Betweenness in Polar Coordinates

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HPL-2000-71
December 21st , 2000*

betweenness, geometry, lines, polar coordinates, trilinear coordinates

We consider straight line drawings. From a graph theoretic point of view these are plane graphs with a set of preferred paths. This graph is expressed in geometric terms through the consideration of the betweenness of the points of intersection along each of the lines. If we fix a reference system for polar coordinates, we can identify a number of equalities and inequalities over the complete set of polar coordinates satisfied by this particular drawing. We show that every solution to these inequalities describes a drawing that is equivalent under graph theoretic considerations, in particular betweenness. We show that this equivalence is independent of the polar reference system.

* Internal Accession Date Only Approved for External Publication

1. Introduction

This work is part of a project that is looking at the relationship between sets of paths in planar graphs, and drawings of the graphs in which those paths are straight lines. Hence, we are interested in properties of straight line drawings in the Euclidean plane.

We are interested in the graph theoretic properties: intersection, concurrency, ordering of intersection (i.e. betweenness), parallelism.

We wish to characterise these properties in terms of the coordinates of the lines in the drawing. We use polar coordinates.

We can express the basic Euclidean propositions about its angles as linear inequalities over the angle ordinates of the lines.

We then examine the triangles formed by every three non-concurrent lines, and the relative angles of every three concurrent lines. Using a formula derived from consideration of exact trilinear coordinates we derive a set of inequalities corresponding to these triangles, and equalities corresponding to the concurrent lines.

We abstract these inequalities by considering them as formulae in a free algebra, and show that all real solutions to these inequalities are equivalent to the original drawing in terms of the graph theoretic properties of interest.

The characterisation we derive has been used as the foundation of a graph-drawing algorithm; Carroll [1],[2]. It also may form a starting point for a more profound characterisation of straight line drawings in the Euclidean plane in terms of their graph theoretic properties.

2. Polar Coordinates

Polar coordinates can be used to describe lines as well as points.

We note that a polar reference system consists of an origin *O*, a semi-line (the axis) starting at *O*, and an orientation around the origin. Lines in a polar system are given a pair of coordinates (d,q). d is the perpendicular distance from the line to the origin. d is always non-negative. q is the oriented angle[1], from the axis to the perpendicular dropped by the line through *O*. q is taken from the interval $[0, 2\pi)$. The line can be given a direction following the polar orientation around the origin. This act of orienting lines depends only on graph theoretic considerations, and the facial boundaries of the drawing.

Throughout this paper, we generally number the lines, and we use (d_i,q_i) as the coordinates of the line l_i.

When two lines intersect we can talk about the first half and second half of each line, as given by the direction of the lines.

It is problematic to determine the angle of the polar coordinates of the origin, or of a line through the origin. For almost all this paper, we will

[1] We express angles in radians.

simply ignore this case and only consider lines, and (finite) sets of lines, which do not pass through the origin. We will revisit this assumption in section 9.

We consider any two non-parallel lines l_1 and l_2, and their point of intersection. We consider the quadrilateral Q formed by the two lines and their two perpendiculars to O. We separate two cases: the normal case in which the polar axis does not intersect the interior of Q, and the wrap-around case in which the polar axis does intersect the interior of Q.

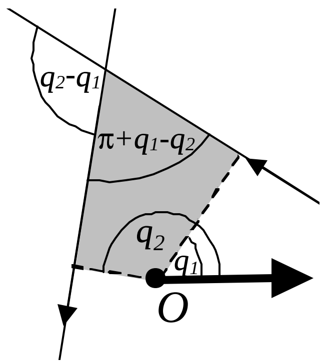

Figure 1: Normal intersection

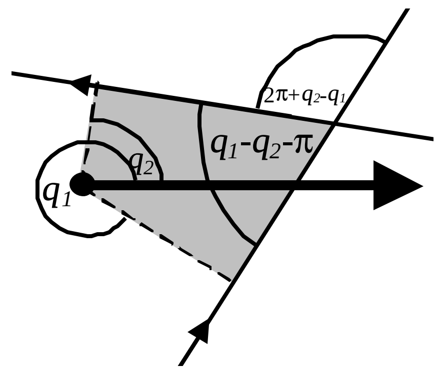

Figure 2: Wrap around intersection

In both cases we see that the origin is in a quarter-plane bounded by the first half of one line l_1 and the second half of the other l_2.

The linear angle inequalities

We note that the two angles marked at the point of intersection in both figures are positive.

Thus the angle ordinates satisfy a pair of linear inequalities.

In the normal case:

$$q_2 - q_1 > 0 \qquad 2.1.1$$

$$p + q_1 - q_2 > 0 \qquad 2.1.2$$

In the wrap-around case:

$$2p + q_2 - q_1 > 0 \qquad 2.1.3$$

$$q_1 - q_2 - p > 0 \qquad 2.1.4$$

By considering every pair of non-parallel lines in the drawing, we generate a set of up to $n(n\text{-}1)$ linear inequalities.

For parallel lines we generate an equality. Either:

$$q_2 - q_1 = 0 \qquad 2.1.5$$

Or:

$$q_1 - q_2 + p = 0 \qquad 2.1.6$$

We ‘simplify’ 2.1.1 and 2.1.2 or 2.1.3 and 2.1.4 as:

$$\sin(q_2 - q_1) > 0 \qquad 2.1.7$$

Similarly we express 2.1.5 or 2.1.6 as:

$$\sin(q_2 - q_1) = 0 \qquad 2.1.8$$

We note that despite the sine term 2.1.7 and 2.1.8 are still expressing linear relationships between the angles. We express both of these results as lemmas:

Lemma 2.1: If l_1 and l_2 intersect with the origin lying in the quarter-plane bounded by the first half of l_1 and the second half of l_2 then $\sin(q_2 - q_1) > 0$.

Lemma 2.2: If l_1 and l_2 are parallel then $\sin(q_2 - q_1) = 0$.

3. Triangles in polar coordinates

We consider a set of three non-concurrent lines, in which no two are parallel. They form a triangle. We take their coordinates to be (d_1, q_1), (d_2, q_2) and (d_3, q_3). Each of the three sides of the triangle divides the plane into two half-planes, giving six half planes to consider in total. Of these the triangle lies in three, and *O* lies in three. If *O* lies inside the triangle then these two sets of half-planes are identical. Alternatively their intersection may have one or two members. We show all three cases in the following figures.

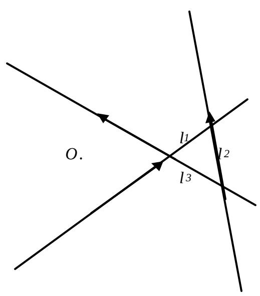

Figure 3: Position 1

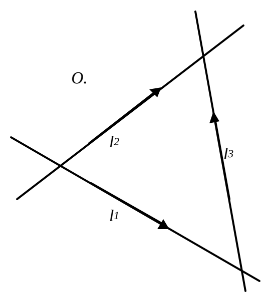

Figure 4: Position 2

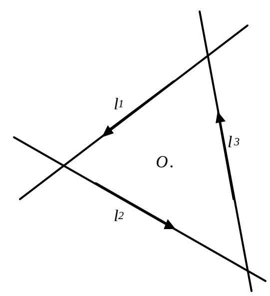

Figure 5: Position

We refer to the relative position of the origin to the triangle by saying that the triangle is in position 1, 2 or 3 depending on whether it shares one, two or three half-planes with the origin.

The Trilinear Constant

Coxeter [3] (p378) asserts that for any point *P* in the plane and any triangle that the following equation holds between the lengths of the sides of the triangles and the exact trilinear coordinates of *P*. (See Weisstein [6] for an introduction to trilinear coordinates).

Lemma 3.1: For all *P*, for any triangle.

$$ax + by + cz = 2\Delta \qquad 3.1.1$$

where, *a b* and *c* are the lengths of the sides of the triangles; *x*, *y*, *z* are the corresponding exact trilinear coordinates[2] of *P*; Δ is the area of the triangle.

[2] I.e. the signed perpendicular distances from *P* to the corresponding side of the triangle.

Proof: (not in [3]) Consider the point A with trilinear coordinates (h_a, 0, 0). The LHS of 3.1.1 evaluates to ah_a and equality holds. Since the same is true at the other two corners, and the three corners are not collinear, the lemma follows by linearity of $ax+by+cz$.

We prefer to express this in terms of the angles *A*, *B*, and *C* of the triangle.

Corollary 3.2: The trilinear constant.

$$x\sin(A)+y\sin(B)+z\sin(C)=\frac{4\Delta^2}{abc}. \qquad 3.2.1$$

Moreover, we don't care about area, and so transform the equation 3.2.1 into an inequality.

Corollary 3.3: The trilinear inequality.

$$x\sin(A)+y\sin(B)+z\sin(C)>0. \qquad 3.3.1$$

For three concurrent lines we can take 3.2.1 to the limit.

Corollary 3.4: Trilinear concurrency.

$$x\sin(A)+y\sin(B)+z\sin(C)=0 \qquad 3.4.1$$

The Betweenness Inequalities

Translating 3.3.1 and 3.4.1 into our framework of polar coordinates we note that:

- The sine function has period 2π; so we can ignore the wrap around issue.
- The distance ordinate is the unsigned distance of *O* to the line, whereas the exact trilinear coordinates use signed distance. The sign of the distance depends on whether *O* lies in the same half plane as the triangle or not, and hence on our notion of position 1, 2 or 3.

We identify the three lines as l_1, l_2 and l_3 as in the diagrams, with polar coordinates (d_1,q_1) etc. Moreover, we order them as in the figures. Then we get one of the following three inequalities depending upon the position of the triangle:

$$-d_1\sin(q_3-q_2)+d_2\sin(q_3-q_1)-d_3\sin(q_2-q_1)>0 \qquad 3.5.1$$

$$d_1\sin(q_3-q_2)-d_2\sin(q_3-q_1)+d_3\sin(q_2-q_1)>0 \qquad 3.5.2$$

$$d_1\sin(q_3-q_2)+d_2\sin(q_1-q_3)+d_3\sin(q_2-q_1)>0 \qquad 3.5.3$$

We have written these inequalitie s so that all of the sine terms are positive, when the linear angle inequalities hold. Particular attention should be paid to the middle term of 3.5.3, where the position 3 triangle differs from the other two (we could alternatively note that this has the same value as the middle term of 3.5.2).

For concurrent lines, position 3 is not possible, and we find:

$$d_1\sin(q_3-q_2)-d_2\sin(q_3-q_1)+d_3\sin(q_2-q_1)=0 \qquad 3.5.4$$

If two of the lines are parallel (but not all three) then one of these formulae still apply. The three degenerate versions are:

$$-d_1 + d_2 > 0 \qquad 3.5.5$$

$$d_1 - d_2 > 0 \qquad 3.5.6$$

$$d_1 + d_2 > 0 \qquad 3.5.7$$

(We take the first two lines to be parallel).

All of these inequalities and equalities 3.5.1 to 3.5.7 can be seen as having a left hand side of:

$$d_1 \sin(q_3 - q_2) + d_2 \sin(q_1 - q_3) + d_3 \sin(q_2 - q_1) \qquad 3.5.8$$

and right hand side 0. The operator is < in 3.5.1 and 3.5.5, = in 3.5.4 and > in the others. Alternatively, by reordering the lines in 3.5.1 and 3.5.5, we find that all the inequalities can be written as:

$$d_1 \sin(q_3 - q_2) + d_2 \sin(q_1 - q_3) + d_3 \sin(q_2 - q_1) > 0 \qquad 3.5.9$$

for some choice of ordering, and the equality can be written as:

$$d_1 \sin(q_3 - q_2) + d_2 \sin(q_1 - q_3) + d_3 \sin(q_2 - q_1) = 0 \qquad 3.5.10$$

4. Free algebras for reasoning about polar coordinates

We are abstracting over the betweenness inequalities, hence it is appropriate to clarify what that means. One approach to the inequalities as formulae that may or may not be true of a set of polar coordinates is to treat each expression as a term in a free algebra over 2 *n* symbols, being the polar coordinates of *n* lines.

Definition 4.1: The free polar algebra over *n* lines FP_n is the free algebra with operations **+** / 3, ´ / 2, **+** / 2, - / 1, **sin** / 1, p / 0 over the set of 2 *n* symbols: $\{d_1,?_1, d_2,?_2 \cdots d_n,?_n\}$.

We use the normal conventions for writing - / 1. We use **+** / 3 whenever we may have used two applications of **+** / 2. This is to by-pass uninteresting (and unchallenging) discussions about associativity and bracketing. (FP_n is not associative; our use of it, however, never adds up more than three items). Thus $a + b - c$ is shorthand for $+(a, b, -(c))$. This algebra is not intended to be anything more than a vehicle for discussing betweenness inequalities.

Definition 4.2: an *n*-line drawing *D* is a finite sequence of *n* straight lines in the Euclidean plane (i.e. a numbered[3] set of lines).

Definition 4.3: Δ_n is the set of all *n*-line drawings in the Euclidean plane.

Given a polar frame of reference, we can define four functions from Δ_n to subsets of the free polar algebra, each corresponding to part of the previous sections:

Definition 4.4: $\Lambda_> : \Delta_n \to FP_n$ maps a drawing to the set of formulae appearing as the LHS of the linear angle inequality 0.7.

[3] We number the sets, so that we can associate specific coordinates (d_i, q_i) with each line, apart from this we treat a drawing as a set.

More formally:

$$\Lambda_{>}(D) = \{\boldsymbol{sin}(\boldsymbol{?_i} - \boldsymbol{?_j}) | 1 \le i, j \le n, \sin(q_i - q_j) > 0\} \qquad 4.1$$

Note that on the one hand we use expressions like $\boldsymbol{sin}(\boldsymbol{?_i} - \boldsymbol{?_j})$ wholly symbolically as an expression of a free algebra, but on the other we evaluate $\sin(q_i - q_j)$ as a real number where q_i, q_j are the angle ordinates of the *i*th and *j*th lines in *D*.

Definition 4.5: $\Lambda_{=} : \Delta_n \to FP_n$ maps a drawing to the set of formulae appearing as the LHS of a linear angle equality 0.8.

More formally:

$$\Lambda_{=}(D) = \{\boldsymbol{sin}(\boldsymbol{?_i} - \boldsymbol{?_j}) | 1 \le i, j \le n, \sin(q_i - q_j) = 0\} \qquad 4.2$$

For completeness we also define:

$$\Lambda_{<}(D) = \{\boldsymbol{sin}(\boldsymbol{?_i} - \boldsymbol{?_j}) | 1 \le i, j \le n, \sin(q_i - q_j) < 0\} \qquad 4.3$$

The contrast between the expressions in *FPn* like $\boldsymbol{?_i} - \boldsymbol{?_j}$ and the real numbers $q_i - q_j$ can be further clarified by introducing the natural evaluation homomorphism $e_D : FP_n \to \Re$. This is defined by **+**/3, ´/2, **+**/2, -/1, **sin**/1, p/0 all having their natural interpretations in $\Re$ and by $e_D(\boldsymbol{?_i}) = q_i$ and $e_D(\boldsymbol{d_i}) = d_i$ where (d_i, q_i) are the coordinates of the *i*th line. We have used this natural evaluation in the definition of $\Lambda_{>}(D)$, $\Lambda_{=}(D)$ and $\Lambda_{<}(D)$.

The trilinear constant expressions T_n are the formulae in *FPn* for the trilinear constants for any three of the *n* lines:

$$\mathrm{T}_n = \{\boldsymbol{d_i sin}(\boldsymbol{?_k} - \boldsymbol{?_j}) + \boldsymbol{d_j sin}(\boldsymbol{?_i} - \boldsymbol{?_k}) + \boldsymbol{d_k sin}(\boldsymbol{?_j} - \boldsymbol{?_i}) | 1 \le i, j, k \le n\} \quad 4.7$$

We define $\mathrm{B}_{>} : \Delta_n \to FP_n$ and $\mathrm{B}_{=} : \Delta_n \to FP_n$ to show the betweenness inequalities and equalities.

Definition 4.6: $\mathrm{B}_{>}(D) = \{c \in \mathrm{T}_n | e_D(c) > 0\}$ 4.8

Definition 4.7: $\mathrm{B}_{=}(D) = \{c \in \mathrm{T}_n | e_D(c) = 0\}$ 4.9

$\mathrm{B}_{<}(D)$ similarly.

We can group these subsets of *FPn* as pairs:

$$\Lambda(D) = \langle \Lambda_{>}(D), \Lambda_{=}(D) \rangle \qquad 4.10$$

$$\mathrm{B}(D) = \langle \mathrm{B}_{>}(D), \mathrm{B}_{=}(D) \rangle \qquad 4.11$$

$$\mathrm{X}(D) = \langle \Lambda_{>}(D) \cup \mathrm{B}_{>}(D), \Lambda_{=}(D) \cup \mathrm{B}_{=}(D) \rangle \qquad 4.12$$

In this way we have formalised the discussion of sets of inequalities and equalities.

Definition 4.8: we say a sequence of coordinates (2*n* real numbers) solve a pair of sets of formulae $\langle g, e \rangle \in FP_n^2$ if every $c \in g$ evaluates to a positive real, and every $c \in e$ evaluates to zero, using the natural homomorphism at those coordinates.

We observe that the coordinates of D solve $\Lambda(D)$, $\mathrm{B}(D)$ and $\mathrm{X}(D)$ by construction.

5. Equivalent drawings

In this section we take D to be an n-line drawing, and $D¢$ to be another n-line drawing whose coordinates also solves $\mathrm{X}(D)$. We hence have numbered the lines of both D and $D¢$ so that line l_1 with coordinates (d_1, q_1) corresponds to l_1' with coordinates (d_1', q_1').

For generality, and to reduce the complexity of the next sections, we use separate polar frames of reference F and $F¢$ for these two drawings. The origin of F is O the origin of $F¢$ *is* $O¢$.

Theorem 5.1: $\mathrm{X}(D) = \mathrm{X}(D')$, and $\mathrm{X}_<(D) = \mathrm{X}_<(D')$

Proof: Consider $c \in \mathrm{T}_n \cup \{\mathbf{sin}(?_i - ?_j) | 1 \le i, j \le n\}$. $c \in \mathrm{X}_>(D)$ or $c \in \mathrm{X}_=(D)$ or $c \in \mathrm{X}_<(D)$ depending as to whether $e_D(c)$ is greater, equal or less than 0. We know that when $c \in \mathrm{X}_>(D)$ then $e_{D'}(c) > 0$, and when $c \in \mathrm{X}_=(D)$ then $e_{D'}(c) = 0$, since $D¢$ solves $\mathrm{X}(D)$. Define $\overline{c}$ as the symbolic negative of c as follows. If

$$c = \boldsymbol{d_i sin}(?_k - ?_j) + \boldsymbol{d_j sin}(?_i - ?_k) + \boldsymbol{d_k sin}(?_j - ?_i), \qquad 5.1.1$$

then

$$\overline{c} = \boldsymbol{d_j sin}(?_k - ?_i) + \boldsymbol{d_i sin}(?_j - ?_k) + \boldsymbol{d_k sin}(?_i - ?_j). \qquad 5.1.2$$

Or if

$$c = \mathbf{sin}(?_i - ?_j), \qquad 5.1.3$$

then

$$\overline{c} = \mathbf{sin}(?_i - ?_j). \qquad 5.1.4$$

Then if $e_D(c) < 0$ we have $e_D(\overline{c}) > 0$ and $\overline{c} \in \mathrm{X}_>(D)$. Hence $e_{D'}(\overline{c}) > 0$, and $e_{D'}(c) < 0$. Thus the partition of $\mathrm{T}_n \cup \{\mathbf{sin}(?_i - ?_j) | 1 \le i, j \le n\}$ into $\mathrm{X}_>(D)$, $\mathrm{X}_=(D)$ and $\mathrm{X}_<(D)$ is identical to its partition into $\mathrm{X}_>(D')$, $\mathrm{X}_=(D')$ and $\mathrm{X}_<(D')$.

Corollary 5.2: $\Lambda(D) = \Lambda(D')$.

Corollary 5.3: $\mathrm{B}(D) = \mathrm{B}(D')$.

Corollary 5.4: $l_i \| l_j$ if and only if $l_i' \| l_j'$.

Proof: $\Lambda_=(D) = \Lambda_=(D')$.

Corollary 5.5: l_i, l_j, and l_k are concurrent if and only if l_i', l_j', and l_k'

Proof: $\mathrm{B}_=(D) = \mathrm{B}_=(D')$.

We can now almost prove the result that gives this paper its title: that *D* and *D¢* have the same betweenness properties.

First, we have to give further consideration to half-planes. Consider a line l_i and a point *v* defined as the intersection of two other lines $l_j \cap l_k = \{v\}$. Then l_j and l_k are not parallel, and neither are l'_j and l'_k (by corollary 5.4). Thus we can also define *v¢* by $l'_j \cap l'_k = \{v'\}$. With these terms:

Lemma 5.6: *v* and *O* lie in the same half-plane of l_i if and only if *v¢* and *O¢* lie in the same half-plane of l'_i.

Proof: We consider the case where *v* and *O* lie in the same half plane. The other case is similar.

Consider

$$c = \boldsymbol{d_i sin}(\boldsymbol{?_k} - \boldsymbol{?_j}) + \boldsymbol{d_j sin}(\boldsymbol{?_i} - \boldsymbol{?_k}) + \boldsymbol{d_k sin}(\boldsymbol{?_j} - \boldsymbol{?_i})$$

and

$$\overline{c} = \boldsymbol{d_j sin}(\boldsymbol{?_k} - \boldsymbol{?_i}) + \boldsymbol{d_i sin}(\boldsymbol{?_j} - \boldsymbol{?_k}) + \boldsymbol{d_k sin}(\boldsymbol{?_i} - \boldsymbol{?_j}).$$

As l_i, l_j, and l_k are not concurrent, $0 \neq e_D(c) = -e_D(\overline{c})$. Thus one of $e_D(c)$ and $e_D(\overline{c})$ is positive, w.l.o.g. c, and $c \in \mathrm{B}_>(D)$. Hence c has been derived[4] directly from the trilinear constant of 3.2.1. Since *O* and *v* lie in the same half-plane, the signed distance of *O* from l_i is positive. Thus $\sin(q_k - q_j) > 0$. From corollary 5.2 $\sin(q'_k - q'_j) > 0$. We have that $e_{D'}(c) > 0$ and that c is the formula for the trilinear constant of the triangle formed by l'_i, l'_j, and l'_k. Hence we also have that the signed distance from *O¢* to l'_i is positive, and the result follows.

The main theorem is that the linear angle inequalities and the betweenness inequalities characterise the betweenness properties of a straight line drawing.

Theorem 5.7: Betweenness in polar coordinates. If l_h, l_i, l_j, and l_k are such that $l_h \cap l_i = \{v_i\}$, $l_h \cap l_j = \{v_j\}$, $l_h \cap l_k = \{v_k\}$ and v_i—v_j—v_k then the same holds for l'_h, l'_i, l'_j, and l'_k.

Proof: Consider the line l_j. The origin lies one side or the other (by general assumption in this paper). v_i and v_k lie in opposite half-planes by hypothesis. From two applications of lemma 5.6, v'_i and v'_k lie in opposite half-planes of l'_j, and we have the result.

[4] Strictly we should consider the parallel case, corresponding to 3.5.5 to 3.5.7 separately.

6. Drawings and Plane Graphs

We briefly explore the relationship between straight line drawings and plane graphs.

We can consider the plane graph $G = (V, E)$ derived from D by taking every line intersection as a vertex and the edges as the distinct finite line-segments formed in $D \setminus V$, i.e. by removing the vertices from the drawing. In other words.

$$V = \{ l_i \cap l_j | 1 \le i \ne j \le n \}$$

$$E = \{ (u,v) \in V^2 | u, v \in l_i, \text{there is no } w \in V, \text{s.t. } u - - w - - v, 1 \le i \le n \}$$

Note that E has been defined in terms of betweenness.

G and D differ in that the lines in D are infinite but the corresponding paths in G are finite. This is shown in the three line Figure 6, where G is shown in bold.

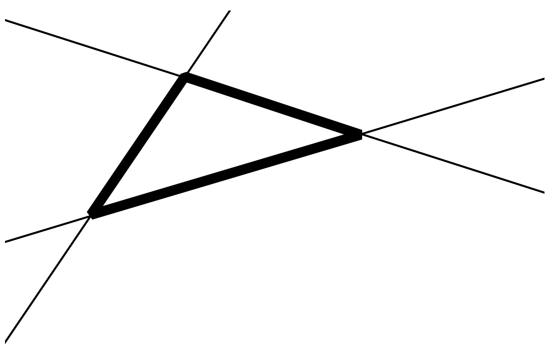

Figure 6 A 3 line drawing and its finite plane graph G

We note that considering only the graph will completely lose lines that do not have any finite line segments in the drawing (these arise either in a drawing of parallel lines or when the line contains only one point of intersection with other lines).

In Carroll [1] we consider a planar graph subject to extended co-linearity constraints. We then form the linear angle inequalities and the betweenness inequalities corresponding to those in this paper. Solving these inequalities then gives the coordinates of a straight line drawing containing a plane graph that satisfies the original extended co-linearity constraints.

7. The Central Face

Before showing that our result is independent of the frame of reference we will consider the facial boundary of the face containing the origin.

We will show that when D and D' are equivalent then the face of D containing the origin of F has facial boundary that maps on to the facial boundary of the face of D' containing the origin of F'. This helps to show the plausibility of there being a natural graph theoretic isomorphism[5] between the graphs derived from D and D', and, more importantly, is used in the proof of the independence of the notion of equivalence from the choice of frame of reference.

Definition 7.1: The *central face* of a drawing D w.r.t. a frame of reference F is that face of D containing the origin of F.

We note that the central face need not be finite.

We will find that the boundary of the central face is characterised by the linear angle and betweenness inequalities.

We use the notation $v_h = v_i — v_j = v_k$ as an extension of the standard betweenness notation (see e.g. Millman and Parker [5]). The symbol '—' indicates that the points either side of it may be identical.

We use $\overline{v_i v_j}$ for the line segment between two points, we use $\overleftarrow{v_i}$ for the *initial* ray from v_i back along a given line counter to the orientation of F, we use $\overrightarrow{v_i}$ for the *final* ray from v_i along a given line in the direction of the orientation of F. Note for both of these the line must be inferred from context.

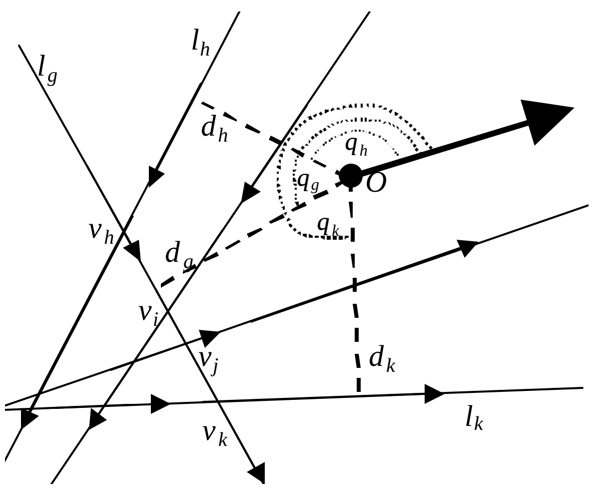

Figure 7 $\overline{v_i v_j}$ is an edge of the central face

Lemma 7.2: Given, l_g, l_i, l_j, with $l_g \cap l_i = \{v_i\}$, $l_g \cap l_j = \{v_j\}$, and v_i, v_j distinct; with v_i preceding v_j in the orientation along l_g as given by the frame of reference F; then the edge $\overline{v_i v_j}$ is an edge of the central face iff:

a) For all l_h intersecting l_g with $l_g \cap l_h = \{v_h\}$ then it is not the case that $v_i — v_h — v_j$,

[5] See Diestel [4] pp 77-79 for a definition of this term.

b) For all l_h intersecting l_g with $l_g \cap l_h = \{v_h\}$ and v_h ══ v_i — v_j then $\sin(q_g - q_h) > 0$

c) For all l_k intersecting l_g with $l_g \cap l_k = \{v_k\}$ and v_i — v_j ══ v_k then $\sin(q_k - q_g) > 0$

d) For all l_h parallel to l_g then $d_h \sin(q_g - q_i) + d_i \sin(q_h - q_g) + d_g \sin(q_i - q_h) > 0$

Proof:

Suppose $\overline{v_i v_j}$ is an edge of the central face then:

a) It is not bisected by another line.

b) We note $\overline{v_i v_j}$ is on the central face so the second half of l_g is a boundary of the quarter-plane formed by l_g and l_h containing the origin. From lemma 2.1, we have $\sin(q_g - q_h) > 0$.

c) Similarly.

d) From b) we have $\sin(q_g - q_i) > 0$. Since l_h is parallel to l_g we have $\sin(q_g - q_h) = 0$ and $\sin(q_g - q_i) = \pm \sin(q_i - q_h)$. Hence the inequation amounts to $d_h + \pm d_g > 0$. If this is false then $d_h \le d_g$ and $q_g = q_h$, i.e. l_h is either the same as l_g, or it is closer to the origin than l_g and on the same side of the origin, hence placing the origin and (v_i, v_j) in separate half-planes.

Conversely, if $\overline{v_i v_j}$ is not an edge of the central face then take l to be a line in D separating (some of) $\overline{v_i v_j}$ from the origin. The origin and $\overline{v_i v_j}$ lie in different half-planes of l. There are four possibilities for where l intersects l_g.

a) l intersects l_g within the interior of $\overline{v_i v_j}$. This contradicts case a) in the lemma statement.

b) l intersects l_g in the ray $\overleftarrow{v_i}$ (including the end-point). The quarter-plane of this intersection containing the origin is not bounded by the edge $\overline{v_i v_j}$, hence it is bounded by the first half of l_g and hence the second half of l. Lemma 2.1 then gives $\sin(q \quad - q_g) > 0$ contradicting b) of the lemma statement.

c) l intersects l_g in the ray $\overrightarrow{v_j}$ (including the end-point). Similarly contradicting c) of the lemma statement.

d) l is parallel to l_g. Then the origin and l_g lie in different half planes of l and d) in the lemma statement is contradicted.

The corresponding results for rays that are edges of the central face have similar proofs.

Lemma 7.3: Given, l_g, l_i, with $l_g \cap l_i = \{v_i\}$, the initial ray $\overleftarrow{v_i}$ along l_g is an edge of the central face iff:

a) For all l_h intersecting l_g with $l_g \cap l_h = \{v_h\}$ then $v_h \in \overrightarrow{v_i}$ only if $v_h = v_i$

b) For all l_k intersecting l_g with $l_g \cap l_k = \{v_k\}$ and $v_k \neq v_i$ then $\sin(q_k - q_g) > 0$

c) For all l_h parallel to l_g then $d_h \sin(q_i - q_g) + d_g \sin(q_h - q_i) + d_i \sin(q_g - q_h) > 0$

Lemma 7.4: Given, l_g, l_i, with $l_g \cap l_i = \{v_i\}$, the final ray $\overrightarrow{v_i}$ along l_g is an edge of the central face iff:

a) For all l_h intersecting l_g with $l_g \cap l_h = \{v_h\}$ then $v_h \in \overleftarrow{v_i}$ only if $v_h = v_i$

b) For all l_k intersecting l_g with $l_g \cap l_k = \{v_k\}$ and $v_k \neq v_i$ then $\sin(q_g - q_k) > 0$

c) For all l_h parallel to l_g then $d_h \sin(q_g - q_i) + d_i \sin(q_h - q_g) + d_g \sin(q_i - q_h) > 0$

All of the conditions in these three lemmas refer to either linear angle or betweenness inequalities or to parallelism or to betweenness. Since D' solves $\mathrm{X}(D)$, we have from section 5 (in particular theorem 5.1, corollary 5.4 and theorem 5.7) that these properties are common to D and D'. Thus:

Theorem 7.5: The central face. For any l_g, l_i, l_j, with $l_h \cap l_i = \{v_i\}$, $l_h \cap l_j = \{v_j\}$ and $l'_h \cap l'_i = \{v'_i\}$, $l'_h \cap l'_j = \{v'_j\}$ then:

a) $\overline{v_i v_j}$ is an edge of the central face of D iff $\overline{v'_i v'_j}$ is an edge of the central face of D'.

b) $\overleftarrow{v_i}$ is an edge of the central face of D iff $\overleftarrow{v'_i}$ is an edge of the central face of D'.

c) $\overrightarrow{v_i}$ is an edge of the central face of D iff $\overrightarrow{v'_i}$ is an edge of the central face of D'.

8. Independence of Frame of Reference

We investigate to what extent the notion of equivalence between D and D' is independent from the choice of polar frames of reference. As before, D' is a drawing, whose coordinates with respect to a frame of reference F' solve

$X(D)$. To clarify we will write $X^F(D)$ to emphasise that the formulae have been defined using the frame of reference F. We consider a second frame of reference $\overline{F}$ for D with inequalities $X^{\overline{F}}(D)$. We systematically use the bar notation to refer to terms with reference to $\overline{F}$. The results of this section show that there is a second frame of reference $\overline{F}'$ for D' such that the coordinates of D' with respect to $\overline{F}'$ solve $X^{\overline{F}}(D)$. Thus we can define an equivalence relation between drawings based on one drawing solving the inequalities derived from the other.

We approach the result of this section in steps.

We first show that it holds if F and $\overline{F}$ differ only in the direction of the axis.

Then we show the result when F and $\overline{F}$ differ only in orientation.

Then we show that we can move the origin within one face while maintaining the orientation and the axis in the same direction.

Next we show that we can move the origin from one face to an adjacent face while maintaining the orientation and the axis in the same direction.

Finally we combine all of these to get the desired result.

Definition: A frame of reference $\overline{F}$ is *as good as* F *for a pair of drawings* (D, D') if for any third frame of reference F' such that the coordinates of D' with respect to F' solve $X^F(D)$ then there exists a fourth frame of reference $\overline{F}'$ such that the coordinates of D' with respect to $\overline{F}'$ solve $X^{\overline{F}}(D)$.

This rather cumbersome definition is simply intended to say that the solubility of $X(D)$ by D' is unchanged if we change from using F to using $\overline{F}$ as our frame of reference for D.

If we simply rotate F to give $\overline{F}$ then the distance ordinates are unchanged, and the angle ordinates are all increased uniformly, but with wrap-around at 2π. We make an equivalent rotation on F' to give $\overline{F}'$. We use symbols $\bar{d}_i, \bar{d}'_i, \bar{q}_i, \bar{q}'_i$ to show the ordinates in the various frames of reference.

Lemma 8.1: For any drawing D if $\overline{F}$ has the same origin and orientation as F then the coordinates of D with respect to $\overline{F}$ solve $X^F(D)$.

Proof : For all the terms c appearing in $X^F(D)$ the angles always appear inside subterms of the form $\mathbf{sin}(?_i - ?_j)$. The evaluation of these subterms is the same with respect to F and $\overline{F}$. The evaluation of the other terms is unchanged. Hence the two evaluations are identical, and the result follows.

Corollary 8.2: For any pair of drawings (D, D'), if $\overline{F}$ has the same origin and orientation as F then $\overline{F}$ is *as good as* F *for the pair* (D, D').

Proof : By theorem 5.1, $\mathrm{X}^{\overline{F}}(D)=\mathrm{X}^{F}(D)$. If we take $\overline{F}'=F'$ then since *D*¢ with respect to *F*¢ solves $\mathrm{X}^{F}(D)$ then *D*¢ with respect to $\overline{F}'$ solves $\mathrm{X}^{\overline{F}}(D)$.

Lemma 8.3: For any pair of drawings (D,D'), if $\overline{F}$ has the same origin and axis as F then $\overline{F}$ is *as good as F for the pair* (D,D').

Proof : If $\overline{F}=F$ there is nothing to prove. The other case is when $\overline{F}$ has the reverse orientation to F. We take $\overline{F}'$ with the same origin and axis as *F*¢ but with the reverse orientation. For all i and j we have:

$$\sin(q_i-q_j)=-\sin(\overline{q}_i-\overline{q}_j) \qquad 8.3.1$$

and:

$$\sin(q_i'-q_j')=-\sin(\overline{q}_i'-\overline{q}_j') \qquad 8.3.2$$

Proof : Thus $\mathrm{X}_{=}^{\overline{F}}(D)=\mathrm{X}_{=}^{F}(D)$ and $\mathrm{X}_{>}^{\overline{F}}(D)=\mathrm{X}_{<}^{F}(D)$; and $\mathrm{X}_{=}^{\overline{F}'}(D)=\mathrm{X}_{=}^{F'}(D)$ and $\mathrm{X}_{>}^{\overline{F}'}(D)=\mathrm{X}_{<}^{F'}(D)$. From theorem 5.1 $\mathrm{X}^{\overline{F}}(D)=\mathrm{X}^{\overline{F}'}(D)$ and the result holds.

Lemma 8.4: For any drawing D, if $\overline{F}$ has the same orientation as F, and the axis of $\overline{F}$ is parallel to the axis of F (including direction), and the origin of $\overline{F}$ is within the central face of D with respect to F then $\mathrm{X}^{\overline{F}}(D)=\mathrm{X}^{F}(D)$.

Proof : The angle co-ordinates of every line is unchanged with the change in frame of reference (i.e. $\overline{q}_i=q_i$). Hence the linear angle inequalities remain unchanged, $\overline{\Lambda(D)}=\Lambda(D)$. The trilinear inequality 3.3.1 and the trilinear concurrency 3.4.1 are independent of the frame of reference, and as discussed in section 3 their expression in polar coordinates depends solely on which half planes of the three lines the origin lieswithin. Since O_F and $O_{\overline{F}}$ both lie within the same face (the central face of D with respect to F) then the polar expression of the trilinear inequalities and the trilinear concurrencies of D are unchanged. Hence $\overline{B(D)}=B(D)$, and we have the result.

Corollary 8.5: For any pair of drawings (D,D'), if $\overline{F}$ has the same orientation as F, and the axis of $\overline{F}$ is parallel to the axis of F (including direction), and the origin of $\overline{F}$ is within the central face of D with respect to F then $\overline{F}$ is *as good as F for the pair* (D,D').

Proof : We take $\overline{F}'$ equal to *F*¢.

Lemma 8.6: For any drawing D, if $\overline{F}$ has the same orientation as F, and the axis of $\overline{F}$ is parallel to the axis of F (including direction), and the origin of $\overline{F}$ is within a face adjacent to the central face of D with respect to F, with l_g separating the two central faces then

- $X_{=}^{\overline{F}}(D) = X_{=}^{F}(D)$ and
- $c \in X_{>}^{\overline{F}}(D)$ iff $c \in X_{>}^{\overline{F}}(D)$ and ***?**$_g$* does not occur in c

  or $c \in X_{<}^{\overline{F}}(D)$ and ***?**$_g$* does occur in c

Proof : This result follows similarly to lemma 8.4, noting that $\overline{q}_g = q_g \pm p$ and that other angles are unchanged, hence $\sin(q_i - q_g) = -\sin(\overline{q}_i - \overline{q}_g) = \sin(\overline{q}_g - \overline{q}_i)$. Also for triangles not involving l_g the relative half-planes of O_F and $O_{\overline{F}}$ with respect to the triangle are unchanged, but for those involving l_g the sign of the term in $\overline{d}_g$ is reversed from that in d_g.

Corollary 8.7: For any pair of drawings (D, D'), if $\overline{F}$ has the same orientation as F, and the axis of $\overline{F}$ is parallel to the axis of F (including direction), and the origin of $\overline{F}$ is within a face adjacent to the central face of D with respect to F, with l_g separating the two central faces then $\overline{F}$ is *as good as* F *for the pair* (D, D').

Proof : We take $\overline{F}'$ with origin within the face adjacent to the central face of D¢ with respect to F¢, separated from it by l'_g. This face exists by virtue of theorem 7.5. We take the axis of $\overline{F}'$ as parallel to that of F¢ (including direction) and $\overline{F}'$ is given the same orientation as F¢. Two applications of lemma 8.6 give this result.

Theorem 8.8: Independence of frame of reference. For any pair of drawings (D, D'), and any pair of frames of reference F and $\overline{F}$ then $\overline{F}$ is *as good as* F *for the pair* (D, D').

Proof : Draw a line (not necessarily straight) from the origin of F to the origin of $\overline{F}$ which avoids any points of intersection of the lines of D. This line passes through a finite number of faces of D. Thus a finite number of applications of corollary 8.7, followed by zero or one applications of each of corollary 8.5, corollary 8.2 and lemma 8.3 give a finite chain of frames of reference each as good as the next. Since 'as good as' is transitive we have the result.

9. Lines passing through the origin

We note that given a frame of reference and a drawing such that a line passes through the origin the above analysis can remain unchanged as long as the angles chosen for the coordinates of each line are consistent with choosing one of the faces of the drawing, with the origin on its boundary, as the central face under that frame of reference. The angles assigned to the coordinates of the lines will be the same as those assigned by any other parallel frame of reference with origin in this face. This does exclude some possible assignment of coordinates, when, for example, three lines intersect at the origin.

10. Conclusion

The graph theoretic properties of a line drawing can be expressed as equations and inequalities concerning the polar coordinates of the lines.

This has been used as the basis of a graph-drawing algorithm that respects extended collinearity constraints.

In the future, by examining the solubility of the betweenness inequalities, we may be able to characterise the complete set of constraints on the graph theoretic properties of straight-line drawings in the Euclidean plane. This examination may start by considering the betweenness inequalities as linear inequalities in the distance coordinates and using the fundamental theorem of linear inequalities, and other results from linear programming, to analyse their solubility.

11. References

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