An Accurate Operator Splitting Scheme for Nonlinear Diffusion Filtering

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HPL-2000-48(R.1)
August 1st, 2000*

nonlinear diffusion filtering, operator-splitting schemes, bilateral filtering

Efficient numerical schemes for nonlinear diffusion filtering based on additive operator splitting (AOS) were introduced in [15]. AOS schemes are efficient and unconditionally stable, yet their accuracy is limited. Future applications of nonlinear diffusion filtering may require better accuracy at the expense of a relatively modest cost in computations and complexity.

In this report we explore second order operator-splitting schemes for nonlinear diffusion filters. An accurate extension of the AOS schemes is suggested, which belongs to the alternating direction implicit (ADI) type methods, coupled with a symmetric splitting. The splitting is both additive and multiplicative, and preserves the accuracy of propagating with the Crank-Nicolson scheme in each dimension.

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* Internal Accession Date Only Approved for External Publication

1 Introduction

There are various applications of nonlinear diffusion filtering [8, 14] in image processing. Such 'filters' can be used for denoising, gap completion and computer aided quality control among many other tasks. These kind of applications demand high processing capabilities. The balance between high accuracy and computational efficiency is therefore an important issue in the design of such filters, that is expected to play an increasing role in future applications.

In this paper, an accurate numerical scheme is proposed which is an extension to Weickert-Romeny-Viergever's additive operator splitting (AOS) schemes [15]. These schemes are efficient and reliable, in the sense that they permit the use of larger time steps, whereas the straight-forward explicit schemes, that were proposed originally in Perona and Malik's classical paper [8], are restricted to small time steps in order to ensure stability. However, the AOS schemes are limited in their accuracy to first order in time even for the linear case. We therefore propose the use of alternating direction implicit (ADI) type schemes[7] which are second order in time for the linear case, analogous to the use of second-order splitting schemes in fluid-dynamics equations [5, 11]. We show that for certain potential image processing applications, the gain in accuracy is noticeable and the ADI schemes can be applied with a single iteration, effectively a large time-step, or very few iterations in order to approximate many iterations with smaller time-steps.

Nonlinear diffusion filtering is a continuous filter, formulated as a partial-differential-equation (PDE). The filter operation is practically performed by solving the nonlinear PDE numerically. Related approaches can be found in the literature [3, 9], in particular bilateral filtering [13], which is a noniterative way of smoothing an image while preserving its edges that is closely related to the geometrical framework in [10]. For illustration, Figure 2 demonstrates two different ways of performing edge-preserving smoothing on the original image in Figure 1. The result of using nonlinear diffusion filtering and the result of bilateral filtering is similar but not identical. In some applications one would prefer to use a single time-step or very few number of iterations for nonlinear diffusion filtering, in the same spirit as in bilateral filtering.

The outline of the paper is as follows. Section 2 presents the continuous model used throughout the paper for applying nonlinear diffusion as a filter. Section 3 illustrates one-dimensional schemes that are important in forming the basis for splitting higher-dimensional schemes across dimensions. In section 4, extensions of these one-dimensional schemes to higher dimensions are discussed and the motivation for using operator splitting schemes is given. Section 5 provides the operator splitting schemes which have been proposed by Weickert *et al.* in [15], all of which are accurate to first order in time for the linear case. Motivation for advancing a step forward by constructing second order operator splitting schemes is given. Consequently, in Section 6, two operator splitting schemes that preserve second-order for the linear case are introduced. The performances of all operator splitting schemes for nonlinear image diffusion are compared. It is also mentioned that a trial to achieve higher accuracy by

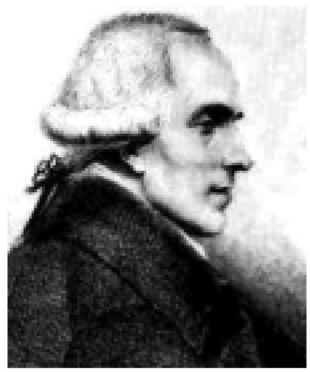

Figure 1: Original image: Laplace.

Figure 2: Edge-preserving smoothing: anisotropic diffusion with 20 time-steps of $\tau = 1.0$ (left) and Gaussian bilateral filtering with a 30×30 window size, $\sigma_D = 5.0$ and $\sigma_R = 30.0$ (right). σ_D and σ_R are bilateral filtering parameters, see [13] for details.

using Richardson's extrapolation [4] fails to produce better results in this context. Section 7 concludes this paper.

2 Nonlinear Diffusion Filtering

Let us first provide a model for nonlinear diffusion in image filtering. We briefly describe the filter proposed by Catté, Lions, Morel and Coll [2]. The CLMC filter is a version of the Perona and Malik model [8] for image selective smoothing that was used in [15] as a benchmark for studying various numerical schemes. The basic equation which governs nonlinear diffusion filtering is

$$\frac{\partial u}{\partial t} = div(g(|\nabla u_\sigma|^2)\nabla u), \tag{1}$$

where $u(x,t)$ is a filtered version of the original image. The original image $f(x)$ is given as the initial condition

$$u(x,0) = f(x), \tag{2}$$

and reflecting boundary conditions are used

$$\frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega, \tag{3}$$

where n is the normal to the image boundary $\partial\Omega$.

The goal of selective smoothing in edge-preserving applications is to reduce smoothing across edges. In order to achieve this goal, the diffusivity g is chosen as a rapidly decreasing function of the gradient magnitude (edge indicator). Specifically, the following form for the diffusivity is suggested in the CLMC filter

$$g(s) = \begin{cases} 1 & (s \le 0) \\ 1 - \exp\left(\frac{-3.315}{(s/\lambda)^4)}\right) & (s > 0), \end{cases} \tag{4}$$

where $\lambda = 10.0$ throughout this paper. In addition, CLMC suggest at each time step a presmoothing mechanism, in which the image u is convolved with a Gaussian of standard deviation σ to obtain u_σ. This can be achieved by solving the linear diffusion filtering ($g \equiv 1$)

$$\frac{\partial u_\sigma}{\partial t} = div(\nabla u_\sigma), \tag{5}$$

for a very small time step of size $T = \sigma^2/2$. This step is called regularization, or presmoothing, and can be approximated by any of the splitting schemes that will be mentioned in the paper. For example, a simple locally one-dimensional (LOD) scheme is a convenient choice. The freedom of choice in selecting any of the splitting schemes occurs because (5) is linear and consequently most splitting schemes that will be mentioned in Sections 5,6 degenerate to the same approximation. In the remaining of this paper, $\sigma = 0.25$ is chosen for the presmoothing.

The continuous image $f(x)$ can be considered as a discrete image, in particular a vector $f \in \mathbb{R}^N$ whose components f_i display the grey values at each pixel. Pixel i corresponds to the location x_i and h is the spatial grid spacing. Discrete times $t_k = k\tau$ are considered, where τ is the time step size. u_i^k denotes an approximation to $u(x_i, t_k)$. Having obtained u_σ, which from now on is referred to as u_i^k, the gradient magnitude can be approximated by a central difference scheme

$$|\nabla u_i^k| = \frac{1}{2} \sum_{p,q \in \mathcal{N}(\mathrm{i})} \left(\frac{u_p^k - u_q^k}{2h} \right)^2, \tag{6}$$

where $\mathcal{N}(\mathrm{i})$ is the set of all neighbors of a pixel i. Boundary pixels have only inner pixels as neighbors, as a result of the boundary conditions. In addition, closely following the notation used in [15], the diffusivities in their discrete form will be denoted by $g_i^k = g(u_i^k)$. In the next sections, numerical schemes are presented for the implementation of nonlinear diffusion filtering.

3 One-Dimensional Schemes

This section briefly describes the one-dimensional explicit and semi-implicit schemes, before we explore the Crank-Nicolson scheme. It is mentioned how all these schemes satisfy discrete nonlinear diffusion scale-spaces criteria, and in particular the accuracy of these schemes is discussed. For more details and theoretical considerations regarding the framework for discrete nonlinear diffusion scale-spaces, the reader is referred to [14, 15].

It follows from (1) that the basic equation which governs one-dimensional nonlinear diffusion filtering is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(g(|\nabla u_\sigma|^2) \frac{\partial u}{\partial x} \right). \tag{7}$$

A simple numerical scheme for solving this equation numerically was suggested by Perona and Malik [8]. It uses the following discretization

$$\frac{u_i^{k+1} - u_i^k}{\tau} = \sum_{j \in \mathcal{N}(\mathrm{i})} \frac{g_j^k + g_i^k}{2h^2} (u_j^k - u_i^k), \tag{8}$$

where $\mathcal{N}(\mathrm{i})$ is the set of two neighbors of i, one neighbor for the boundary pixels. A compact way of writing this scheme is

$$\frac{u^{k+1} - u^k}{\tau} = A(u^k) u^k, \tag{9}$$

where u^k is a signal vector of size N and $A(u^k) = (a_{ij}(u^k))$ is an $N \times N$ matrix whose elements are given by

$$a_{ij}(u^k) = \begin{cases} \frac{g_i^k + g_j^k}{2h^2} & j \in \mathcal{N}(\mathrm{i}), \\ -\sum_{n \in \mathcal{N}(\mathrm{i})} \frac{g_i^k + g_n^k}{2h^2} & j = i, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Isolating u^{k+1} on the left hand side, we obtain

$$u^{k+1} = (I + \tau A(u^k))u^k. \tag{11}$$

This scheme is known as an *explicit scheme*, since u^{k+1} is obtained explicitly from u^k without a matrix inversion. This scheme is simple, straight-forward, and computationally cheap because only matrix-vector multiplications are required. However, it is conditionally stable and therefore limited to small time steps. A rigorous way to analyze numerical schemes in our context is to verify that they satisfy six criteria in order to create a discrete scale-space [15]. The matrix on the right hand side (in this case, $I + \tau A(u^k)$) needs to satisfy continuity in its argument, symmetry, unit row sum, nonnegativity, positive diagonal, and irreducibility. It follows that these conditions are satisfied for (11) if $\tau < \frac{1}{2}$, assuming $h = 1$ (see [15] for exact details). This means that implementation of (11) is restricted by small time steps, and even though each iteration by itself is computationally cheap, as a whole, the efficiency for applying the filter can be improved. The improvement comes by a different numerical scheme

$$\frac{u^{k+1} - u^k}{\tau} = A(u^k)u^{k+1}, \tag{12}$$

Rearranging terms, so that u^{k+1} is on the left hand side and u^k is on the right hand side, we obtain

$$(I - \tau A(u^k))u^{k+1} = u^k. \tag{13}$$

This scheme is known as a *semi-implicit scheme*, since u^{k+1} is obtained implicitly from u^k by inverting a matrix. The use of *fully implicit schemes*, in which $A = A(u^{k+1})$, is more complicated to solve and therefore not considered in our case. Although a matrix inversion is in general an expensive $O(N^3)$ operation, the matrix in Equation (13) is tridiagonal, which can be inverted efficiently using the Thomas algorithm which is $O(N)$. Furthermore, the scheme is unconditionally stable, and in the rigorous framework one can verify that it satisfies all six criteria. However, both the explicit scheme and the semi-implicit scheme are only first order in time. A scheme which is a combination of (11),(13) and is second order in time for the linear case is the Crank-Nicolson scheme

$$(I - \tau A(u^k))u^{k+1} = (I + \tau A(u^k))u^k. \tag{14}$$

4 Higher-Dimensional Schemes

This section builds upon the one-dimensional semi-implicit scheme (13) and the Crank-Nicolson scheme to construct schemes for higher dimensions. It follows from (1) that the basic equation which governs m-dimensional nonlinear diffusion filters is

$$\frac{\partial u}{\partial t} = \sum_{l=1}^{m} \frac{\partial}{\partial x_l} \left(g(|\nabla u_\sigma|^2) \frac{\partial u}{\partial x_l} \right). \tag{15}$$

And, a straight forward extension to the one-dimensional semi-implicit scheme (13) is

$$\left(I - \tau \sum_{l=1}^{m} A_l(u^k) \right) u^{k+1} = u^k, \tag{16}$$

where the matrix $A_l(u^k)$ corresponds to the derivatives along the l-th coordinate axis. It is shown in [15] that the m-dimensional semi-implicit scheme unconditionally satisfies all the requirements of the discrete scale-space. However, the accuracy of the scheme in (16) is limited because it is built upon a one-dimensional scheme that is only first order in time. An extension to second-order in the linear case, is built upon the Crank- Nicolson

$$\left(I - \tau \sum_{l=1}^{m} A_l(u^k) \right) u^{k+1} = \left(I + \tau \sum_{l=1}^{m} A_l(u^k) \right) u^k. \tag{17}$$

It is worthwhile noticing that the only drawback when moving to higher dimensions is in the efficiency of (17): the matrix $\sum_{l=1}^{m} A_l(u^k)$ is no longer tridiagonal and therefore the matrix inversion at each time step is costly. This occurrence in higher-dimensional diffusion equations has been known since the early days of numerical solutions to parabolic PDEs. The work of Peaceman and Rachford [7] is a famous example for overcoming this problem by *splitting methods* [5, 6, 18]. For simplicity, let us consider $m = 2$ (two-dimensions) for the time being, noting that it is possible to extend splitting methods to three and higher dimensions. In addition, let us assume the case of a linear diffusion equation, $g = \alpha$, where α is constant. We start from the two-dimensional linear diffusion equation

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right). \tag{18}$$

The scheme in (17) now reads

$$\left(I - \alpha\tau \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right) \right) u^{k+1} = \left(I + \alpha\tau \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right) \right) u^k. \tag{19}$$

However, this scheme amounts to inverting a non-tridiagonal matrix at each time step, which is inefficient. The *alternating direction implicit* (ADI) scheme [7] suggests approximating the scheme in (19) the following way

$$\left(I - \alpha\tau\frac{\partial^2 u}{\partial x_1^2}\right)\left(I - \alpha\tau\frac{\partial^2 u}{\partial x_2^2}\right)u^{k+1} = \left(I + \alpha\tau\frac{\partial^2 u}{\partial x_1^2}\right)\left(I + \alpha\tau\frac{\partial^2 u}{\partial x_2^2}\right)u^k. \tag{20}$$

It is now possible to perform two half time steps, splitting the two dimensions such that in each half time step one of the two dimensions is treated implicitly

$$\begin{aligned}\left(I - \alpha\tau\frac{\partial^2 u}{\partial x_1^2}\right)u^{k*} &= \left(I + \alpha\tau\frac{\partial^2 u}{\partial x_2^2}\right)u^k \\ \left(I - \alpha\tau\frac{\partial^2 u}{\partial x_2^2}\right)u^{k+1} &= \left(I + \alpha\tau\frac{\partial^2 u}{\partial x_1^2}\right)u^{k*},\end{aligned} \tag{21}$$

where $k*$ is an intermediate time step. The ADI scheme is both efficient, since at each time step a tridiagonal matrix inversion is performed, and accurate to second order in time. Our goal is to seek a splitting scheme for the nonlinear case (16), that will be as good as the ADI scheme for the linear case. More precisely, it should amount to inverting tridiagonal matrices, unconditionally satisfy all discrete scale-space requirements, and retain the time accuracy which was achieved before the splitting by starting from accurate one- dimensional schemes.

5 Operator Splitting Schemes

Before we introduce second-order accurate splitting schemes for solving (17), let us review the first-order accurate splitting schemes which have been proposed in [15]. The simplest splitting scheme that might be considered is the *locally one-dimensional* (LOD) scheme

$$\prod_{l=1}^{m}\left(I - \tau A_l(u^k)\right)u^{k+1} = u^k, \tag{22}$$

which belongs to the general class of *multiplicative operator splitting* schemes. It is the most efficient and straight-forward for implementation. However, the main drawback of the LOD scheme is that the system matrix in (23) is non-symmetric, which violates one of the criterions for discrete diffusion scale-spaces as proposed in [15]. Because of the non-commutativity of the operators A_l, the order of applying these operators can affect the final result. For example, the filtered two-dimensional image will not be the same after a rotation by 90 degrees. Figure 3 illustrates this disadvantage and motivates the search for a symmetric splitting which does not suffer from this deficiency. It is worthwhile noticing that the splitting suggested in (22), when applied to any of the one-dimensional schemes discussed in section 3, results in a multi-dimensional scheme that is first-order accurate in time.

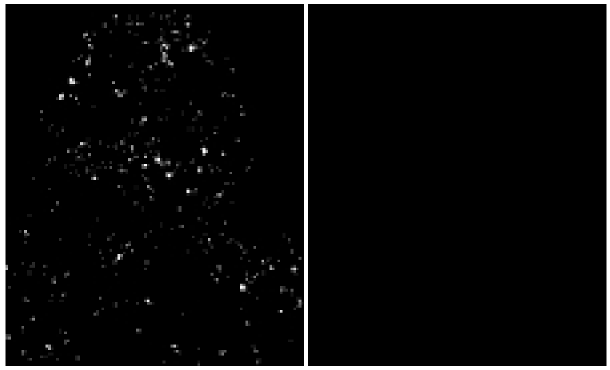

Figure 3: The difference between applying first the operator which corresponds to the x-axis and then the operator which corresponds to the y-axis and vice versa, on the original image in Figure 1. The LOD scheme (left) is sensitive to the order, whereas the AOS scheme (right) is independent of the order. Nonlinear diffusion filtering was performed with 20 time-steps of $\tau = 1.0$

The splitting operator scheme proposed in [15] as the method of choice, the *additive operator splitting* (AOS), is

$$u^{k+1} = \frac{1}{m} \sum_{l=1}^{m} \Big(I - m\tau A_l(u^k) \Big)^{-1} u^k. \tag{23}$$

Unlike the LOD scheme the AOS scheme is symmetric, see Figure 3, and unconditionally satisfies all discrete diffusion scale-space requirements. It is almost as efficient as the LOD scheme; instead of applying the operators in a pipeline, one calculates the operators independently and then sums them up at each time step. It is therefore a reliable and efficient scheme. However, similar to the LOD scheme it is first-order accurate in time. Moreover, it is less accurate then the LOD scheme since operators of type $(I - m\tau A_l)^{-1}$ that are used in the AOS scheme, describe one-dimensional diffusions with a step size $m\tau$, whereas operators of the type $(I - \tau A_l)^{-1}$ that are used in the LOD scheme possess smaller time steps when stepping in one dimension.

Let us illustrate how in some potential applications, the better accuracy of the LOD scheme can be noticeable. We compare the AOS and the LOD schemes' performances on the veneer image in Figure 4. Figure 7 is the reference image, after applying nonlinear diffusion filtering with 256 time steps of $\tau = 0.78$ each. We now keep the time constant ($T = 200$) and decrease the number of iterations while increasing the duration τ of each time step accordingly. In the reference image, Figure 7, the LOD and the AOS schemes results are practically identical. As we increase the time steps, the results start to deviate from the reference by a certain amount which is related to the accuracy of the scheme. Figures 5 and 6 demonstrate nonlinear diffusion filtering approximated by two time steps of $\tau = 100$ and eventually one time step of $\tau = 200$, the LOD scheme is found to be more accurate than the AOS scheme, as it is closer to the reference filtered image. These results motivate us to look for second-order accurate schemes, as well as symmetric accurate ones, that will lead to higher accuracy compared to the AOS schemes.

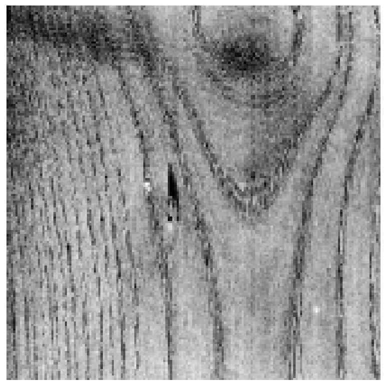

Figure 4: Original image of a veneer, taken from [17] with permission.

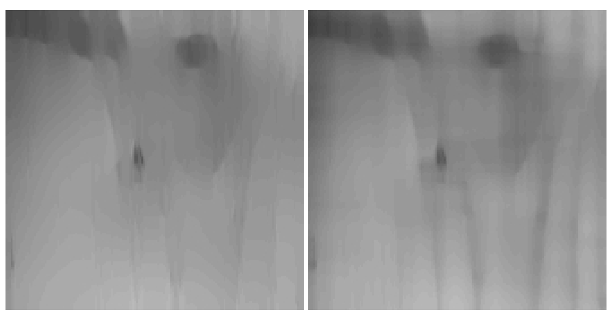

Figure 5: Checking accuracy: LOD vs. AOS, nonlinear diffusion filtering with one time-step of $\tau = 200.0$.

6 Accurate Operator Splitting Schemes

In this section, we propose accurate operator splitting schemes. For simplicity, we will choose the two dimensional case which corresponds to images, noting that these schemes can easily be extended to three and higher dimensions. We then compare the performance of the additive-multiplicative operator splitting (AMOS) schemes and the AOS schemes, and to bilateral filtering.

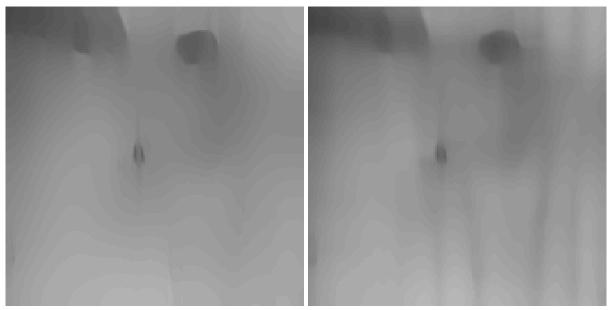

Figure 6: Checking accuracy: LOD vs. AOS, nonlinear diffusion filtering with two time-steps of $\tau = 100.0$.

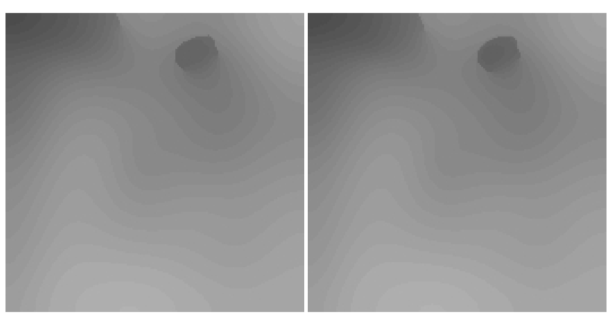

Figure 7: Reference image: LOD vs. AOS, nonlinear diffusion filtering with 256 time-steps of $\tau = 0.78$.

Following "Strang Splitting" [5, 11] which is known from the equations of fluid dynamics, our first proposal is to use a scheme that belongs to the family of multiplicative operator splitting schemes in order to increase the order of accuracy achieved by the LOD scheme (23). The new scheme that is based on Strang splitting is different from the LOD scheme, and we refer to it as the MOS (Multiplicative Operator Splitting). If we use the semi-implicit scheme in each one of the dimensions, for simplicity in writing, the MOS scheme is given by

$$\left(I - \frac{\tau}{2} A_1(u^k)\right)\left(I - \tau A_2(u^k)\right)\left(I - \frac{\tau}{2} A_1(u^k)\right) u^{k+1} = u^k. \tag{24}$$

where we wrote the semi-implicit version of the MOS scheme, without the explicit right-hand-side that appears in the Crank-Nicolson version of the MOS scheme. The MOS scheme for the linear case preserves second order accuracy provided the Crank-Nicolson scheme is used in each one of the dimensions, instead of the semi-implicit scheme as in (24) which is only first-order accurate. This is an advantage over the LOD and AOS schemes, which do not preserve the second order accuracy of the Crank-Nicolson. However, the splitting is not symmetric and the non-symmetry which was also a deficiency in the LOD scheme has not been recovered. In practice, results after a rotation by 90 degrees as in Figure 3 appear cleaner with the MOS scheme compared to the LOD scheme, but there is no guarantee the symmetry criterion in the list of criteria for discrete diffusion scale-spaces will be satisfied. Therefore, after several trials in which the rotation by 90 degrees was still noticeable, we abandoned the MOS scheme. A second order accurate split operator scheme which unconditionally meets the scale-spaces criteria is desired.

Motivated by ADI [7] which was mentioned in Section 4 as a favorable splitting scheme for the linear diffusion equation, we wish to combine the merits of the AOS scheme as a symmetric scheme, together with the family of multiplicative operator splittings (to which the ADI belongs, as well as the first order LOD and the first and second order MOS schemes). Multiplicative operator splittings are known in general to be more accurate than the AOS schemes. We therefore propose another scheme, also mentioned by Strang in [11, 12], which is both additive and multiplicative operator splitting (AMOS)

$$u^{k+1} = \frac{1}{2}\left[\left(I - \tau A_1(u^k)\right)^{-1}\left(I - \tau A_2(u^k)\right)^{-1} + \left(I - \tau A_2(u^k)\right)^{-1}\left(I - \tau A_1(u^k)\right)^{-1}\right] u^k \tag{25}$$

As in (4), Equation (5) applies the AMOS scheme to the semi-implicit scheme. Such a combination is known in the literature [4] as the approximate factorization implicit (AFI) scheme, which is first order accurate in time. However, even in the case where it is built upon the semi-implicit scheme, the AMOS scheme is expected to be more accurate than the AOS scheme (see discussion on LOD) while preserving symmetry. Furthermore, it is possible to achieve second order accuracy by applying the AMOS scheme on the Crank-Nicolson scheme. At each time step, two calculations are performed

$$\begin{aligned} \left(I - \tau A_1(u^k)u\right) u^{k*} &= \left(I + \tau A_1(u^k)u\right) \quad u^k \\ \left(I - \tau A_2(u^k)u\right) u^{k+1} &= \left(I + \tau A_2(u^k)u\right) \quad u^{k*}, \end{aligned} \tag{26}$$

and

$$\begin{aligned} \left(I - \tau A_2(u^k)\right) u^{k*} &= \left(I + \tau A_2(u^k)\right) \quad u^k \\ \left(I - \tau A_1(u^k)\right) u^{k+1} &= \left(I + \tau A_1(u^k)\right) \quad u^{k*}. \end{aligned} \tag{27}$$

After the time step is completed, the two results are averaged together which ensures a symmetric splitting. Although the directions are not alternating in each of the two calculations, i.e. the forward and backward Euler are performed on the same direction, in effect this scheme belongs to the family of alternating direction implicit (ADI) type methods. In our experiments, alternating the directions as in the classical ADI that works well on linear parabolic PDEs, produced some artifacts when applied to nonlinear diffusion. Therefore, we refer to (26),(27) as ADI, whereas (25) is AFI. The ADI scheme in (26),(27) leads to visually more accurate results as can be seen in Figures 8,9,10. We take 512 time steps of 0.05 as a reference, then decrease the number of iterations to check the deviation from the reference. First, we observe that the ADI scheme acts as a slightly better filter than the AOS scheme already in the reference image calculation, Figure 8. As we decrease the number of iterations, we observe that the deviation from the converged result is smaller with the ADI scheme than with the AOS scheme. Filtering effect becomes stonger in the ADI scheme, while preserving fine details, which is an indication that the ADI scheme is visually more accurate than the AOS scheme, Figures 9,10.

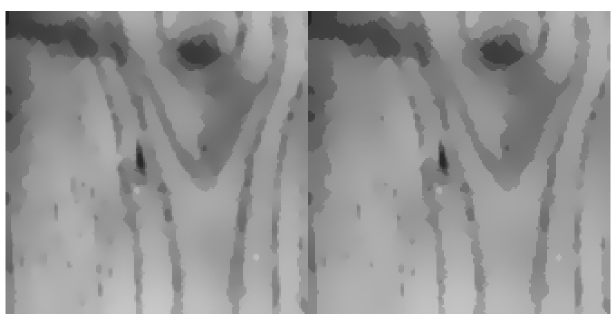

Figure 8: Reference: AOS vs. ADI, nonlinear diffusion filtering with 512 time steps of $\tau = 0.05$.

Quantitative examination of the deviations from the reference is calculated as follows. Starting from the original image in Figure 11, which is a texture image taken from a neutron diffraction experiment, we first observe in Figure 12 that simple normalized bilateral filtering can not reach the same degree of filtering as in nonlinear diffusion, since the choice of time propagation is flexible in the PDE approach, whereas it is fixed by the normalization factor of the bilateral filter. Figures 13,14,15 show the comparison in terms of accuracy between the AOS, AFI and ADI schemes, which are discussed next. In terms of speed, the AOS and AFI schemes in actual simulations indicate that the AFI scheme takes roughly 1.5

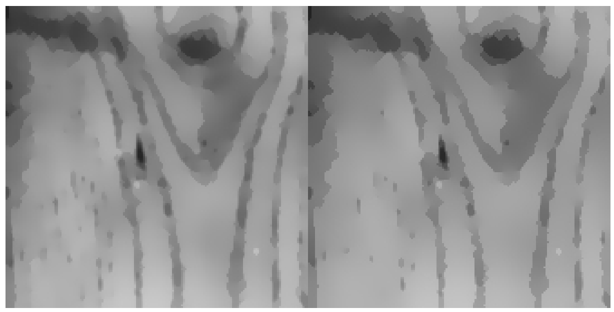

Figure 9: Reference: AOS vs. ADI, nonlinear diffusion filtering with 32 time steps of $\tau = 0.875$.

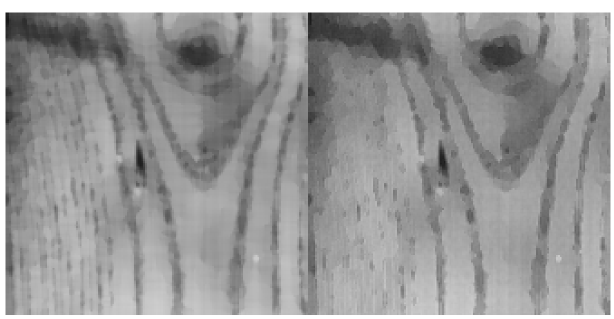

Figure 10: Reference: AOS vs. ADI, nonlinear diffusion filtering with one time step of $\tau = 28.0$.

the time it takes the AOS scheme to perform the filtering. The ADI scheme is roughly a factor of 2 to 3 longer in processing the images in Figures 4,11, relative to the AOS scheme. We note that simply increasing the time step with the AOS scheme by this ratio does not produce the fine filtering that is achieved with the ADI scheme.

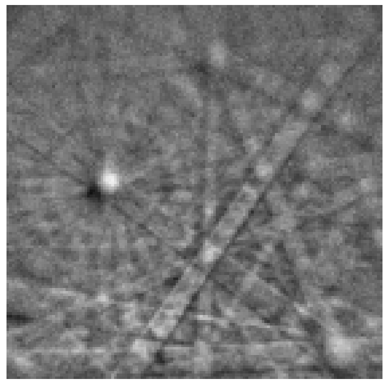

Figure 11: Original texture image.

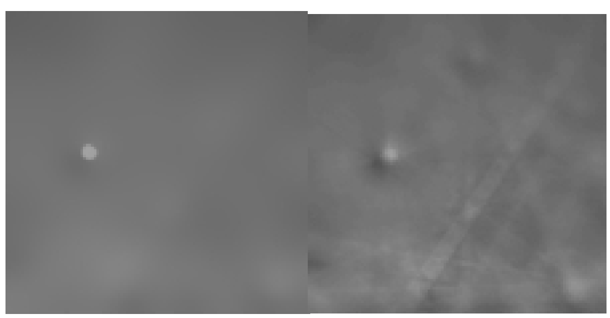

Figure 12: Reference: AOS, 2000 time steps of $\tau = 0.1$ (left) and Gaussian bilateral filtering with a 60×60 window size, $\sigma_D = 10.0$ and $\sigma_R = 80.0$ (right). σ_D and σ_R are bilateral filtering parameters, see [13] for details.

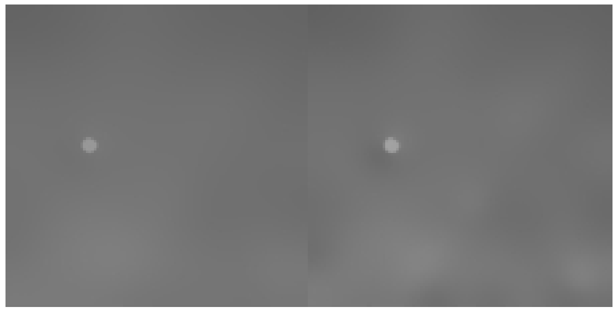

Figure 13: Reference: ADI vs. AOS, nonlinear diffusion filtering with 2000 time steps of $\tau = 0.1$.

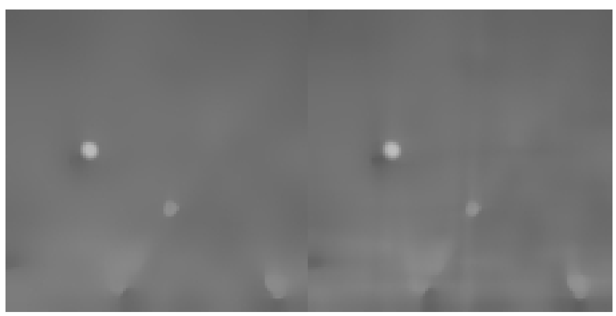

Figure 14: AFI vs. AOS, nonlinear diffusion filtering with four time step of $\tau = 50.0$.

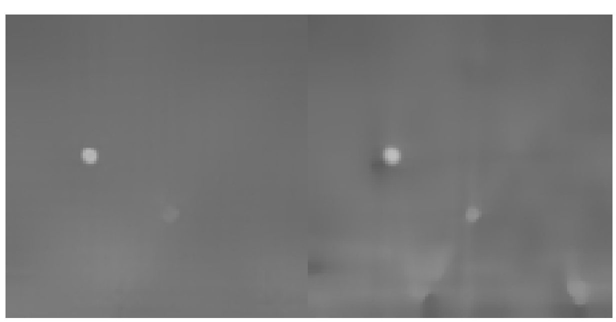

Figure 15: ADI vs. AOS, nonlinear diffusion filtering with four time step of $\tau = 50.0$.

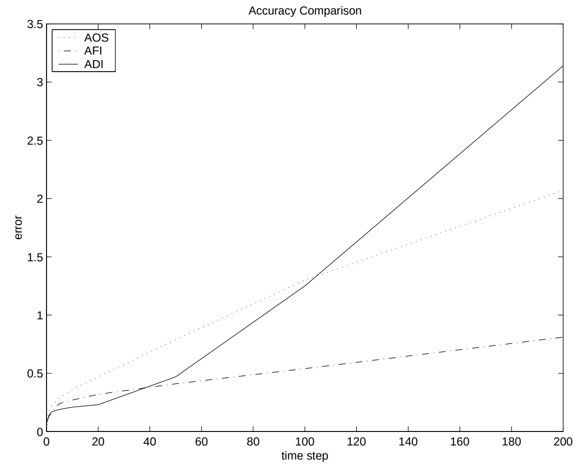

Figure 16: Comparison of error estimation for different time steps: AOS, AFI, ADI

Table I: l_2 norm error estimation			
τ	AOS	AFI	ADI
0.25	0.09	0.06 %	0.08 %
0.5	0.13	0.1 %	0.11 %
1.0	0.17	0.14 %	0.13 %
2.0	0.22	0.17 %	0.17 %
5.0	0.29	0.24 %	0.19 %
10	0.36 %	0.27 %	0.21 %
20	0.47 %	0.32 %	0.23 %
50	0.79 %	0.41 %	0.47 %
100	1.3 %	0.54 %	1.25 %
200	2.07 %	0.81 %	3.14 %

In Table I, the relative l_2 norm errors are calculated for the example in Figures 13,14,15 as follows. Let v denote the reference solution: AOS, $\tau = 0.1$, in the case of the AOS and AFI schemes, and ADI, $\tau = 0.1$, in the case of the ADI scheme. Let u denote the approximate solution in each of the schemes. The relative error percentages are calculated by

$$error = \frac{\|u - v\|_2}{\|v\|_2}. \tag{28}$$

Note that the small relative error percentage values do not reflect the strength of the deviations and accuracies, since the large propagation time (T=200) produces very smooth images, where the differences between the schemes appear only in small regions near prominent features within the original image. Also the comparison with the ADI scheme is done for a separate reference frame, since even with a small time step the ADI scheme acts as a better filter, see Figures 8,13, and hence its reference to measure deviations should be different. From Table I and the plot in Figure 16 it can be observed that up to a time step of $\tau = 50.0$, the ADI scheme is the most accurate, which is expected because the Crank-Nicolson is used as its building block. With very large time steps of more than $\tau = 50.0$, the AFI scheme is the most accurate scheme, probably because the higher order error terms affect the accuracy of the ADI scheme. The Crank-Nicolson scheme stops acting as the most accurate one-dimensional scheme, and among the schemes which are based on the semi-implicit scheme as their building block, the AFI scheme will produce more accurate results than the AOS scheme since the AMOS scheme is a more accurate splitting scheme than the AOS scheme at the expense of some increase in computations. Finally, we tried to obtain better accuracy out of the results in Figure 15 by using Richardson's extrapolation [4] for our case

$$R_I(\tau/2) = \frac{4R(\tau/2) - R(\tau)}{3}, \tag{29}$$

where $R_I(\tau/2)$ denotes an improved result, using a time grid with a spacing of $\tau/2$ or coarser. $R(\tau/2)$ and $R(\tau)$ are the results of applying nonlinear diffusion filtering for time steps $\tau/2$ and τ, respectively. Our trials (with $\tau = 50$) failed to show an improvement of $R_I(\tau/2)$ relative to $R(\tau/2)$. An improvement is not guaranteed to begin with, since our equation is nonlinear and the solution is non-smooth.

7 Conclusions

In this paper, accurate splitting operator schemes were proposed for performing nonlinear diffusion filtering. They are gradually constructed by reviewing other schemes which are relevant and have been suggested in this context. Comparing two proposed schemes which satisfy all scale-space criteria of Weickert *et al's* [15] LOD and AOS schemes, it is found that higher order of accuracy can be visually inspected and might become a desirable feature in some future applications.

The two splitting methods which unconditionally satisfy all discrete scale-space criteria are Weickert *et al's* AOS scheme and our proposed scheme, the AMOS scheme. Both are reliable, simple and parallalizable [16] splittings for implementation. The AOS scheme is more efficient than the AMOS scheme in its first-order form, the AFI scheme, by approximately a factor of 1.5, and the AMOS scheme in its second-order form, the ADI scheme, by a factor of 2 to 3, depending on the efficiency of the implementation. Although the AOS remains the simplest and most efficient choice for implementation, in the arsenal of numerical schemes for performing nonlinear diffusion filtering the AMOS scheme can be considered as an extension for applications that require high accuracy. Multiplicative operator schemes are in general more accurate than their additive counterparts, and the combination of the two in the AMOS schemes ensures both symmetry and better accuracy at the expense of an increase in execution time.

In addition, it is interesting to note that nonlinear diffusion filtering can produce a convincing edge-preserving smoothing, as in Figure 10, with a single large time step. The close relationship between nonlinear diffusion and bilateral filtering was described in [1], and in this paper an attempt to use nonlinear diffusion noniteratively, by using accurate operator splitting schemes, produced some more insights.

Acknowledgments

We thank Moshe Israeli from the computer science department at the Technion, Israel Institute of Technology, for his valuable comments. We also thank Joachim Weickert from the department of mathematics and computer science at the University of Mannheim, Germany, for kindly providing us the original image of the veneer in Figure 4.

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