The Sharpness of Circular Saws

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We draw triangles around the outside of a convex polygon to make a figure resembling a circular saw. We show that the product of the sines of the convex angles of the saw is less than the product of the sines of the concave angles of the saw. We consider the set of inequalities in polar co-ordinates that describe the betweenness constraints of a straight line drawing.

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1. Introduction

This work is part of a project that is looking at the relationship between sets of paths in planar graphs, and drawings of the graphs in which those paths are straight lines. Hence, we are interested in properties of straight line drawings.

One approach to the general problem of drawing a graph subject to co-linearity constraints is to first determine the relative angles between the lines; and then determine their relative position. This depends upon understanding the geometric constraints on angles, independent of distance and parallelism (which is not a graph theoretic property). Essentially we wish to explore the relationship between betweenness and angles.

Some constraints on angles are demonstrated by Euclid [4], in particular his propositions I-13, I-16 and I-32. These elementary observations result in linear inequalities and equalities relating the various angles in the figure, which are detailed in Carroll [1],[2].

We have found, however, one class of drawing which imposes a non-linear constraint on the relative sizes of its angles: the circular saw.

2. What is a circular saw

A circular saw is a figure that is drawn around a convex polygon. Here is one drawn around a triangle, and one around a square:

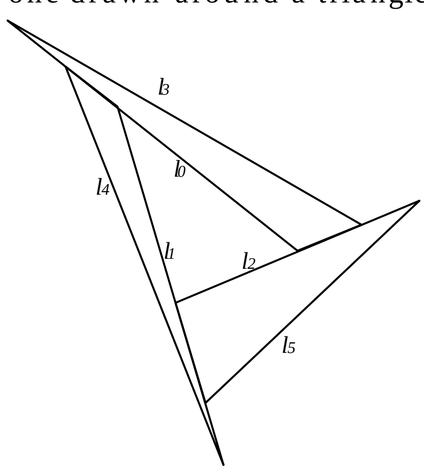

Figure 1 A 3-Circular Saw

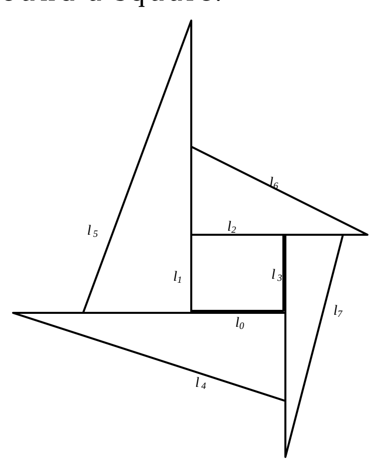

Figure 2: A 4-Circular Saw

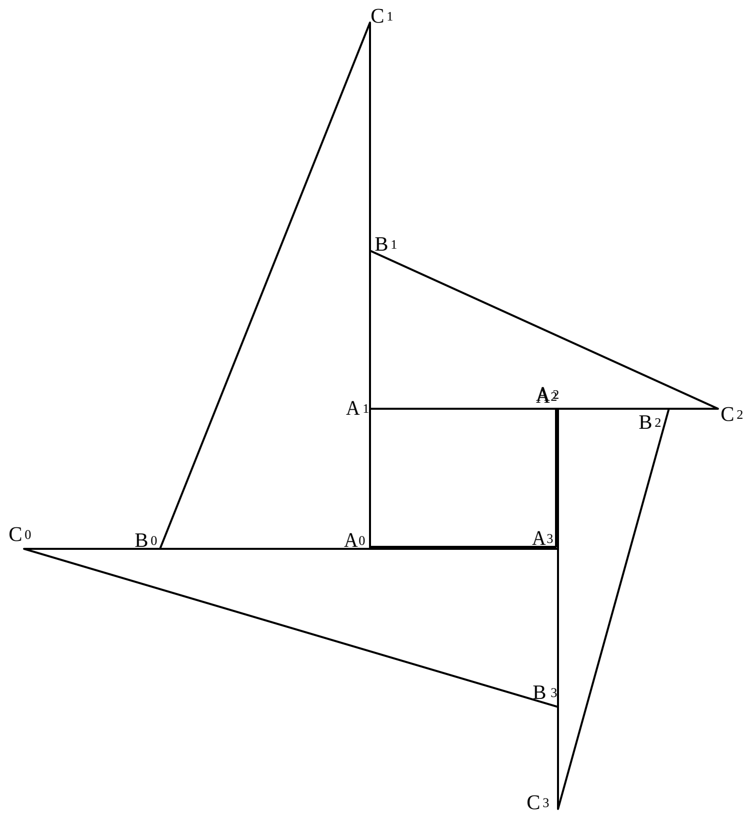

Figure 3 A 4-Circular Saw with labelled vertices

The key properties are that the central figure is a convex polygon, and a triangle is drawn on each of its sides. Each of these triangles has a base that extends a second side of the central polygon, and a point that sticks out past the base of the next triangle around the outside. We can describe this more formally in terms of betweenness.

Defintion: An *n-circular saw* is $3n$ points A_0, B_0, C_0, A_1, B_1, C_1, … A_{n-1}, B_{n-1}, C_{n-1} for some $n = 3$, where:

A_0, A_1, … A_{n-1} form a convex polygon.	2.1
For all i: A_{i-1}, A_i, B_i, C_i are collinear,	2.2
with[1] $A_{i-1} \Relbar A_i \text{ — } B_i \Relbar C_i$	2.3

Here the notation $A_{i-1} \Relbar A_i \text{ — } B_i \Relbar C_i$ is an extension of the standard betweenness notation (see e.g. Millman and Parker [5]). The symbol ‘$\Relbar$’ indicates that the points either side of it may be identical.

Thus $A_{i-1} \Relbar A_i \text{ — } B_i \Relbar C_i$ expands to:

[1] Subscript arithmetic modulo n.

$$A_{i-1} — A_i — B_i — C_i \tag{2.4}$$

Or:

$$A_{i-1}=A_i \;\&\; A_i — B_i — C_i \tag{2.5}$$

Or:

$$A_{i-1} — A_i — B_i \;\&\; B_i = C_i \tag{2.6}$$

Or:

$$A_{i-1}=A_i \;\&\; B_i = C_i \tag{2.7}$$

Given this definition we state the circular saw theorem.

Theorem: The circular saw theorem. In a circular saw we have:

$$\prod_{i=0}^{n-1} \sin\left(\angle A_i C_{i+1} B_i\right) \le \prod_{i=0}^{n-1} \sin\left(\angle C_{i+1} B_i C_i\right) \tag{2.8}$$

with equality if and only if every betweenness condition 2.3 of the circular saw is satisfied with double equality 2.7.

Proof: Using the sine formula on the triangle $A_i B_i C_{i+1}$ we have:

$$\frac{\sin\left(\angle A_i C_{i+1} B_i\right)}{A_i B_i} = \frac{\sin\left(\angle C_{i+1} B_i A_i\right)}{C_{i+1} A_i} = \frac{\sin\left(\angle C_{i+1} B_i C_i\right)}{C_{i+1} A_i} \tag{2.9}$$

Taking the product over all i we have:

$$\prod_i \frac{\sin\left(\angle A_i C_{i+1} B_i\right)}{A_i B_i} = \prod_i \frac{\sin\left(\angle C_{i+1} B_i C_i\right)}{C_{i+1} A_i} \tag{2.10}$$

Rearranging:

$$\prod_i \frac{\sin\left(\angle A_i C_{i+1} B_i\right)}{\sin\left(\angle C_{i+1} B_i C_i\right)_i} = \prod_i \frac{A_i B_i}{C_{i+1} A_i} \tag{2.11}$$

By 2.3 we have:

$$A_i B_i \le C_i A_{i-1} \tag{2.12}$$

with equality if and only if 2.7 holds.

Thus:

$$\prod_i \frac{A_i B_i}{C_i A_{i-1}} \le 1 \tag{2.13}$$

i.e.

$$\prod_i \frac{A_i B_i}{C_{i+1} A_i} \le 1 \tag{2.14}$$

with equality if and only if 2.7 holds for every i.

Combining with 2.11 we have the result.

3. References

[1] J.J. Carroll, *Betweenness in Polar Coordinates*. Hewlett-Packard Laboratories Report, No. HPL-2000-71, Bristol, U.K., 2000.

[2] J.J. Carroll, *Drawing Straight Lines*. Hewlett-Packard Laboratories Report, No. HPL-2000-72, Bristol, U.K., 2000.

[3] Reinhard Diestel, *Graph Theory* , New York, 1997.

[4] Euclid, The Elements, Book I, c 300 BC, [English translation: D. Joyce, http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html, 1996].

[5] Richard S. Millman and George D. Parker, Geometry: A Metric Approach with Models, 2nd Ed., New York, 1991 (1st Ed. 1981).