particular resolution (where the objective is to minimise the code length n) and the theoretical question of finding, for fixed n and k, length n, spread k single-track circuit codes of the highest possible resolution. As well as reporting many good codes for small parameters n, k of practical interest, we give a flexible construction for spread k single-track circuit codes that achieves high resolutions with reasonable lengths n. We then use this construction to describe families of good codes.

In fact, our construction uses an extension of the methods introduced in [10]. In essence, we take a length n, spread k circuit code and embed its codewords in longer codewords to obtain a spread k single-track circuit code whose length is slightly greater than n. The extra components that we add in our embedding are used to guarantee that the resulting code is single-track. So from good spread k circuit codes, we can construct good spread k single-track circuit codes.

As particular examples of our construction methods we report length 12, spread 2 and length 15, spread 4 single-track codes with 360 codewords and construct a length 20, spread 2 single-track code with 1000 codewords. This last code is optimal in the sense that no spread 2 single-track code with 1000 codewords can have length less than 20.

We also construct, for every even k, a large family of spread k single-track codes. Our main result here is:

Theorem 1: Let $k \geq 2$ be even and let $P(n,k)$ denote the maximum period of a length n, spread k circuit code. Then there exists an (n, nt, k)-STCC for every $n > \max(12, \frac{k^3}{2} - k^2 + k + 2)$ and every even t in the range

$$k \;\leq\; t \;\leq\; P(n - k\lfloor\sqrt{2(n-k-2)/k}\rfloor - 2k, k).$$

Similar families can also be obtained for odd k using the techniques of our paper.

Our paper is organised as follows. In Section II we introduce some basic notation and give formal definitions for the codes that we consider. We also give a characterisation of spread k circuit codes and single-track circuit codes in terms of their *coordinate sequences*. In Section III, we derive simple necessary conditions on the parameters of a single-track circuit code. We also give some upper bounds on the number of codewords in a length n, spread k single-track circuit code. These are based on the bounds for general circuit codes that already exist in the literature. In Section IV we report the results of a computer search for single-track circuit codes with small lengths and spreads. In the Sections V and VI, we give our construction for spread k single-track circuit codes and then in Section VII, some refinements of this construction method with some detailed examples. We use our method to obtain families of even spread single-track circuit codes in Section VIII. Finally, we close by proposing a number of open problems and areas for future research.

II. Coordinate Sequences

We begin with some definitions and notations. Suppose $n \geq 1$ and $k \leq n$. For binary n-tuples W_1, W_2, the usual Hamming distance between W_1 and W_2 is denoted by $d_H(W_1, W_2)$. We also use C to denote a list

$W_0, \ldots, W_{p-1}$ of p binary n-tuples. We are interested in lists in which all n-tuples are different and adjacent n-tuples differ in exactly one position, i.e. have Hamming distance 1. Such a list corresponds to a path of vertices in the n-dimensional binary cube. If in addition the first and last n-tuple of C differ in exactly one position, then we say that C is a *cyclic path.*

Definition 2: Let C be a cyclic path consisting of p binary n-tuples $W_0, W_1, \ldots, W_{p-1}$. Then C is said to be a length n, period p cyclic Gray code with codewords $W_0, W_1, \ldots, W_{p-1}$.

For W_i, W_j in a cyclic path C, we define

$$d_C(W_i, W_j) = \min\{(i-j) \bmod p, (j-i) \bmod p\}.$$

Thus d_C represents the distance in the cyclic path C between codewords.

Definition 3: A length n, period p, spread k circuit code (or (n,p,k)-CC) is a cyclic path C of p binary n-tuples $W_0, W_1, \ldots, W_{p-1}$ with the property that for all $0 \le i, j < p$,

$$d_H(W_i, W_j) < k \Rightarrow d_C(W_i, W_j) < k. \tag{1}$$

It is clear from the above definition that an (n,p,k)-CC is also a (n,p,k')-CC for every $1 \le k' \le k$. Moreover, an $(n,p,1)$-CC is simply a length n, period p cyclic Gray code: it is easy to see that when $k=1$, condition (1) simply states that the codewords are distinct.

Definition 4: Let C be a cyclic path consisting of p binary n-tuples $W_0, W_1, \ldots, W_{p-1}$. Write $W_i = [w_i^0, w_i^1, \ldots, w_i^{n-1}]$. We define component sequence j of C, denoted C^j, to be the binary periodic sequence

$$w_0^j, w_1^j, \ldots, w_{p-1}^j$$

consisting of component j of each of the codewords of C $(0 \le j < n)$.

We can now give a formal definition of single-track circuit codes.

Definition 5: Let C be a (n,p,k)-CC with component sequences C^j, $0 \le j < n$. Then C is said to be a length n, period p, spread k single-track circuit code, (or (n,p,k)-STCC) if sequence C^j is a cyclic shift of sequence C^0 for each $1 \le j < n$.

We will find it convenient to work with the coordinate sequences of these codes [11]:

Definition 6: Let C be a cyclic path consisting of p binary n-tuples $W_0, W_1, \ldots, W_{p-1}$. Let s_i $(0 \le i < p-2)$ denote the unique component in which W_i and W_{i+1} differ and let s_{p-1} denotes the unique component in which w_{p-1} and w_0 differ. So $0 \le s_i < n$ for each i. The sequence s with terms

$$s_0, s_1, \ldots, s_{p-1}$$

is called the (cyclic) coordinate sequence of C.

It is clear that, given the first n-tuple W_0 and the coordinate sequence $s = s_0, s_1, \ldots, s_{p-2}$ of a cyclic path, then the path itself can easily be reconstructed: we simply begin with W_0 and generate subsequent codewords by changing components according to the terms of the coordinate sequence. In fact, any choice $[w_0, w_1, \ldots, w_{n-1}]$ of the first codeword W results in a cyclic path $C(W)$ which has coordinate sequence s. Moreover, component sequence j of $C([w_0, w_1, \ldots, w_{n-1}])$ is the complement of component sequence j of $C([0, 0, \ldots, 0])$ if $w_j = 1$.

We have the following results characterising coordinate sequences of (n,p,k)-CCs and (n,p,k)-STCCs:

Theorem 7: Let C be a cyclic path of p binary n-tuples with coordinate sequence $s = s_0, s_1, \ldots, s_{p-1}$. Then C is an (n,p,k)-CC if and only if:

i) Every symbol j, with $0 \le j < n$, occurs an even number of times in s.

ii) Every subsequence $s_i, s_{i+1}, \ldots, s_{i+r-1}$ of s, with $k \le r \le p-k$ and subscripts taken modulo p, contains at least k symbols with an odd number of occurrences each.

Lemma 8: If $s = s_0, s_1, \ldots, s_{p-1}$ is the coordinate sequence of an (n,p,k)-STCC then, s satisfies properties i) and ii) in Theorem 7 and:

iii) For each symbol j with $1 \le j < n$, the positions where symbol j occurs in s are a cyclic shift of the positions where symbol 0 occurs in s.

Conversely, if s is any sequence satisfying properties i),ii) and iii) above, then there exists a choice for the first codeword W such that the resulting cyclic path is an (n,p,k)-STCC.

The proofs of these two results follow closely the proofs of Theorem 2 and Lemma 3 of [10] and are omitted.

III. Necessary Conditions and Bounds on STCCs

We can now derive some necessary conditions on the period of (n,p,k)-STCCs. A first condition follows form Theorem 7 and Lemma 8, by using a simple counting argument as in the proof of [10, Lemma 4]:

Lemma 9: Suppose there exists an (n,p,k)-STCC. Then p is an even multiple of n and $2n \le p \le 2^n$.

Furthermore, if $P(n,k)$ denotes the maximum possible period p of an (n,p,k)-CC then, $P(n,k)$ is certainly also an upper-bound for the maximum period of an (n,p,k)-STCC. We therefore now give a brief summary of what is known about $P(n,k)$. This information will be helpful in proving the optimality of some of the small codes constructed in the next section, as well as for judging the performance of our later general construction. Mostly this information is in the form of upper bounds, though the exact value of $P(n,k)$ is known for a variety of small parameters.

We mentioned already that spread 1 circuit codes are cyclic Gray codes, so $P(n,1) = 2^n$ for every n [10, Lemma 11].

For spread 2 codes, the best upper bounds are to be found in a series of papers [5], [8], [15], [16], [19], [21] with the best bound for large n being [22]:

$$P(n,2) \le 2^{n-1}\left(1 - \frac{1}{89n^{1/2}} + O(\frac{1}{n})\right).$$

The exact value of $P(n,2)$ is known only for $n \le 7$ (the values are 4,6,8,14 and 26 for $n =$ 2,3,4,5 and 6 respectively [3] and 48 for $n = 7$ [14]) while the best general construction methods known at the moment [1] show that:

$$P(n,2) \ > \ \frac{77}{256}\cdot 2^n \quad \text{for all } n.$$

A table of the highest known periods for spread 2 circuit codes of length $n \le 20$ is also given in [1].

For spreads $k \ge 3$, the known results are much less comprehensive. For a number of small values of n and k, $P(n,k)$ is known exactly [12], [4], [17]. The best possible codes are also known when k is large compared to n: it is shown in [6], [20] that

$$\begin{aligned}
P(n,k) &= 2n \quad \text{for } n < \lceil \tfrac{3k}{2} \rceil + 2;\\
P(\lceil \tfrac{3k}{2} \rceil + 2, k) &= 4k+4 \quad \text{for } k \text{ odd};\\
P(\lceil \tfrac{3k}{2} \rceil + 2, k) &= 4k+6 \quad \text{for } k \text{ even};\\
P(\lceil \tfrac{3k}{2} \rceil + 3, k) &= 4k+8 \quad \text{for } k \ge 9 \text{ odd}.
\end{aligned}$$

The following upper bounds on $P(n,k)$ for general n and k can be found in [2]:

$$P(n,2t+1) \le \frac{2^n}{\binom{n}{t} - 2\binom{n-1}{t-1}} \quad \text{for } n > 2t+1 \tag{2}$$

and

$$P(n,2t+2) \le \frac{2^n - 2Q(n)}{\binom{n}{t} - 2\binom{n-1}{t-1}} \quad \text{for } n > 2t+2 \tag{3}$$

where $Q(n)$ is a polynomial in n of degree $t+1$ with $1/(t+1)!$ as leading coefficient. This latter bound was improved, roughly by a factor of 2, in [7]. Lower bounds on $P(n,k)$ can be obtained from the constructions in [13], [20], the actual bounds being rather complicated to state. What is more important are the tables of code parameters given in [13], [4] and the updated table in [17].

IV. High Period STCCs for Small Parameters

We consider the construction of (n,p,k)-STCCs with lengths n up to 17 and spreads k up to 6. For each pair (n,k), we concentrate on finding a code with period as high as possible. However, the method described in this section can be adapted to produce single-track codes of period less than this highest period, as long as the period satisfies the conditions of Lemma 9. This is an important point for the practical use of single-track circuit codes, since it is usually desired to use a code of a specified resolution in a particular application. The results of this section are summarised by the data presented in Table 1.

A spread 1 circuit code is of course a Gray code. For $n \leq 16$, many good (and several optimal) single-track Gray codes were reported in [9, Table I]. Using exactly the same construction method, we have also obtained an optimal length 17, period 131070 single-track Gray code.

We now concentrate on circuit codes for spreads $k \geq 2$. For every n, there is a trivial $(n, 2n, n)$-STCC in which the code has first codeword $[0, 0, \ldots, 0]$ and coordinate sequence

$$0, 1, 2, \ldots, n-1, 0, 1, \ldots, n-1.$$

From the upper bounds presented in Section III and the conditions of Lemma 9, these codes are in fact optimal $(n, 2n, k)$-CCs for every k large enough to satisfy $n < \lceil \frac{3k}{2} \rceil + 2$. Using the tables of optimal circuit codes in [13], [4], it is possible to show that these codes are optimal for some other values of n and k too.

When k is small relative to n, it is possible to construct codes with significantly higher periods than those given by the trivial codes. In [17], a construction method for single-track circuit codes generalising the approach taken in [9] was given. This construction yielded 19 single-track circuit codes (with spreads $k \geq 3$) that are *superior* to the best previously known circuit codes. This is a reflection of the weakness of existing constructions for circuit codes rather than an inherent superiority of STCCs. We used the same method to construct spread 2 single-track circuit codes for $6 \leq n \leq 17$. While not surpassing the best known spread 2 codes, the single-track codes that we found are competitive with them. The periods that we obtained can be found in Table 1 below.

To illustrate our contention that the methods discussed above can be modified to give codes with periods that are of practical importance, we give in Appendix A the coordinate sequences for a $(12, 360, 2)$-STCC and a $(15, 360, 4)$-STCC. These were found using the same computer programs as those used to generate the high-period codes reported above.

Length	Spread					
	1	2	3	4	5	6
2	4*	4*	–	–	–	–
3	6*	6*	6*	–	–	–
4	8*	8*	8*	8*	–	–
5	30*	10*	10*	10*	10*	–
6	60*	24*	12*	12*	12*	12*
7	126*	42*	14*	14*	14*	14*
8	240	80	16	16*	16*	16*
9	504*	162	54	18	18*	18*
10	960	320	80	20	20*	20*
11	2046*	594	154	22	22	22*
12	3960	960	288	96	24	24
13	8190*	1898	442	182	26	26
14	16128	3528	700	280	28	28
15	32730	6630	1280	450	210	30
16	65504	12512	2176	672	288	32
17	131070*	22406	3842	1088	476	204

TABLE I

Number of codewords in best known length n, spread k single-track circuit codes. Asterisk denotes a single-track circuit code known to be optimal.

V. Constructing STCCs from Base Coordinate Sequences

In this section we give a general method for constructing coordinate sequences of spread k STCCs. We begin by introducing a little more notation.

Definition 10: Let $b = b_0, b_1, \ldots, b_{t-1}$ be a sequence with t terms from $0, 1, \ldots, n-1$. Then the occurrence vector for b,

$$e(b) = [e_0, e_1, \ldots, e_{n-1}],$$

is the vector with e_j equal to the number of occurrences of symbol j in b. Notice that $\sum_{j=0}^{n-1} e_j = t$.

Definition 11: Let $b = b_0, b_1, \ldots, b_{t-1}$ be a sequence with terms from $0, 1, \ldots, n-1$ and with occurrence vector $e(b)$. Then b is called a spread k base coordinate sequence if b has the following properties:
i) Every subsequence $b_i, \ldots, b_{i+r-1}$ of b, with $r \geq k$ and $0 \leq i < i+r-1 < t$, contains at least k symbols with an odd number of occurrences each.
ii) For some integer δ, $0 \leq \delta < k$, the δ symbols $v_0, v_1, \ldots, v_{\delta-1}$ at the beginning of b and the $k-\delta$ symbols $v_\delta, v_{\delta+1}, \ldots, v_{k-1}$ at the end of b are all different, and $e(b)$ satisfies:

$$\begin{aligned} &e_{v_i} = 1, \quad e_{v_i-1} = \cdots = e_{v_i-k} = 0, \qquad 0 \leq i < \delta, \\ &e_{v_i} = 1, \quad e_{v_i+1} = \cdots = e_{v_i+k} = 0, \qquad \delta \leq i < k. \end{aligned}$$

(subscripts modulo n).
iii) $\sum_{j=0}^{n-1} e_j = t$ is even.
iv) For every ϵ with $k+1 \leq \epsilon \leq n-(k+1)$, there exist k distinct integers $l_0, l_1, \ldots, l_{k-1}$ with $0 \leq l_i < t$ such that $e_{l_i} = e_{l_i+\epsilon} = 0$ and $e_{l_i+1} + e_{l_i+2} + \cdots + e_{l_i+\epsilon-1}$ is odd (subscripts being reduced modulo n).
Note that property i) in Definition 11 implies that every interval $b_i, \ldots, b_{i+r-1}$ of b with $r \leq k$ must contain r distinct symbols.

Construction 12: Suppose b is a sequence with t terms from $0, 1, \ldots, n-1$. For $0 \leq j \leq n-1$, let the sequence $b(j)$ be defined by

$$b(j) = b_0 - j, b_1 - j, \ldots, b_{t-1} - j$$

where terms are taken modulo n. We construct the sequence s of length nt as

$$s = b, b(1), \ldots, b(n-1),$$

i.e. by concatenating the sequences $b(j)$ for $0 \leq j < n$. We refer to $b(j)$ as block j of sequence s.

Theorem 13: Let b be a spread k base coordinate sequence with t terms from $0, 1, \ldots, n-1$, and let s be obtained from b according to Construction 12. Then s is the coordinate sequence of an (n, nt, k)-STCC.

Proof: The proof is very similar to that of [10, Theorem 8] and once again the following observation is crucial to each of the steps of our proof: from the definition of $b(j)$, for every i, j and σ, symbol i occurs in the same positions in block j as symbol $i-\sigma$ does in block $j+\sigma$ (here and from now on we work modulo n with symbols and block numbers). It follows from this that symbol $i-j$ occurs e_i times in block j.

We first show that s has the cyclic shift property of Lemma 8. Choosing $\sigma = i$, the above observation shows that symbol i occurs in the same positions in block j as symbol 0 does in block $j+i$. That the positions where symbol i occurs in s are just a shift of the positions where symbol 0 occurs is then obvious.

Thus we are left to check that s has properties i) and ii) in Theorem 7. Again by our observation above (with $\sigma = -j$), symbol i plays the same role in block j as symbol $i+j$ does in block 0. Symbol i therefore occurs e_{i+j} times in block j and $\sum_{j=0}^{n-1} e_{i+j} = t$ times in s. So, by property iii) in Definition 11, s has property i) in Theorem 7. For property ii), it is enough to show that, for every $k \leq r \leq nt-k$, all the length r subsequences $s_i, s_{i+1}, \ldots, s_{i+r-1}$ of s contain at least k distinct symbols with an odd number of occurrences each. We split the argument into a number of cases, depending on the number of boundaries between blocks that our subsequence $s_i, s_{i+1}, \ldots, s_{i+r-1}$ covers. In what follows, we refer to the δ symbols that begin any block and the $k-\delta$ symbols that end any block as being *special* terms. A key point in our proof is that if v is a special term at the end of a block j, then v does not occur at all in any of the k following blocks $j+1, j+2, \ldots, j+k$. Likewise, if v is a special term at the beginning of block j, then v does not occur at all in any of the k preceding blocks $j-1, j-2, \ldots, j-k$. These facts follow from our crucial observation and property ii) in Definition 11.

Suppose first that the subsequence is contained entirely within some block j. Because block j is obtained by subtracting j (modulo n) from each term of the spread k base coordinate sequence b, it follows trivially from property i) in Definition 11, that every subsequence of block j contains at least k symbols with an odd number of occurrences each.

Suppose now that the subsequence covers just one boundary between two blocks, say block j and block $j+1$. From the above observation and the fact that b has property ii) in Definition 11, it is easily seen that the $k-\delta$ symbols $v_\delta - j, \ldots, v_{k-1} - j$ occur as the last terms of block j and the δ symbols $v_0 - (j+1), \ldots, v_{\delta-1} - (j+1)$ occur as the first terms of block $j+1$, but that none of these k symbols occur anywhere else in the blocks j and $j+1$. If the subsequence includes all of these k special terms, it clearly has the required property that it contains k symbols with an odd number of occurrences each. Suppose then that the subsequence does not contain all the special terms in block j. Then it certainly contains no symbols from block j except some special terms. In turn, the special terms of block j that the subsequence does contain do not appear in block $j+1$. Using these symbols together with the fact that property i) in Definition 11 also holds for the sequence $b(j+1)$, it follows that the subsequence does contain at least k symbols with an odd number of occurrences each. A similar argument applies in the case where the subsequence does not contain all the special terms in block $j+1$.

Suppose now that the subsequence covers exactly ϵ boundaries between blocks, where $\epsilon \leq k$. We apply a similar argument to that used in the previous paragraph. The subsequence includes terms from just $\epsilon+1$ consecutive blocks, say blocks $j, j+1, \ldots, j+\epsilon$. If the subsequence contains the last $k-\delta$ terms of block j and the first δ terms of block $j+\epsilon$, then it follows from the key point about special terms that the subsequence contains at least k distinct symbols ($v_\delta - j, \ldots, v_{k-1} - j$ and $v_0 - (j+\epsilon), \ldots, v_{\delta-1} - (j+\epsilon)$) with an odd number of occurrences each. Suppose then that the subsequence does not contain all of these k special terms. Suppose it does not contain all the special terms at the end of block j and consider the $k-\delta$ special terms at the end of block $j+1$. These are all contained within the subsequence, but it is easy to see (using the key point about special terms) that they occur just once each in the subsequence. Likewise, if the subsequence does not contain all of the δ special terms at the beginning of block $j+\epsilon$, then the δ special terms at the beginning of block $j+\epsilon-1$ occur just once each in the subsequence. So by considering special terms at the end of blocks j and $j+1$ and at the beginning of blocks $j+\epsilon$ and $j+\epsilon-1$, we can find k distinct symbols that occur exactly once each in the subsequence.

Suppose now that the subsequence covers exactly ϵ boundaries between blocks, where $k+1 \leq \epsilon \leq n-(k+1)$. Assume that the first boundary covered is between block j and block $j+1$, so that the last one is between block $j+\epsilon-1$ and block $j+\epsilon$. From property iv) in Definition 11, there exist k intervals $e_{l_i}, e_{l_i+1}, \ldots, e_{l_i+\epsilon}$ in the occurrence vector for b that both begin and end with zeroes and have odd sum. Again using our crucial observation, for each i, symbol $l_i - j$ occurs e_{l_i} times in block j, e_{l_i+1} times in block $j+1$ and so on. Since $e_{l_i} = e_{l_i+\epsilon} = 0$ and the subsequence contains every term of the blocks $j+1, \ldots, j+\epsilon-1$, we see that the subsequence contains symbol $l_i - j$ an odd number of times. So it contains at least k symbols with an odd number of occurrences each.

Finally suppose that the subsequence $s_i, s_{i+1}, \ldots, s_{i+r-1}$ covers at least $n-k$ boundaries between blocks. The complementary subsequence $s_{i+r}, s_{i+r+1}, \ldots, s_{i-1}$ is then of length at least k and covers at most k boundaries between blocks. Since we have already established that such a subsequence contains at least k symbols with an odd number of occurrences each and since every symbol occurs t (an even number) of times in all of s, we conclude that $s_i, s_{i+1}, \ldots, s_{i+r-1}$ must also contain at least k symbols with an odd number of occurrences each. □

VI. A Construction for Base Coordinate Sequences

We begin by showing how to construct occurrence vectors satisfying some of the properties of Definition 11. This construction uses a generalisation of the idea behind [10, Construction 9].

Construction 14: Let $k \geq 1$ be fixed. Suppose $m \geq k$, $r \geq 1$ and n satisfies

$$mr + k + 2 \ \leq \ n \ \leq \ 2mr + k + 2$$

We choose $r-1$ integer vectors $f_1, f_2, \ldots, f_{r-1}$ of length $m-k$, each vector having components whose sum is even (when $m = k$, each vector is empty and has sum zero). We also choose one integer vector f_r of length $n-(mr+k+1)$ having components whose sum is odd. Finally, we construct the vector $e = [e_0, e_1, \ldots, e_{n-1}]$ of length n as

$$e = [1, 0, \ldots, 0, f_1, 0, \ldots, 0, f_2, \ldots, 0, \ldots, 0, f_{r-1}, 0, \ldots, 0, f_r, 0, \ldots, 0]$$

where k zeros precede f_j, $1 \leq j \leq r$, and m zeros follow f_r. Notice that the vector e contains at least $m + kr$ zeros.

Lemma 15: The vectors e of Construction 14 have properties iii) and iv) in Definition 11.

Proof: Suppose vector e is obtained according to Construction 14. That e has even sum is clear from the even-sum/odd-sum property of the vectors f_j. So property iii) of Definition 11 holds. We further claim that, for every ϵ with $k+1 \leq \epsilon \leq n-(k+1)$, there exist k intervals $e_{l_i}, e_{l_i+1}, \ldots, e_{l_i+\epsilon}$ of $e(b)$, $0 \leq i \leq k-1$, that satisfy $e_{l_i} = e_{l_i+\epsilon} = 0$ and $e_{l_i+1} + e_{l_i+2} + \cdots + e_{l_i+\epsilon-1}$ odd.

Consider first the intervals in which e_{l_i} is one of the m zeros following f_r and $e_{l_i+\epsilon}$ is one of the kr zeros that precede the f_j's. By inspection it can be seen that these intervals account for k valid intervals for each ϵ with $k+1 \leq \epsilon \leq mr+1$, and for $k-i$ valid intervals for each $\epsilon = mr+1+i$ with $1 \leq i \leq k-1$. Interchanging the roles of starting and ending zeros in the above argument, it is also easy to see that e has k valid intervals for each ϵ with $n-(mr+1) \leq \epsilon \leq n-(k+1)$, and $k-i$ valid intervals for each $\epsilon = n-(mr+1+i)$ with $1 \leq i \leq k-1$.

So we certainly have k intervals of the required type for every ϵ, except possibly for ϵ with $mr+2 \leq \epsilon \leq n-(mr+2)$. For these cases, we write $\epsilon = mr+1+i$ with $1 \leq i \leq k-1$ (as $n-(mr+2) \leq mr+k$) and obtain $k-i$ valid intervals from the first set of intervals and at least i valid intervals from the second set of intervals. It follows that e does contain k valid intervals for every ϵ with $k+1 \leq \epsilon \leq n-(k+1)$ and, therefore, that e has property iv) in Definition 11. $\square$

The above construction can be used in a variety of different ways as an ingredient to produce spread k single-track circuit codes from spread k circuit codes. Next we will describe in full detail the most straightforward of these methods and give a detailed example. In the next section, we will go on to discuss some refinements of our method, illustrating with examples.

Theorem 16: Let $k \geq 1$ be fixed and suppose that an (s,t,k)-CC with $s \geq k$ exists. Then there exist (n, nt, k)-STCCs of length $n = m + k(r+1) + s$ for every choice of $r \geq \max(1, \lfloor \frac{k}{2} \rfloor)$ and $m \geq k + \min(2, r)$ satisfying:

$$mr + 2k + 3 \ \leq \ n = m + k(r+1) + s \ \leq \ 2mr + k + 2.$$

Proof: In view of Theorem 13, we only need to construct a spread k base coordinate sequence with t terms from $0, 1, \ldots, n-1$. From the specification of parameters in the theorem above, it follows that n satisfies $mr + 2k + 3 \leq n \leq 2mr + k + 2$ and therefore that m, r and n fulfil the conditions of Construction 14. So we take e to be a vector of length n, obtained according to Construction 14 and thus satisfying properties iii) and iv) in Definition 11 (by Lemma 15).

Because $n-(mr+k+1) \geq k+2$, the vector f_r in e has length at least $k+2$. Also, since $m \geq k+2$ for $r \geq 2$, each vector f_i, $1 \leq i \leq r-1$, has length at least 2. For each j with $1 \leq j \leq r$, we denote by p_j the position of the initial component of f_j in e and by q_j the position of the final component of f_j in e. We now distinguish two cases depending on the parity of k. We consider in detail the case where k is odd and give a sketch for the case where k is even.

Consider k odd. We recall that $e_0 = 1$ and specify that, for $k > 1$, the vector e has additional 1's in positions p_r and q_r and in positions $p_1, p_2, \ldots, p_g, q_1, q_2, \ldots, q_g$ where $g = \lfloor \frac{k}{2} \rfloor - 1$: this is possible because $r \geq \max(\lfloor \frac{k}{2} \rfloor, 1)$ guarantees that $g \leq r-1$ and because from the previous paragraph the f_j all have length at least 2. Thus, a total of k 1's and $m + kr$ 0's are assigned to e, with each of the 1's either followed or preceded by k zeros. This leaves $n - m - kr - k = s$ as yet unspecified entries in e of which at least k lie in the vector f_r. We label all these unspecified entries by $e_{v_0}, e_{v_1}, \ldots, e_{v_{s-1}}$, where $v_0 < v_1 < \cdots < v_{s-1}$.

Now let a be the coordinate sequence of an (s,t,k)-CC with $s \geq k$ (and consequently $t \geq 2k$). We assume that a is on symbols $0,1,\ldots,s-1$. Every symbol occurs an even number of times in a, and the last k terms of a are all distinct (otherwise a fails to satisfy condition ii) of Lemma 7). By permuting the symbols of a if necessary, we can arrange that the last k terms of a are

$$a_{t-k} = s-k,\ a_{t-k+1} = s-k+1,\ \ldots,\ a_{t-1} = s-1.$$

We then derive a new sequence d from a by deleting these last k terms and replacing every occurrence of symbol j in a by symbol v_j. From the fact that a is the coordinate sequence of a circuit code, we can deduce that the occurrence vector for d has odd values in positions $v_{s-k}, v_{s-k+1}, \ldots, v_{s-1}$ and even values in all the positions $v_0, v_1, \ldots, v_{s-k-1}$. We now modify d by appending symbol 0 and, if $k > 1$, by prepending symbols $p_1, \ldots, p_g$ and p_r and appending symbols $q_1, \ldots, q_g$, and q_r. The result is our final sequence b with t terms.

It is not hard to see from the modifications to a that the occurrence vector for b equals a vector e from Construction 14 with some valid choice for the vectors $f_1, f_2, \ldots, f_r$: in particular, f_r has odd sum because f_r contains an odd number of odd entries (in positions $v_{s-k}, v_{s-k+1}, \ldots, v_{s-1}$) and, if $k > 1$, a 1 in positions p_r and q_r, while each vector f_i, $1 \leq i \leq r-1$, either contains exactly two odd entries (two 1's) or no odd entry at all and so has even sum. So b and $e(b)$ have properties iii) and iv) in Definition 11. That the sequence b has property ii) is also clear. Finally, property i) holds for b because of the way in which b was derived from a (itself the coordinate sequence of a spread k circuit code), namely, by deleting the last k symbols from a and appending and prepending k symbols that appear nowhere else in b. Thus b is a spread k base coordinate sequence with t terms.

When k is even, a very similar procedure applies. Here we specify that in addition to $e_0 = 1$ the vector e always has a 1 in position p_r (but not necessarily in position q_r) and in positions $p_1, p_2, \ldots, p_g, q_1, q_2, \ldots, q_g$. We label the unspecified entries in e again by $e_{v_0}, e_{v_1}, \ldots, e_{v_{s-1}}$ and perform the same operation of deleting the last k terms of the coordinate sequence a of an (s,t,k)-CC (with possibly permuted symbols) and mapping the terms of the resulting sequence into the symbols $v_0, v_1, \ldots, v_{s-1}$ to obtain a new sequence d. In this case, vector f_r always contains a 1 (in position p_r) and an even number of additional odd entries (in positions $v_{s-k}, v_{s-k+1}, \ldots, v_{s-1}$), so that its sum is still odd. Finally, we modify d by prepending the symbols $p_1, \ldots, p_g$ and p_r and appending the symbols $0, q_1, \ldots, q_g$ to d to obtain the sequence b, a spread k base coordinate sequence with t terms. □

Example 1: We know that a trivial $(3,6,2)$-CC with coordinate sequence $a = 0,1,2,0,1,2$ exists. The parameter set $s=3$, $t=6$, $k=2$, $n=10$, $r=1$ and $m=3$ satisfies the conditions of Theorem 16. We follow through the details of the proof of this theorem, first using Construction 14 to get

$$e = [1,0,0,f_1,0,0,0],$$

where f_1 has length $n-(mr+k+1) = 4$ and odd sum. The proof in the even case tells us that e should have a 1 in position $p_1 = 3$ and that symbols e_4, e_5 and e_6 are as yet unspecified. Then for d, we obtain the sequence $4,5,6,4$ and finally for b, the spread 2 base coordinate sequence $3,4,5,6,4,0$. Notice that the corresponding occurrence vector equals

$$e(b) = [1,0,0,1,2,1,1,0,0,0].$$

So there exists a $(10,60,2)$-STCC whose coordinate sequence is

$$\begin{gathered}3,4,5,6,4,0,2,3,4,5,3,9,\\ 1,2,3,4,2,8,0,1,2,3,1,7,\\ 9,0,1,2,0,6,8,9,0,1,9,5,\\ 7,8,9,0,8,4,6,7,8,9,7,3,\\ 5,6,7,8,6,2,4,5,6,7,5,1.\end{gathered}$$

VII. Refinements of the Construction Method for STCCs

The proof of Theorem 16 contains the kernel of a general technique for obtaining spread k single-track circuit codes: use a vector e from Construction 14 as a template to control the way that symbols from a short length s, period t circuit code a are embedded in the symbols of a larger length n, period nt single-track circuit code. However, if we want to achieve a particular final period nt, then it may be quite difficult to find a suitable coordinate sequence a. The key variant of the technique in the above proof that we introduce now is the use of *truncated* coordinate sequences of circuit codes.

Suppose $a = a_0, \dots, a_{l-1}$ is the coordinate sequence of an (s, l, k)-CC (necessarily, l is even and every symbol $0, 1, \dots s-1$ occurs an even number of times in a). Now suppose $t \le l$ is even and consider the truncated coordinate sequence

$$a' = a_0, a_1, \dots, a_{t-1-k},$$

with $t-k$ terms. From condition ii) in Theorem 7, it follows that the number of symbols occurring an odd number of times in a' is at least k and has the same parity as k.

We choose parameters $r \ge \max(\lfloor k/2 \rfloor, 1)$ and $m \ge k + \min(2, r)$ so that

$$mr + k + 2 \le n = m + k(r+1) + s \le 2mr + k + 2.$$

We then take a vector e of length n from Construction 14. As in the proof of Theorem 16, our aim is to map the symbols of a' into the unspecified positions in e so that all the vectors f_i, $1 \le i \le r-1$, have even sum and so that f_r has odd sum. Recall that we denote by p_j and q_j the first and last positions of the vectors f_j, $1 \le j \le r$, in e and write $g = \lfloor \frac{k}{2} \rfloor - 1$.

Suppose k is even. We set $e_{p_r} = 1$ and $e_{p_1} = \cdots = e_{p_g} = e_{q_1} = \cdots = e_{q_g} = 1$ and let $2h$ denote the number of symbols occurring an odd number of times in a'. We arbitrarily place these $2h$ symbols into $h \ge \frac{k}{2}$ pairs, which we call *even-occurrence pairs.* We can achieve our aim by ensuring that the two symbols of each of these h pairs are always mapped together into a single vector f_j. By simple counting of positions in e, it is not hard to show that we can do this so long as the number h of even-occurrence pairs satisfies

$$h \;\le\; (r-1)\lfloor \tfrac{|f_i|}{2} \rfloor + \lfloor \tfrac{|f_r|-1}{2} \rfloor - (\tfrac{k}{2} - 1),$$

where $|f_i|$ and $|f_r|$ denote the lengths of the vectors f_i and f_r, respectively. In particular if $|f_i| = m - k$ is even, then there is always enough space in e to assign the h even-occurrence pairs.

Similarly, when k is odd, we set $e_{p_r} = e_{q_r} = 1$ and $e_{p_1} = \cdots = e_{p_g} = e_{q_1} = \cdots = e_{q_g} = 1$ and let $2h+1$ denote the number of symbols occurring an odd number of times in a'. Then it is not hard to see we can ensure that, for each of the h even-occurrence pairs, the two symbols are mapped together into a single vector f_j (and that the last symbol occurring an odd number of times is mapped into f_r), provided that $m-k$ is even (i.e. that m and k have the same parity) and that $|f_r| \ge 3$.

As in the proof of Theorem 16, we let d denote the new sequence obtained after mapping the symbols of a into the free positions in e. We prepend the symbols $p_1, \dots, p_g$ and p_r and append the symbols $0, q_1, \dots, q_g$ (and q_r when k is odd) to d to obtain a sequence b which, as can be verified by essentially the same steps as in the proof of Theorem 16, is a spread k base coordinate sequence. The code resulting after an application of Construction 12 is an (n, nt, k)-STCC.

We illustrate the procedure described above by the following examples.

Example 2: We aim to construct a $(20, 1000, 2)$-STCC. From Section III, there is no $(10, 1000, 2)$-STCC and so, using Lemma 9, the smallest possible length n for which a period $nt = 1000$, spread $k = 2$ single-track circuit code can exist is $n = 20$ (implying $t = 50$). In this sense, this code is optimal.

We take $n = 20$, $t = 50$, $m = 4$, $r = 2$ and $s = 10$. These parameters certainly satisfy $mr + 4 \le n \le 2mr + 4$ and $n = m + 2(r+1) + s$ Our vector e from Construction 14 then has the form

$$[1, 0, 0, f_1, 0, 0, f_2, 0, 0, 0, 0]$$

where f_1 has length $m - k = 2$ and f_2 has length $n - (mr + k + 1) = 9$ and begins with a 1 at position $p_2 = 7$ (since $k = 2$ is even). In the notation of the proof of Theorem 16, we have

$$l_0 = 3, l_1 = 4, l_2 = 8, l_3 = 9, l_4 = 10, \dots, l_9 = 15.$$

From Example 1 we know that there exists a $(10, 60, 2)$-STCC with period $l = 60$ (larger than $t = 50$) which of course represents a valid choice for a $(10, 60, 2)$-CC. The first $t - k = 48$ terms of the coordinate sequence of this code are:

$$\begin{aligned} a' = \quad & 3,4,5,6,4,0,2,3,4,5,3,9, \\ & 1,2,3,4,2,8,0,1,2,3,1,7, \\ & 9,0,1,2,0,6,8,9,0,1,9,5, \\ & 7,8,9,0,8,4,6,7,8,9,7,3, \end{aligned}$$

with occurrence vector

$$e(a') = [6,5,5,6,5,3,3,4,5,6].$$

The 6 symbols in a' with an odd number of occurrences can be placed in 3 even-occurrence pairs: the pairs of symbols we take are:

$$\{1,2\}, \{4,5\}, \{6,8\}.$$

Now we have to map these even-occurrence pairs and the remaining symbols onto the symbols l_i so that each even-occurrence pair is mapped onto a pair of symbols both lying in a single f_j. The mapping we choose is as follows

$$\begin{aligned} 1,2 &\rightarrow 8,9, \\ 4,5 &\rightarrow 10,11, \\ 6,8 &\rightarrow 12,13, \\ 0,3,7,9 &\rightarrow 3,4,14,15. \end{aligned}$$

Applying this mapping to the symbols of a' gives us the sequence d:

$$\begin{aligned} d = \quad & 4,10,11,12,10,3,9,4,10,11,4,15 \\ & 8,9,4,10,9,13,3,8,9,4,8,14, \\ & 15,3,8,9,3,12,13,15,3,8,15,11, \\ & 14,13,15,3,13,10,12,14,13,15,14,4. \end{aligned}$$

Finally, to obtain our spread 2 base coordinate sequence b with $t = 50$ terms, we prepend $p_2 = 7$ and append 0 to d:

$$\begin{aligned} b = \quad & 7,4,10,11,12,10,3,9,4,10,11,4, \\ & 15,8,9,4,10,9,13,3,8,9,4,8,14, \\ & 15,3,8,9,3,12,13,15,3,8,15,11,14, \\ & 13,15,3,13,10,12,14,13,15,14,4,0. \end{aligned}$$

Notice that the occurrence vector of b is

$$e(b) = [1,0,0,6,6,0,0,1,5,5,5,3,3,5,4,6,0,0,0,0]$$

in accordance with the properties of e required in Construction 14. Thus, using b in Construction 12 results in a $(20, 1000, 2)$-STCC.

Example 3: We aim to construct a spread $k = 3$ single-track circuit code with $nt = 360$ codewords and with length n as small as possible. The best we can do with the method in this section is to construct an $(18, 360, 3)$-STCC. Recall that Appendix A gives the coordinate sequence of a $(15, 360, 4)$-STCC which is also a $(15, 360, 3)$-STCC, so that the code in this example is not optimal. Nevertheless, the example illustrates how high spread single-track circuit codes with reasonable lengths can be constructed.

Thus, we want to choose parameters s, n, r and m (with the same parity as k) satisfying $mr + k + 2 \leq n = m + k(r+1) + s \leq 2mr + k + 2$ with the properties that $t = 360/n$ is an even integer as large as possible and that an (s, l, k)-CC with $l \geq t$ exists. Clearly, the aim of maximising t (or equivalently, of minimising the length n) conflicts with that of finding an (s, l, k)-CC: as n decreases, so does the maximum value of s over all choices of m and r, while $l \geq t$ increases.

We try each divisor of 360 for n in turn. The smallest for which our method is successful is $n = 18$ (yielding $t = 20$) and the choice $m = 5$, $r = 1$ maximises the value of s at $s = 7$. There exists a $(7, 24, 3)$-CC with coordinate sequence

$$a = 6, 5, 4, 3, 6, 2, 1, 5, 6, 3, 0, 2, 6, 5, 4, 3, 6, 2, 1, 5, 6, 3, 0, 2,$$

so we can take $l = 24$ and have $l \geq t$. We then take the first $t - k = 17$ terms of this sequence to obtain:

$$a' = 6, 5, 4, 3, 6, 2, 1, 5, 6, 3, 0, 2, 6, 5, 4, 3, 6$$

with occurrence vector $e(a') = [1, 1, 2, 3, 2, 3, 5]$.

According to Construction 14, our final sequence b should have occurrence vector $[1, 0, 0, 0, f_1, 0, 0, 0, 0, 0]$, in which f_1 is a vector of length $n - (mr + k + 1) = 9$ beginning with a 1 in position $p_1 = 4$ and ending with a 1 in position $q_1 = 12$ (as $k = 3$ is odd). Positions $5, 6, 7, 8, 9, 10$ and 11 remain unspecified in $e(b)$. We need to map the symbols of a' into these symbols so that the vectors f_j of $e(b)$ have the appropriate parity. In this example, we have just one vector f_1, and we can use the mapping

$$0, 1, 2, 3, 4, 5, 6 \quad \rightarrow \quad 5, 6, 7, 8, 9, 10, 11.$$

Applying this mapping to a' gives us the sequence d:

$$d = 11, 10, 9, 8, 11, 7, 6, 10, 11, 8, 5, 7, 11, 10, 9, 8, 11.$$

Finally, to obtain our spread 3 base coordinate sequence with $t = 20$ terms, we prepend 4 and append 0 and 12 to d:

$$b = 4, 11, 10, 9, 8, 11, 7, 6, 10, 11, 8, 5, 7, 11, 10, 9, 8, 11, 0, 12.$$

VIII. A Family of Even Spread STCCs

Example 2 shows how a family of single-track circuit codes with a flexible range of parameters can be obtained: by truncating the best known length 10, spread 2 circuit code (which has 340 codewords, [17]), we obtain $(20, 20t, 2)$-STCCs for every even t with $2 \leq t \leq 340$, the maximum period nt here being 6800. We now prove Theorem 1, a general result of this type for even spread codes. A similar result can also be obtained for odd spread codes.

Proof: (of Theorem 1) Let $k \geq 2$ be even and let $m \geq \max(4, \frac{k^2}{2})$ be the unique multiple of k (same parity as k) such that

$$\tfrac{2}{k}m(m-k) + k + 2 \ < \ n \ \leq \ \tfrac{2}{k}m(m+k) + k + 2.$$

Note that this implies $n > \max(12, \frac{k^3}{2} - k^2 + k + 2)$.

We start by considering the case where n satisfies

$$\tfrac{2}{k}m(m-k) + k + 2 \ < \ n \ < \ \tfrac{2}{k}m^2 + k + 2. \tag{4}$$

In this case it is easy to verify that $k\lfloor\sqrt{2(n-k-2)/k}\rfloor = 2m - k$. Thus, in view of Theorem 13 we need only construct spread k base coordinate sequences with t terms for every even t satisfying $k \leq t \leq P(n-2m-k, k)$. By choosing $r = \frac{m}{k}$ in Construction 14, we obtain a vector e of length n, with $r \geq \lfloor\frac{k}{2}\rfloor$ and $m \geq k+2$, for every n in the range above. After placing the additional 1's, this vector contains $m + kr = 2m$ 0's and k 1's, and so has $n - 2m - k$ unspecified positions. The lower bound on m and the lower bound on n in (4) ensure that $n - 2m - k > 0$. Now we take a to be the coordinate sequence of a length $n - 2m - k$, spread k circuit code with $P(n - 2m - k, k)$ codewords. Let a' denote the length $t - k$ truncated version of a. We use a' in the procedure described in the section above to produce our base coordinate sequence b: because m has the same parity as k, we can always map the symbols of a' onto the unspecified symbols in e whilst ensuring that the sum of the components of each f_j has the correct parity. Finally, using b in Construction 12, we obtain a code with the required parameters.

The case where n satisfies

$$\tfrac{2}{k}m^2 + k + 2 \;\leq\; n \;\leq\; \tfrac{2}{k}m(m+k) + k + 2 \tag{5}$$

is dealt with analogously after noting that $k\lfloor\sqrt{2(n-k-2)/k}\rfloor = 2m$ for every n in this range. We simply choose $r = \frac{m+k}{k}$ and follow the same sequence of steps as in the first case above. This time, we need spread k base coordinate sequences with t terms for every even t satisfying $k \leq t \leq P(n-2m-2k, k)$. The vector e has $m + kr + k = 2m + 2k$ fixed positions and $n - 2m - 2k$ unspecified positions, so that the lower bound on m and the lower bound on n in (5) ensure that $n - 2m - 2k > 0$. □

The next corollary follows immediately from the result of [1] that $P(n, 2) > \frac{77}{256} \cdot 2^n$ for all n.

Corollary 17: Suppose $n \geq 13$. Then there exists an $(n, nt, 2)$-STCC for every even t with

$$2 \;\leq\; t \;\leq \frac{77}{2^{12}} \cdot 2^{n-2\lfloor\sqrt{n-4}\rfloor} + 2.$$

For certain lengths n the bound of the corollary can be improved slightly by using odd values of m, choosing $r = (m-1)/2$ or $r = (m+1)/2$ and using Theorem 16. We omit the details.

IX. Conclusions and Open Problems

We have pointed out the potential advantages of using single-track circuit codes in certain types of analogue-to-digital conversion application. We have shown that this advantage can be realised in practice, by exhibiting many optimal and good single-track codes for small parameters n and k. We have also significantly generalised the methods introduced in [10] to give flexible constructions for spread k single-track codes that allow us to achieve our aim of finding codes with a specified resolution for a reasonable length n. We have illustrated our methods with a number of example codes having 360 and 1000 codewords.

It is worth noting that all the (n, nt, k)-STCCs constructed in this paper have the property that, given any particular codeword, the n cyclic shifts of that codeword are distinct and appear at equally spaced intervals throughout the code: that this is so is a consequence of the use of base coordinate sequences and necklaces to construct codes. So all our codes can be regarded as being composed of a code on necklaces concatenated with appropriate cyclic shifts of those necklaces (c.f. the constructions of [9], [17], [18]). In fact our general constructions for base coordinate sequences can be regarded as being a way of transforming a 'standard' circuit code into a code on necklaces by embedding many zero coordinates into the codewords. Since necklace methods have also been very successful in constructing good circuit codes for small parameters [17] and optimal and near-optimal single-track Gray codes [9], [18], it might be expected that these methods could also be harnessed to construct general families of good single-track circuit codes. A useful starting point would be to attempt to adapt the recursive constructions of [9], [18] for single-track Gray codes to produce STCCs.

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Appendix A

We give the first codewords and coordinate sequences for a $(12, 360, 2)$-STCC and a $(15, 360, 4)$-STCC. These codes were found using a generalisation of [9, Theorem 4].

$(12, 360, 2)$**-STCC:**

$$W_0 = [000000110011]$$

$$\begin{aligned} s = \; & 6,7,8,11,5,8,2,3,4,5,1,0,10,3,11,2,10,0,11,8,3,1,6,3,11,5,9,11,4,8,\\ & 5,6,7,10,4,7,1,2,3,4,0,11,9,2,10,1,9,11,10,7,2,0,5,2,10,4,8,10,3,7,\\ & 4,5,6,9,3,6,0,1,2,3,11,10,8,1,9,0,8,10,9,6,1,11,4,1,9,3,7,9,2,6,\\ & 3,4,5,8,2,5,11,0,1,2,10,9,7,0,8,11,7,9,8,5,0,10,3,0,8,2,6,8,1,5,\\ & 2,3,4,7,1,4,10,11,0,1,9,8,6,11,7,10,6,8,7,4,11,9,2,11,7,1,5,7,0,4,\\ & 1,2,3,6,0,3,9,10,11,0,8,7,5,10,6,9,5,7,6,3,10,8,1,10,6,0,4,6,11,3,\\ & 0,1,2,5,11,2,8,9,10,11,7,6,4,9,5,8,4,6,5,2,9,7,0,9,5,11,3,5,10,2,\\ & 11,0,1,4,10,1,7,8,9,10,6,5,3,8,4,7,3,5,4,1,8,6,11,8,4,10,2,4,9,1,\\ & 10,11,0,3,9,0,6,7,8,9,5,4,2,7,3,6,2,4,3,0,7,5,10,7,3,9,1,3,8,0,\\ & 9,10,11,2,8,11,5,6,7,8,4,3,1,6,2,5,1,3,2,11,6,4,9,6,2,8,0,2,7,11,\\ & 8,9,10,1,7,10,4,5,6,7,3,2,0,5,1,4,0,2,1,10,5,3,8,5,1,7,11,1,6,10,\\ & 7,8,9,0,6,9,3,4,5,6,2,1,11,4,0,3,11,1,0,9,4,2,7,4,0,6,10,0,5,9 \end{aligned}$$

$(15, 360, 4)$**-STCC:**

$$W_0 = [000011011100101]$$

$$\begin{aligned} s = \; & 14,12,8,10,9,11,3,12,1,7,4,9,6,14,8,1,5,13,4,12,7,14,9,10,\\ & 13,11,7,9,8,10,2,11,0,6,3,8,5,13,7,0,4,12,3,11,6,13,8,9,\\ & 12,10,6,8,7,9,1,10,14,5,2,7,4,12,6,14,3,11,2,10,5,12,7,8,\\ & 11,9,5,7,6,8,0,9,13,4,1,6,3,11,5,13,2,10,1,9,4,11,6,7,\\ & 10,8,4,6,5,7,14,8,12,3,0,5,2,10,4,12,1,9,0,8,3,10,5,6,\\ & 9,7,3,5,4,6,13,7,11,2,14,4,1,9,3,11,0,8,14,7,2,9,4,5,\\ & 8,6,2,4,3,5,12,6,10,1,13,3,0,8,2,10,14,7,13,6,1,8,3,4,\\ & 7,5,1,3,2,4,11,5,9,0,12,2,14,7,1,9,13,6,12,5,0,7,2,3,\\ & 6,4,0,2,1,3,10,4,8,14,11,1,13,6,0,8,12,5,11,4,14,6,1,2,\\ & 5,3,14,1,0,2,9,3,7,13,10,0,12,5,14,7,11,4,10,3,13,5,0,1,\\ & 4,2,13,0,14,1,8,2,6,12,9,14,11,4,13,6,10,3,9,2,12,4,14,0,\\ & 3,1,12,14,13,0,7,1,5,11,8,13,10,3,12,5,9,2,8,1,11,3,13,14,\\ & 2,0,11,13,12,14,6,0,4,10,7,12,9,2,11,4,8,1,7,0,10,2,12,13,\\ & 1,14,10,12,11,13,5,14,3,9,6,11,8,1,10,3,7,0,6,14,9,1,11,12,\\ & 0,13,9,11,10,12,4,13,2,8,5,10,7,0,9,2,6,14,5,13,8,0,10,11 \end{aligned}$$