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Invariant Signatures from Polygonal Approximations of Smooth Curves

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In this paper we propose to use invariant signatures of polygonal approximations of smooth curves for projective object recognition. Similar signatures have been proposed previously as simple and robust signatures. However, they were known to be sensitive to the curve sampling scheme and density, and worked well mainly for intrinsically polygonal shapes. This paper proposes a re-sampling method for arbitrary polygonal approximations of smooth curves. The proposed re-sampling provides for weak-affine invariant parameterization and signature. Curve templates characterized by a scale space of these weak-affine invariant signatures together with a metric based on a modified Dynamic Programming algorithm can accommodate projective invariant indexing.

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1 Introduction

An invariant signature of a planar curve is a unique description of that curve that is invariant under a group of viewing transformations. Namely, all curves which are transformations of each other, have the same signature, whereas the signatures of all other curves are different. Invariant signatures may be used to index [2, 3, 6, 8, 11, 13, 20], or to detect symmetries [5, 17] of planar curves under viewing transformations.

Invariant signatures of planar curves are the subject of many research papers [1, 2, 3, 8, 12, 13, 19, 20], to name a few. Generally speaking, in order to describe planar curves under a group of transformations, one has to employ two independent local descriptors, which are invariant under the required group of transformations. Namely, two numbers, which are well defined on small curve segments and change whenever the curve is changed, unless the change is a transformation under which these numbers should be invariant. The above is expressed independently in many ways:

- For every curve point there are two numbers which specify a point in R^2. The full curve defines therefore a unique (usually continuous) arrangement of points in R^2, which characterizes the curve and all its transformations [2, 6].
- Let one invariant descriptor be used for invariant reparameterization (the simplest invariant reparameterization is usually called the arclength of the transformation group). Let the other descriptor be the signature value on the invariant position determined by the parameterization. This description is a scalar function over S^1, which describes the curve up to an unknown initial curve point (or equivalently the phase in S^1).
- When the two descriptors have a differential formulation, the curve is a solution of the differential equations and the required initial conditions. Two independent descriptors are needed since each limits the locus of the 'next curve point' to a one dimensional manifold (the 'next point' is the intersection of these manifolds).

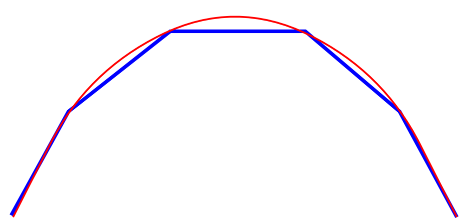
Let us consider the boundary condition mentioned in the last item. Since the description is invariant under a group of transformations, one should be able to reconstruct all the instances of that transformation from the same signature (though necessarily from different boundary conditions). Table 1 details the common groups of viewing transformation, and the associated free parameters. In order to accommodate the dimensionality of the transformation groups we necessarily need more boundary conditions for more complicated groups.

Group	Transl.	Rotation	Scale	Aspect R.	Skew	Tilt	Total
Translation	2	-	-	-	-	-	2
Euclidean	2	1	-	-	-	-	3
Similarity	2	1	1	-	-	-	4
Weak-Affine	2	1	-	1	1	-	5
Affine	2	1	1	1	1	-	6
Projective	2	1	1	1	1	2	8

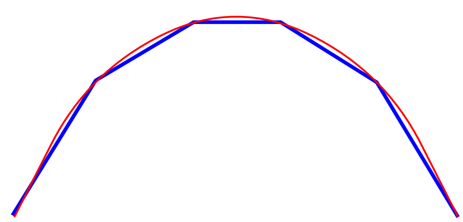
Table 1: Degrees of freedom for common viewing transformation groups.

It is indeed a known fact that the differential signatures of more complicated groups of transformations have higher degrees of derivatives (see e.g. [3]). This, in turn causes practical problems when one tries to implement the theory for complicated groups.

An accepted solution to the high derivative problem is to trade derivatives for features, which are integral in nature, and thus more stable. The integral features are local applications of global geometric invariants [1, 20]. They are sometimes called semi-differential invariants [12], and in other cases they are formulated as stable numerical schemes for differential invariants [6, 7]. For example, setting a grid of equally spaced points on the curve, as in Figure 1a, is the integral equivalent to the Euclidean arclength described differentially[1] as $ds_E = \sqrt{X_p^2 + Y_p^2}$. Alternatively, setting a grid such that the area enclosed between edges and curve segments is equal, as in Figure 1b, is the integral equivalent to the weak-affine invariant arclength $dS_a = \sqrt[3]{X_p Y_{pp} - Y_p X_{pp}}$.



a



b

Figure 1: Invariant grids on a curve: a. Euclidean arclength, b. affine arclength

[1]We use the following notations $X_p = \frac{dX}{dp}$, $X_{pp} = \frac{d^2X}{dp^2}$.

In this paper we address two problems:

- The simplest parametric curve description is polygons. Polygonal signatures have been addressed before [5, 9], however, usually one had to resort to assumptions that are often unrealistic in practical cases. Signatures need to address the situation of polygonal approximations of smooth curves, rather then the easier case of intrinsically polygonal shapes. They should not depend on invariant sampling as in [7], and be robust to non stationary sampling, i.e. different sampling density in different, and possibly even within the same approximations.
- Both differential an non-differential descriptors for transformation groups that are more complicated than the weak-affine group are complex and unstable. With the notable exception of the robust methods by Weiss [19], who fits canonical curves to discrete curve segments. This method can however not be implemented for sparsely sampled curves - i.e. polygons. Another effort to approximate complex invariants by simpler ones [11], is more in the spirit of this paper. Although we do not approximate complex invariants, but rather accommodate the difference between the required invariance and the available (simpler) signature by a tailored metric.

In the sequel we propose to solve these problems. Specifically we propose a re-sampling method for polygonal shapes that is invariant under weak affine transformations. Namely, two polygonal versions of the same curve, or its weak affine transform, with arbitrarily different sampling schemes can be up-sampled so that the new sampling density on both curves is similar, see Figure 2. Consequently the polygonal signatures of the up-sampled polygons are similar. In addition we propose a method to use our weak affine signature, to index or detect symmetries in curves under affine and projective transformations.

In the next section we detail the proposed polygonal re-sampling. In Section 3 we present the second invariant descriptor used for signature value. In Section 4 we propose a metric for matching two signatures. We start with matching for invariance to weak affine transformations in 4.1, and gradually elaborate to accommodate invariant indexing and symmetry detection under affine 4.2, and projective 4.3 transformations. Section 5 is a brief Summary of this paper.

Figure 2: Polygonal up-sampling of arbitrarily pre-sampled curves.

2 Re-Sampling

Calabi et al. [6, 7] have proposed a sampling method for smooth curves that converges to the weak affine arclength. In this section we present a method to sample polygonal approximations of curves in a manner that is consistent with the weak affine arclength.

Note that a curve that has already been sampled in a manner that is consistent with the weak affine arclength may be re-sampled by polygonal up/down sampling. On the other hand if a curve has been sampled in another manner it can not be simply re-sampled to be consistent with the weak-affine arclength. Specifically, it cannot be up-sampled using the standard weak affine invariant sampling methods designed for smooth curves.

It as already been suggested [1, 7] to sample curves in a manner invariant to weak affine transformations by setting sampling points so that the area enclosed between the curve and line segments connecting them is constant[2]. The problem is that this and other area measures proposed previously for weak affine arclength are, to the best of our knowledge, not well defined for polygons, particularly for up-sampling purposes.

The solution proposed in this paper is based on the fact that one can use any area that is defined well on the given curve. Specifically we propose the following scheme defined well for polygonal shapes as well as smooth shapes, see Figure 3a. Given a point S on the curve, the next point S' is found by first determining an anchor point A based on an enclosing area of predefined size $\epsilon > 0$, and then determining S' such that the line segment based on A and sweeping the curve starting at S covers an area of $\alpha\epsilon$ for some $\alpha \in [0, 1]$. Note that both A and S' are, by definition, invariant under weak affine transformations.

[2] The use of the cube root of the area is important only if one needs to preserve the linearity of the arclength. For example, in order to sample a curve twice as dense, one needs to use an area that is 8 times smaller.

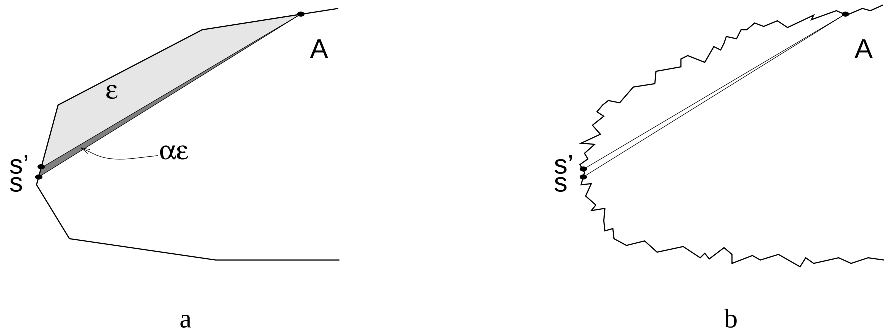


Figure 3: Polygonal re-sampling invariant to weak affine transformations.

When $\epsilon \to 0$ the resulting sub-sample density is proportional to the weak-affine arclength, nevertheless, one can use any predefines $\epsilon > 0$ and still keep the invariance to weak affine transformations.

The parameter ϵ is, in a certain sense, a scale parameter, filtering out small curve perturbations, so that perturbations as in Figure 3b are filtered out. Since our main interest is polygonal shapes, it should be noted that polygons have intrinsic artifacts (the vertices), whose influence we need to filter out. A thumb rule to select ϵ is therefore an area large enough, so that any boundary segment it delimits contains at least two vertices (note that for any $\epsilon > 0$ delimited boundary segments contain at least one vertex). Since the re-sampling needs to be invariant to a wide range of original polygonal approximations, we need to determine ϵ considering the worst expected sampling density in a given application. Let us note again that ϵ can be arbitrarily large and still be invariant to weak affine transformations. If for a large ϵ one still needs a dense sub-sampling, one can always resort to small α.

3 Signature Value

In this section we describe the proposed second invariant, the signature value. Like the first invariant used for invariant arclength, the signature value is based on a local application of global geometric invariants. As already proposed by others, [1] we advance an invariant curve segment on the curve in both directions, and define the signature value to be the area of the triangle defined by the current, forward, and backward points. Note, that it is not recommended to use the invariant arclength proposed in the previous section as a measure for the forward/backward advance unless the invariant distance advanced to either sides is larger than $1/\alpha$. Otherwise, polygon artifacts might influence the signature value (e.g. all the invariant points might be located on the same polygon edge).

The proposed signature uses the anchor point described in he previous section as the forward invariant point, and a symmetrically obtained point as the backwards one. The signature value is the area ratio between the triangle area and ϵ.

Note that:

- The proposed signature value does not approximate the invariant curvature.
- For $\epsilon \to 0$ on smooth curves the absolute value of the signature converges to 6, however since in our implementations ϵ is constrained from below, this is not the usual case with the proposed signature. The reader is referred to Figure 4 for an example of the descriptive nature of the signature.
- Like the invariant arclength, the proposed signature value is not full affine invariant because of the need to determine ϵ.

4 Signature Matching

In this section we discuss the metric used to compare signatures. Traditionally, signatures are compared using the l^2 distance or similar measures. We argue that in case of small perturbations in the original shape (as may be expected in our - pre-sampled curves) one may have to resort to more complicated distance measures. We also show that it is possible to use the proposed metric in order to overcome, the otherwise complicated problem, of invariance to affine and projective transformations.

4.1 Weak Affine

Since our subject is curves that have already been sampled by arbitrary methods, two polygonal approximations of the same curve are, strictly speaking, different curves. Therefore, we cannot expect their invariant signatures to be identical. Although, if the scale parameter ϵ is chosen appropriately[3], we expect the signature functions of two polygon instances of the same curve to be *similar* in both the value and the arclength dimensions.

Value perturbations are trivially dealt with by standard metrics (e.g. l^2). However, to deal with arclength perturbations we will usually have to resort to more complicated metrics. Figure 4 depicts different polygonal approximations of the same smooth curve, and the corresponding weak-affine invariant signatures. The first (Red) polygon has 276 vertices, the

[3] See discussion in the end of Section 2.

second (Blue) has 125, and the last (Black) has 55 vertices. Notice that the last approximation is slightly too sparse for the chosen ϵ^3, nevertheless, the proposed metric will handle this case well.

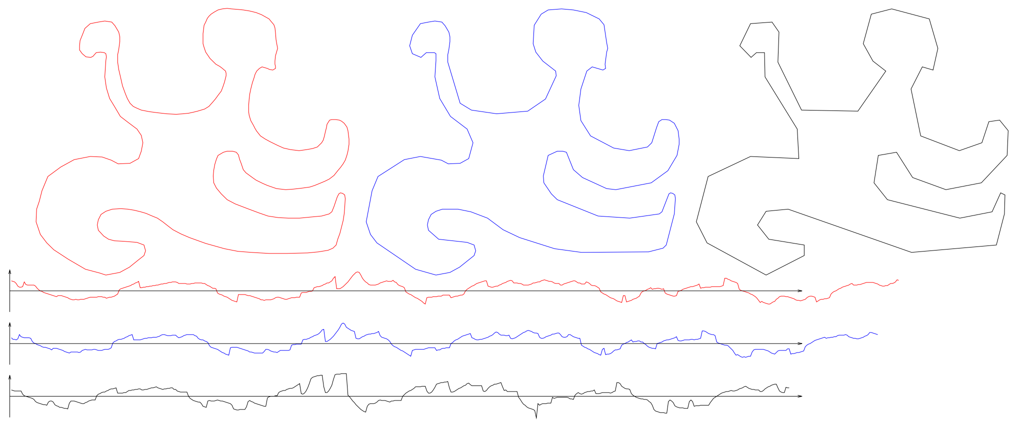


Figure 4: Polygonal approximations of a smooth curve and corresponding weak-affine invariant signatures.

We propose to employ a composite measure based on standard metrics (e.g. l^2) both in the value and in the arclength dimensions. The proposed metric is the well known warp metric[4] used in many fields of engineering, see e.g. [10, 14, 15, 16, 18]. It is based on the following minimization problem:

Given two signature functions $V_{\mathrm{Q}}(i), \quad i \in \{1,2,\ldots\ L_{\mathrm{Q}}\}$ for the query curve and $V_{\mathrm{T}}(i), \quad i \in \{1,2,\ldots\ L_{\mathrm{T}}\}$ for the template, we look for the optimal warp or reparameterization function $\Psi: \quad \{1,2,\ldots\ L_{\mathrm{Q}}\} \to \{1,2,\ldots\ L_{\mathrm{T}}\}$ to minimize the composite distance function

$$D\left(V_{\mathrm{Q}}(\cdot), V_{\mathrm{T}}(\cdot)\right) = \min_{\Psi} \left\{ \Sigma_{i=0}^{L_{\mathrm{Q}}} \left(V_{\mathrm{Q}}(i) - V_{\mathrm{T}}(\Psi(i))\right)^2 + \Lambda \cdot \left(\nabla\Psi(i)\right)^2 \right\}$$

subject to constraints on Ψ (we use: $\Psi(0) = 0$, and $0 \leq \nabla\Psi(i) \leq 2$).

The optimization problem described above is solved in $O(L_{\mathrm{Q}} \cdot L_{\mathrm{T}})$ time by the *Dynamic Programming algorithm*. Dynamic programming is based mainly on the following recursion[5]:

Given a series of solutions to the warping problems of a given part of the query signature $V_{\mathrm{Q}}(i), \quad i \in \{1,2,\ldots\ k\}$, to all the sub-parts of the template signature $V_{\mathrm{T}}(i), \quad i \in \{1,2,\ldots\ j\}$, with $j \in \{1,2,\ldots\ L_{\mathrm{T}}\}$, we can solve the series of warping problems from a longer part of the

[4] Known also as Dynamic Warping, reparameterization, and Viterbi algorithm.

[5] Refer to [15] for a detailed description of Dynamic Programming.

query $V_{\mathrm{Q}}(i), \ \ i \in \{1, 2, \dots \ k+1\}$, to each of the sub-parts of $V_{\mathrm{T}}(\cdot)$. Simply select one of the optimal paths of the given solution, and extend it by a single further match to minimize the total warp error composed of: (1) The total error at the given path (2) The warp error due to the required step (3) The additional match. Figure 5 depicts the recursive completion process.

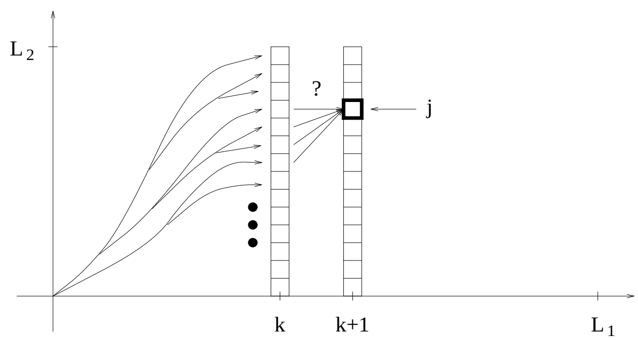


Figure 5: Recursive path extension in Dynamic Programming.

The recursive process is initiated according to the boundary conditions. In our implementation we assume $\Psi(0) = 0$, and thus $\{V_{\mathrm{Q}}(0)\}$ matches $\{V_{\mathrm{T}}(0)\}$, and none of the other sub-parts of V_{T}. In real applications one can not take initial matching for granted. A more realistic initial condition for signatures on S^1 is described in [21].

The recursion is terminated similarly, according to the boundary conditions. If we insisted that $\Psi(L_{\mathrm{Q}}) = L_{\mathrm{T}}$, we would have chosen the resulting match of the full signature of V_{Q} to the full signature of V_{T}. However, we chose to apply a more relaxed boundary condition, as in [16], and to select the best match of the full signature V_{Q} to either of the longer sub parts of V_{T}.

4.2 Full Affine

The weak affine signature is invariant to full affine transformations only up to the scale parameter ϵ. Note that all the other descriptors are derived from ϵ via area ratios, which are in turn invariant to full affine transformations [1]. Thus, representing a plane curve by a set of signatures representing a range of ϵ values, instead of a single signature corresponding to a specific ϵ_0, makes it possible to identify the planar curve under affine transformations.

The practical question is naturally, how many signatures to keep ? The range of ϵ should be determined by the range of scales relevant to the application in mind. Specifically, if we expect the affine scale to be in the range of $\times 0.5$ to $\times 2$ relative to the template, we will need signatures with scale parameters in the range $0.25 \times \epsilon_0$ to $4 \times \epsilon_0$. As for the number of ϵ values in the range, they have to be selected so that the signatures of intermediate scales will be similar to one of the represented signatures.

Although we have not proven the following conjecture, we found out empirically, that signatures change slowly with ϵ. Moreover, the fact that we use a warp metric reduces the influence of arclength deformations, leaving us mainly with the relatively gradual value changes. Figure 6 depicts the way signatures change with ϵ.

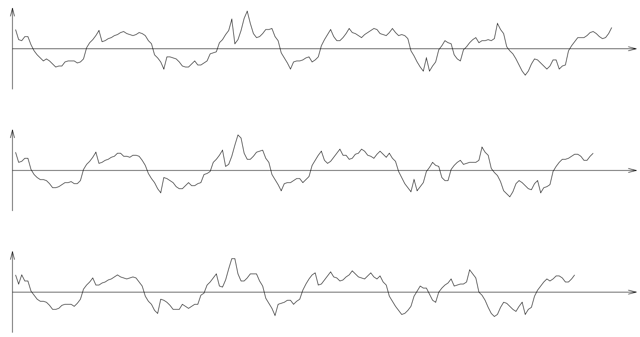


Figure 6: Weak-affine invariant signatures for different ϵ values ($\epsilon = 1100,\ 1200,\ 1300.$)

4.3 Projective

In this section we show how a modified warp distance can accommodate shape indexing under projective transformations.

$$\begin{pmatrix} \hat{X} \\ \hat{Y} \end{pmatrix} = \frac{1}{1 + w_x x + w_y y} A \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

Given a small neighborhood in the plane the projective transformation can be approximated by an affine transformation with a space varying scale parameter corresponding to

$$\epsilon(X, Y) = \left(\frac{\det(A)}{1 + w_x X + w_y Y} \right)^2$$

Note that for continuous curves $\epsilon(X, Y)$ changes continuously on the curve.

Before we detail the proposed projective invariant matching, let us recall the affine matching described in Section 4.2, where we proposed the following procedure:

Calculate the warp distance from the signature V_{Q} of the query curve to a set of signatures $V_{\mathrm{T}}^{\epsilon_m}$ corresponding to the template curve, and a set of ϵ values. The query is considered an affine transformation of the template if at least one of the warp distances is below a predetermined threshold. Evidently, it is sufficient to consider the smallest warp distance over all m. Thus, the proposed algorithm is equivalent to an algorithm combining the dynamic programming algorithms of the different warps to a single algorithm matching $\{V_{\mathrm{Q}}\}$ to a huge set of states $\{V_{\mathrm{T}}^{\epsilon_m}(j)\} \quad \forall j,\ \forall m$. Note that:

1. The initialization should facilitate equal conditions for matching $\{V_{\mathrm{Q}}(0)\}$, to the initial states $\{V_{\mathrm{T}}^{\epsilon_m}(0)\}$ of each of the template's signatures.
2. The recursion should restrict path extension to within the same m.
3. The warp representing the template is the path corresponding to the best final state.

Now consider the case of projective invariant matching. The only difference to the above algorithm is the need to enable the recursion step to extend paths across similar scales.

A technical complication related to the projective case is the need to synchronize the actual location the template's signatures relate to over scales, so that when paths are extended across scales they will do so at points corresponding to the same location on the curve. In our implementation we solved this problem by over-sampling the signatures so that $V_{\mathrm{T}}^{\epsilon_m}(j)$ corresponds to the same location on the curve for every j, and keeping a map $N^m(j)$, so that the next signature point after $V_{\mathrm{T}}^{\epsilon_m}(j)$ is $V_{\mathrm{T}}^{\epsilon_m}(j + N^m(j))$ (rather than $V_{\mathrm{T}}^{\epsilon_m}(j+1)$).

Another practical problem is that the number of states, and the corresponding computational complexity implied by the above description is huge. Here we use a state pruning method similar to [16] based on a conjecture that the states' values are skewed from the start, and there is no practical need to keep track of all of them. We therefore reduce the number of states we track, and prune the states recursively.

Figure 7 describes a match of the signature of a projective transformation of a curve to the template signature set. The signature of the template has been slightly lowered, otherwise it would have been difficult to distinct the two signatures. The values of the template signature were compiled by tracking the best-path selected by the algorithm, and taken from weak-affine signatures corresponding to different scales. This match quality would not have been possible had we limited the algorithm to any single scale.

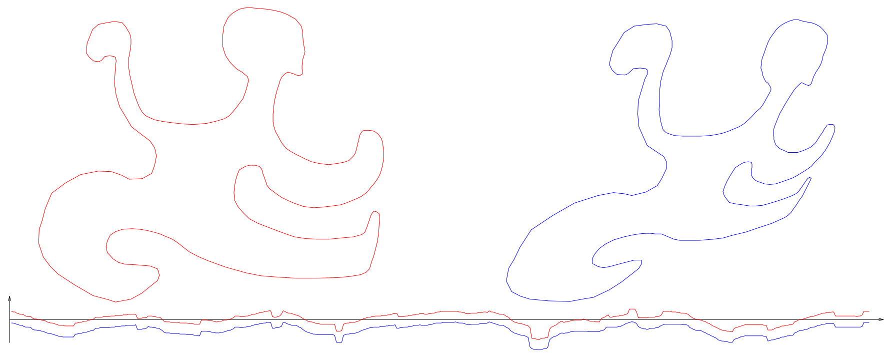


Figure 7: A weak-affine signature of the projectively transformed curve, and the warped template signature (lowered to allow distinction).

5 Summary

In this paper we have presented a weak-affine invariant re-sampling method for polygonal approximations of smooth curves. The weak-affine signature of the resulting polygon is invariant to the original curve sampling method.

We proposed to use a signature scale space similar to the one described in [4], and argued that a metric based on a modified Dynamic Programming algorithm accommodates projective invariant indexing.

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