Inf-Semilattice Approach to Self-Dual Morphology

Henk J.A.M. Heijmans, Renato Keshet
HP Laboratories Israel
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E-mail renato@hpl.hp.com

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Self-duality is a desired property in many applications, and, in particular, image filtering, where it is expressed by identical treatment of bright and dark objects. It is however not a straightforward property of most operators in classical mathematical morphology (defined on complete lattices). With the recent extension of mathematical morphology to inf-semilattices, an alternative approach for self-dual morphology became possible. In this report, an inf-semilattice theory for self-dual morphology is developed, and two approaches for deriving self-dual operators are proposed.

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1 Introduction

The classical way of defining self-duality is as follows. An operator ψ on a set of real functions $\mathcal{L}$ is called self-dual when $\psi(f) = -\psi(-f)$, for all function f in $\mathcal{L}$. Another type of self-duality is obtained when $\psi(f) = N - \psi(N - f)$, for functions mapping into the interval $[0, N]$. The concept of self-duality is accurately defined within the report.

Self-duality is a desired property in many applications, and, in particular, image filtering, where it is expressed by identical treatment of bright and dark objects. It is however not a straightforward property of most operators in classical mathematical morphology (defined on complete lattices). This is a consequence of the inherent duality of complete lattices, which causes operators to appear usually in pairs – one of the operators dealing with bright and the other with dark objects only. Nevertheless, it is possible to design self-dual operators in classical morphology. The work of Serra in [8, chapter 8] and Heijmans in [5] are examples of self-dual morphology. On the other hand, these approaches lack of a simple intuition, and suffer a bit from complicated notation.

With the recent extension of mathematical morphology to inf-semilattices [6, 7], an alternative approach for self-dual morphology became possible. In this report, an inf-semilattice theory for self-dual morphology is developed, and two approaches for deriving self-dual operators are proposed: Through reference semilattices and through lattice ordered groups. The former is also studied from the point of view of translation invariance. Some examples are provided.

Apart of providing new image processing operators, the proposed approach has the benefit of being more intuitive than the classical one, and has the potential of providing simpler notation for self-dual processing. At this point, however, most of the operators originated by the proposed approach are different from those obtained in the classical theory, and also from those proposed in [6, 7]. Therefore, a unifying theory for self-dual morphology is still missing.

2 Adjunctions on Posets

2.1 Operators on posets

Consider two posets (partially ordered sets) $\mathcal{L}$ and $\mathcal{M}$ with partial orderings $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{M}}$, respectively. If no confusion about the partial orderings seems possible, we will delete the subindices $\mathcal{L}$ and $\mathcal{M}$ indicating the underlying space.

An operator (i.e., mapping) $\psi : \mathcal{L} \to \mathcal{M}$ is called *increasing* (or *isotone*) if $x \leq_{\mathcal{L}} y$ implies $\psi(x) \leq_{\mathcal{M}} \psi(y)$. It is called *decreasing* if $x \leq_{\mathcal{L}} y$ implies $\psi(y) \leq_{\mathcal{M}} \psi(x)$. If ψ is a bijective operator between $\mathcal{L}$ and $\mathcal{M}$ such that both ψ and its inverse ψ^{-1} are increasing, then ψ is called an *isomorphism* (*automorphism* if $\mathcal{L} = \mathcal{M}$). A bijective operator ψ for which both ψ and its inverse ψ^{-1} are decreasing is called a *dual isomorphism* (respectively, *dual automorphism* if $\mathcal{L} = \mathcal{M}$). An isomorphism ψ on $\mathcal{L}$ with $\psi \neq \mathsf{id}_{\mathcal{L}}$ and $\psi^2 = \mathsf{id}_{\mathcal{L}}$, where $\mathsf{id}_{\mathcal{L}}$ denotes the identity operator on $\mathcal{L}$, is called an $\circ$-*negation*. A dual isomorphism with $\psi^2 = \mathsf{id}_{\mathcal{L}}$ is called an *involution* or $*$-*negation*. In many cases (e.g. if $\mathcal{L}$ is a chain) there do not exist $\circ$-negations on $\mathcal{L}$. The mapping $\psi : \mathbb{R} \to \mathbb{R}$ given by $\psi(t) = -t$ is an $*$-negation.

In the sequel, the following notation will be used. If ψ is an $\circ$-negation on $\mathcal{L}$, then we denote $\psi(x)$ by x°, for $x \in \mathcal{L}$, when no confusion is possible as to which $\circ$-negation is meant. Similarly, if ψ is a $*$-negation we denote $\psi(x)$ by x^*. If both $\mathcal{L}$ and $\mathcal{M}$ possess an $\circ$-negation and ψ is an operator between $\mathcal{L}$ and $\mathcal{M}$, then the $\circ$-negative of ψ is defined as the operator between $\mathcal{L}$ and $\mathcal{M}$ given by

$$\psi^\circ(x) = (\psi(x^\circ))^\circ, \quad x \in \mathcal{L}. \tag{2.1}$$

Similarly, if $\mathcal{L}$ and $\mathcal{M}$ possess a $*$-negation, then the $*$-negative of ψ is the operator between $\mathcal{L}$ and $\mathcal{M}$ given by

$$\psi^*(x) = (\psi(x^*))^*, \quad x \in \mathcal{L}. \tag{2.2}$$

When using the notation ψ°, resp ψ^*, it is tacitly assumed that there do exist $\circ$-negation, resp. $*$-negations, on the underlying sets $\mathcal{L}$ and $\mathcal{M}$ It is easy to verify that both ψ° and ψ^* are increasing iff ψ is increasing. Furthermore,

$$(\psi^\circ)^\circ = \psi \quad \text{and} \quad (\psi^*)^* = \psi,$$

for every operator $\psi : \mathcal{L} \to \mathcal{M}$.

If $x \mapsto x^\circ$ is an $\circ$-negation on $\mathcal{L}$ and ψ is an operator between $\mathcal{L}$ and $\mathcal{M}$, then ψ is called $\circ$-*selfdual* if

$$\psi = \psi^\circ.$$

The concept of an $*$-selfdual operator is defined analogously. The *range* of an operator ψ is denoted by $\mathrm{ran}(\psi)$, that is $\mathrm{ran}(\psi) = \{\psi(x) \mid x \in \mathcal{L}\}$.

2.2 Adjunctions on posets

2.1. Definition. Assume that $\mathcal{L}$ and $\mathcal{M}$ are posets and that $\varepsilon : \mathcal{L} \to \mathcal{M}$ and $\delta : \mathcal{M} \to \mathcal{L}$ are operators. The pair (ε, δ) is called an *adjunction between* $\mathcal{L}$ and $\mathcal{M}$ if

$$\delta(y) \leq_{\mathcal{L}} x \iff y \leq_{\mathcal{M}} \varepsilon(x), \quad x \in \mathcal{L},\ y \in \mathcal{M}.$$

If $\mathcal{L} = \mathcal{M}$, then (ε, δ) is called an adjunction *on* $\mathcal{L}$.

We list some basic properties. The proofs are rather straightforward (see also [4, Chapter 3]) and therefore omitted.

2.2. Proposition. *If (ε, δ) is an adjunction then both ε and δ are increasing. Furthermore*

$$\delta\varepsilon \le \mathsf{id}_{\mathcal{L}} \text{ and } \varepsilon\delta \ge \mathsf{id}_{\mathcal{M}}$$
$$\varepsilon\delta\varepsilon = \varepsilon \text{ and } \delta\varepsilon\delta = \delta$$

2.3. Proposition. *If $\psi : \mathcal{L} \to \mathcal{M}$ is an isomorphism, then (ψ, ψ^{-1}) is an adjunction.*

2.4. Proposition. *If $(\varepsilon_1, \delta_1)$ is an adjunction between $\mathcal{L}$ and $\mathcal{M}$ and $(\varepsilon_2, \delta_2)$ is an adjunction between $\mathcal{M}$ and $\mathcal{N}$, then $(\varepsilon_2\varepsilon_1, \delta_1\delta_2)$ is an adjunction between $\mathcal{L}$ and $\mathcal{N}$.*

2.5. Proposition. *Assume that (ε, δ) is an adjunction between the posets $\mathcal{L}$ and $\mathcal{M}$.*

(*a*) *Suppose that $\mathcal{L}, \mathcal{M}$ both have a $\circ$-negation. Then $(\varepsilon^\circ, \delta^\circ)$ is an adjunction between $\mathcal{L}$ and $\mathcal{M}$.*

(*b*) *Suppose that $\mathcal{L}, \mathcal{M}$ both have a $*$-negation. Then $(\delta^*, \varepsilon^*)$ is an adjunction between $\mathcal{M}$ and $\mathcal{L}$.*

The next result states that the pairing between dilations and erosions is unique.

2.6. Proposition. *Let ε be an operator from $\mathcal{L}$ into $\mathcal{M}$, let $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathcal{M}$, and assume that* $\operatorname{ran}(\varepsilon) \subseteq \mathcal{M}_1 \cap \mathcal{M}_2$. *Let δ_i be an operator from $\mathcal{M}_i$ into $\mathcal{L}$ such that (ε, δ_i) is an adjunction between $\mathcal{L}$ and $\mathcal{M}_i$, for $i = 1, 2$. Then $\delta_1(y) = \delta_2(y)$ for $y \in \mathcal{M}_1 \cap \mathcal{M}_2$.*

Proof. Assume that $y \in \mathcal{M}_1 \cap \mathcal{M}_2$, then

$$\delta_1(y) \le x \iff y \le \varepsilon(x) \iff \delta_2(y) \le x\,,$$

for every $x \in \mathcal{L}$. Choosing $x = \delta_1(y)$ at the left yields $\delta_2(y) \le \delta_1(y)$. Similarly, choosing $x = \delta_2(y)$ at the right gives $\delta_1(y) \le \delta_2(y)$. Thus we arrive at our conclusion. □

We point out that the Duality Principle, which says that $\mathcal{L}$ provided with the relation $x \le' y$ iff $y \le x$ is a poset as well, implies an analogue of Proposition 2.6 concerning the uniqueness of the erosion that forms an adjunction with a given dilation.

In general, a subset of the poset $\mathcal{L}$ does not have a supremum (least upper bound) nor infimum (greatest lower bound) in general. In this respect, the following results are remarkable.

2.7. Proposition. *If* (ε, δ) *is an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$ *then* $\{x \in \mathcal{L} \mid y \leq \varepsilon(x)\}$ *has infimum* $\delta(y)$*, for every* $y \in \mathcal{M}$*. Dually,* $\{y \in \mathcal{M} \mid \delta(y) \leq x\}$ *has supremum* $\varepsilon(x)$*, for every* $x \in \mathcal{L}$*.*

Proof. First, since $\delta(y) \leq x$ if $y \leq \varepsilon(x)$, we get that $\delta(y)$ is a lower bound of $\{x \in \mathcal{L} \mid y \leq \varepsilon(x)\}$. Now suppose that a is a lower bound of this set, then in particular, $a \leq \delta(y)$ since $y \leq \varepsilon\delta(y)$. This proves the result. □

2.8. Proposition. *Assume that* (ε, δ) *is an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$*.*

(*a*) *Suppose that the family* $\{x_i \mid i \in I\}$ *has an infimum* a*, then* $\{\varepsilon(x_i) \mid i \in I\}$ *has infimum* $\varepsilon(a)$ *in* $\mathcal{M}$*.*

(*b*) *Suppose that the family* $\{y_i \mid i \in I\}$ *has a supremum* b*, then* $\{\delta(y_i) \mid i \in I\}$ *has a supremum* $\delta(b)$ *in* $\mathcal{L}$*.*

Proof. We prove (a), then (b) follows by duality. If $\{x_i \mid i \in I\}$ has an infimum a, then by the increasingness of the operator ε, $\varepsilon(a) \leq \varepsilon(x_i)$ for $i \in I$. Now b is a lower bound of $\{\varepsilon(x_i) \mid i \in I\}$ if $b \leq \varepsilon(x_i)$, that is $\delta(b) \leq x_i$ for $i \in I$. But then $\delta(b) \leq a$, hence $b \leq \varepsilon(a)$. We conclude that $\varepsilon(a)$ is the infimum of $\{\varepsilon(x_i) \mid i \in I\}$. □

2.9. Definition. (**Erosions on posets**)
An operator ε between the posets $\mathcal{L}$ and $\mathcal{M}$ is called an *erosion* if for all families $\{x_i\} \subseteq \mathcal{L}$ it is true that

(*i*) $\bigwedge x_i$ exists in $\mathcal{L}$ iff $\bigwedge \varepsilon(x_i)$ exists in $\mathcal{M}$;

(*ii*) $\bigwedge \varepsilon(x_i) = \varepsilon(\bigwedge x_i)$ in case (i) holds.

A dilation δ is defined analogously with the infimum replaced by supremum. It is easy to see that dilations and erosions are increasing operators. We give an example.

2.10. Example. Let $\mathcal{L}$ be the integers with partial ordering $\preceq$ defined by $m \preceq n$ if $m \leq n$ and $m + n$ even. Hence $2 \preceq 6$ and $-3 \preceq 7$ but $1 \not\preceq 2$. Every erosion on $\mathcal{L}$ must have the property that $\varepsilon(\mathbb{Z}_e) \cap \varepsilon(\mathbb{Z}_o) = ?$, where $\mathbb{Z}_e, \mathbb{Z}_o$ are the even and odd integers, respectively. For, if $\varepsilon(n) = \varepsilon(m) = k$, where n is odd and m is even, then the infimum of n and m does not exist whereas the infimum of $\varepsilon(n)$ and $\varepsilon(m)$ equals k. Note, however, that the condition $\varepsilon(\mathbb{Z}_e) \cap \varepsilon(\mathbb{Z}_o) = ?$ is far from being sufficient for ε to be an erosion, even under the additional assumption that ε is increasing. Two simple examples of erosions are $\varepsilon(n) = n + k$ and $\varepsilon(n) = n^3 + k$, where k is a fixed integer.

2.11. Proposition. *Let* ε *be an erosion between the posets* $\mathcal{L}$ *and* $\mathcal{M}$*. Define* $\mathcal{M}[\varepsilon] \subseteq \mathcal{M}$ *as*

$$\mathcal{M}[\varepsilon] = \{y \in \mathcal{M} \mid \text{ the set } \{x \in \mathcal{L} \mid y \leq \varepsilon(x)\} \text{ has infimum in } \mathcal{L}\}, \tag{2.3}$$

and $\delta : \mathcal{M}[\varepsilon] \to \mathcal{L}$ *as*

$$\delta(y) = \bigwedge\{x \in \mathcal{L} \mid y \leq \varepsilon(x)\}. \tag{2.4}$$

Then $\mathrm{ran}(\varepsilon) \subseteq \mathcal{M}[\varepsilon]$ *and* (ε, δ) *is an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}[\varepsilon]$*.*

Proof. We first show that $\mathrm{ran}(\varepsilon) \subseteq \mathcal{M}[\varepsilon]$. Take $x_0 \in \mathcal{L}$, we show that $\{x \in \mathcal{L} \mid \varepsilon(x_0) \leq \varepsilon(x)\}$ has an infimum. Because of Definition 2.9(i) this is true iff $\{\varepsilon(x) \mid x \in \mathcal{L} \text{ and } \varepsilon(x_0) \leq \varepsilon(x)\}$ has an infimum. But obviously, this infimum exists and equals $\varepsilon(x_0)$.

To prove that (ε, δ) is an adjunction between $\mathcal{L}$ and $\mathcal{M}[\varepsilon]$, we must demonstrate that $\delta(y) \leq x$ iff $y \leq \varepsilon(x)$ for $x \in \mathcal{L}$ and $y \in \mathcal{M}[\varepsilon]$. First assume that $\delta(y) \leq x$, that is $\bigwedge\{x' \in \mathcal{L} \mid y \leq \varepsilon(x')\} \leq x$. Since ε is an erosion, we find

$$\varepsilon\delta(y) = \bigwedge\{\varepsilon(x') \mid y \leq \varepsilon(x')\} \leq \varepsilon(x)\,.$$

It follows immediately from this expression that $y \leq \varepsilon\delta(y)$, hence $y \leq \varepsilon(x)$. On the other hand, $y \leq \varepsilon(x)$ in combination with (2.4) yields that $\delta(y) \leq x$. □

The dilation given by (2.4) will sometimes be denoted by $\Delta(\varepsilon)$ to emphasise the dependence on ε.

2.3 Adjunctions on Complete Lattices

The theory of adjunctions on complete lattices has played an important role in mathematical morphology over the past ten years or so [4, 9]. In this section we will briefly recall some of the major results, in particular those that are not generally valid in the poset framework.

First of all, it is obvious that the definition of erosion and dilation given in Definition 2.9 can be simplified as follows: the operator ε between the complete lattices $\mathcal{L}$ and $\mathcal{M}$ is an *erosion* if

$$\varepsilon(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} \varepsilon(x_i),$$

for every family $\{x_i \mid i \in I\}$ in $\mathcal{L}$. Note that $\varepsilon(\top) = \top$ by this definition. To see this, one has to choose the collection $\{x_i\}$ empty and use that the infimum of the empty set is $\top$. A similar definition holds for dilation.

2.12. Proposition. (*a*) *To every erosion ε between the complete lattices $\mathcal{L}$ and $\mathcal{M}$ there corresponds a unique dilation $\delta : \mathcal{M} \to \mathcal{L}$ given by*

$$\delta(y) = \bigwedge \{x \in \mathcal{L} \mid y \leq \varepsilon(x)\}, \quad y \in \mathcal{M}, \tag{2.5}$$

such that (ε, δ) is an adjunction.

(*b*) *To every dilation δ between the complete lattices $\mathcal{M}$ and $\mathcal{L}$ there corresponds a unique erosion $\varepsilon : \mathcal{L} \to \mathcal{M}$ given by*

$$\varepsilon(x) = \bigvee \{y \in \mathcal{M} \mid \delta(y) \leq x\}, \quad x \in \mathcal{L}, \tag{2.6}$$

such that (ε, δ) is an adjunction.

Note that the set $\mathcal{M}[\varepsilon]$ introduced in Proposition 2.11 equals $\mathcal{M}$ in this case.

There is a great deal of literature where it is explained that the theory of adjunctions on complete lattices provides the appropriate framework for various different approaches in mathematical morphology. The best known examples are binary and grey-scale morphology, respectively. We will briefly discuss both cases below.

Binary (i.e., black-and-white) images can be modeled mathematically by the complete Boolean lattice $\mathcal{L} = \mathcal{P}(E)$ comprising all subsets of an underlying universal set E, usually $\mathbb{R}^2$ or $\mathbb{Z}^2$ or a finite subset of one of these sets. A binary morphological operator is then nothing but an operator on $\mathcal{P}(E)$. The complement operator, mapping a set X onto its complement X^{c} is a $*$-negation. The $*$-negative of an operator ψ is then given by

$$\psi^*(X) = (\psi(X^{\mathrm{c}}))^{\mathrm{c}}, \quad X \subseteq E .$$

Grey-scale images can be modeled as elements of the power set $\mathcal{T}^E$, where $\mathcal{T}$ is the set of grey-values and where E has the same interpretation as before. Note that we are back in the binary case if $\mathcal{T} = \{0, 1\}$. If $\mathcal{T}$ carries a partial ordering such that it has a complete lattice structure, then $\mathcal{T}^E$ endowed with the pointwise partial ordering also becomes a complete lattice. In many practical cases, $\mathcal{T}$ is totally ordered (i.e., a chain). Typical choices for $\mathcal{T}$ are $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, $\overline{\mathbb{R}}_+ = [0, \infty]$, $\overline{\mathbb{Z}}$, $\overline{\mathbb{Z}}_+$, and $\{0, 1, \ldots, N\}$, where $N \geq 1$ is an integer. Often, it is straightforward to provide $\mathcal{T}$ with a $*$-negation, which can then be extended to $\mathcal{T}^E$ by applying it pointwise. A typical $*$-negation on $\overline{\mathbb{R}}$ (and also on $\overline{\mathbb{Z}}$) is given by $t \mapsto -t$. On $\overline{\mathbb{R}}_+$ we have a $*$-negation $t \mapsto 1/t$ (with $1/0 = \infty$ and $1/\infty = 0$) and on $\{0, 1, \ldots, N\}$ we have $t \mapsto N - t$. It is, however, easy to show that there exists no $*$-negation on $\overline{\mathbb{Z}}_+$.

Finally we point out that $\mathcal{T}^E$ is not totally ordered in general, even if $\mathcal{T}$ is.

2.4 Morphology, Operator Types, and Self-Duality

It is typical for mathematical morphology, in the binary as well as in the grey-scale case, that operator types always occur in pairs, dilation and erosion, opening and closing, etc. By an operator type we mean a family of operators on $\mathcal{L}$ which can be completely specified in terms of the underlying partial ordering. For example the operator type called "opening" refers to operators ψ which are increasing, idempotent, and anti-extensive. We emphasise that it is not necessarily true that operators exist pairwise (like erosions and dilations in an adjunction on a complete lattice; see Proposition 2.12). The pairwise occurrence of operator types is a consequence of the Duality Principle for posets, and is in no way constructive; in particular, it does not imply that both operator types in such pairings play an equal role on a given poset $\mathcal{L}$. An important illustration of this fact follows later when we discuss erosions and dilations on an inf-semilattice. As we shall see, dilations and erosions play a very asymmetric role there.

Of great importance in this respect is the (non-) existence of a $*$-negation on the underlying poset $\mathcal{L}$. In fact, the existence of a $*$-negation *does* provide a constructive tool for transforming an operator type into its dual. For example, if $\mathcal{L}$ is a poset with $*$-negation ν and if ψ is an operator of type "opening", then $\psi^* = \nu\psi\nu$ is of type "closing".

2.13. Definition. A poset $\mathcal{L}$ for which there exists a $*$-negation $\nu : \mathcal{L} \to \mathcal{L}$ is called a *$*$-negation poset* or *self-dual poset.*

Up to this point, our considerations have not been referring to the physical world that our model is supposed to describe. What is missing is the observation that the $*$-negation that is being used, should map an image onto another image that may be considered as its physical negative. We don't want to go into this matter very deeply here, as we think that most readers will have some intuition for the meaning of the "physical negative" of an image (bright parts of the original image corresponding to dark parts of its negative and vice versa).

We have now reached the point that we are able to explain the phrase "self-dual morphology" in the title of our paper. If the operator ν that maps an image $x \in \mathcal{L}$ to its physical negative $x^* = \nu(x)$ is a $*$-negation, then an erosion ε on $\mathcal{L}$ will not be self-dual in the sense that

$$\varepsilon(\nu(x)) = \nu(\varepsilon(x)), \tag{2.7}$$

because the operator $\varepsilon^* = \nu\varepsilon\nu$ is of type "dilation".[1] Note however, that there may exist self-dual operators in this case; refer to [5] for construction methods of such operators

[1] Some cautionary remark is in order here: there do exist operators ψ which are at the same time erosion and dilation and which do satisfy $\psi^* = \psi$ with respect to some $*$-negation.

If, on the other hand, the physical negation operator ν is a $\circ$-negation, then $\varepsilon^\circ = \nu\varepsilon\nu$ is an erosion iff ε is one, and in this case self-duality of ε is within reach; see Section 4 for specific examples. In the complete inf-semilattice framework discussed later, erosions and dilations (and also openings and closings) play a completely different role. In view of the fact that the infimum, but not the supremum, of any subcollection of elements exists, this is not very suprising.

3 Complete Inf-Semilattices

3.1 Definitions and examples

An important instance of a poset which, in general, does not allow a $*$-negation is the inf-semilattice that will be introduced now.

3.1. Definition. A poset $\mathcal{L}$ is called an *inf-semilattice* if for every two elements $x, y \in \mathcal{L}$ their infimum $x \,\mathrm{f}\, y$ exists. It is called a *complete inf-semilattice*, or briefly, *cisl* if every non-empty subset $\mathcal{K}$ of $\mathcal{L}$ has an infimum (greatest lower bound) $\bigcurlywedge \mathcal{K} \in \mathcal{L}$. The least element of a cisl is denoted by $\perp$, i.e. $\perp = \bigwedge \mathcal{L}$.

From now on a partial ordering on a complete inf-semilattice will be denoted by $\preceq$. It is easy to see that a (complete) inf-semilattice $\mathcal{L}$ which possesses a $*$-negation $x \mapsto x^*$ is actually a (complete) lattice. Namely, the supremum of x, y is given by

$$x \,\mathrm{g}\, y = (x^* \,\mathrm{f}\, y^*)^*.$$

Two simple examples of an inf-semilattice are represented by the Hasse diagrams:

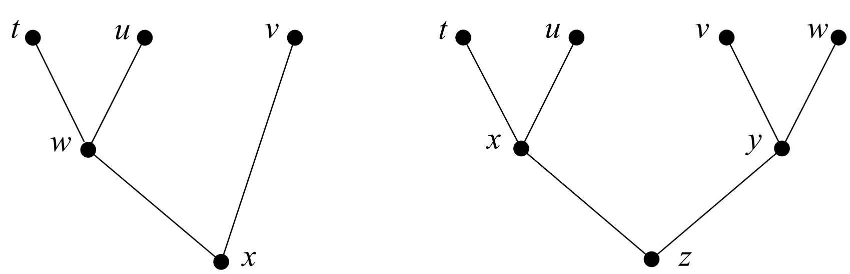


Figure 1: *Two Hasse diagrams representing a cisl.*

In the diagram at the left we have $w \preceq t, u$ and $x \preceq v, w$, and by transitivity also $x \preceq t, u$. Note for example that $t \,\mathrm{f}\, v = x$. The inf-semilattice at the right has 5 $\circ$-negations, namely:

(1) $t \longleftrightarrow u$ (i.e., only t and u are interchanged);

(2) $v \longleftrightarrow w$;

(3) $t \longleftrightarrow u$, $v \longleftrightarrow w$;

(4) $t \longleftrightarrow v$, $u \longleftrightarrow w$, $x \longleftrightarrow y$;

(5) $t \longleftrightarrow w$, $u \longleftrightarrow v$, $x \longleftrightarrow y$.

The inf-semilattice at the left of Fig. 1 has only one $\circ$-negation, namely the operator that interchanges t and u and leaves all other elements unaltered.

The next example of a cisl will play a prominent role in the remainder of this paper. Define the partial ordering $\preceq$ on $\mathbb{R}$ as follows:

$$s \preceq t \ \text{ if } \ 0 \leq s \leq t \ \text{ or } \ t \leq s \leq 0. \tag{3.1}$$

Thus $\mathbb{R}$ can be considered as the concatenation of two chains $(\mathbb{R}_-, \geq)$ and $(\mathbb{R}_+, \leq)$ intersecting at the origin, which is the least element of the poset thus defined. We

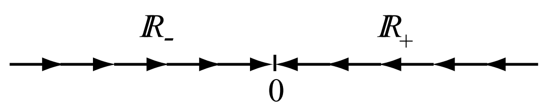


Figure 2: *The cisl $\mathbb{R}_0$ is a concatenation of two chains. The arrows point in the direction of smaller elements.*

denote $\mathbb{R}$ provided with this partial ordering by $\mathbb{R}_0$. There exists one $\circ$-negation on $\mathbb{R}_0$, namely the operator $t \mapsto -t$.

It s not difficult to understand how the previous cisl-ordering can be extended to the complex plane $\mathbb{C}$. Consider $\mathbb{C}$ as an (infinite) union of chains $\mathbb{C}_\alpha = \{re^{i\alpha} \mid r \geq 0\}$ ordered by the magnitude of the modulus. Thus, given two elements $w, z \in \mathbb{C}$, we have

$$w \preceq z \text{ if } \arg w = \arg z \text{ and } |w| \leq |z|\,. \tag{3.2}$$

Here $\arg z$ denotes the argument of z. Evidently, the mappings $z \mapsto -z$ and $z \mapsto e^{i\alpha}\overline{z}$ (where $\alpha \in \mathbb{R}$) are $\circ$-negations.

One final example of a cisl that we want to mention here is the family of all finite subsets of an infinite set E provided with the set inclusion as partial ordering.

We state some basic results concerning cisl's. The proof of the first result is straightforward.

3.2. Proposition. *Let E be a nonempty set and assume that $(\mathcal{T}_p, \preceq_p)$ is a cisl, for every $p \in E$. Then the set $\mathcal{L}$ comprising all mappings $x : E \to \bigcup_{p\in E} \mathcal{T}_p$ with $x(p) \in \mathcal{T}_p$ ordered by*

$$x \preceq y \text{ if } \forall\, p \in E \;:\; x(p) \preceq_p y(p),$$

defines a cisl.

An important special case is obtained if $(\mathcal{T}_p, \preceq_p)$ is the same for all p, in which case $\mathcal{L} = \mathcal{T}^E$ is called the *power cisl*. Observe that, in the latter case, every $\circ$-negation $\nu_{\mathcal{T}}$ on $\mathcal{T}$ easily extends to a (pointwise) $\circ$-negation $\nu_{\mathcal{L}}$ on $\mathcal{L}$ given by $\nu_{\mathcal{L}}(x)(p) = \nu_{\mathcal{T}}(x(p))$.

As an example, assume that $E = \mathbb{R}$ and $\mathcal{T} = \mathbb{R}_0$. In Fig. 3 we illustrate the cisl ordering on $\mathcal{T}^E$ and the corresponding infimum. The next result, the proof of which

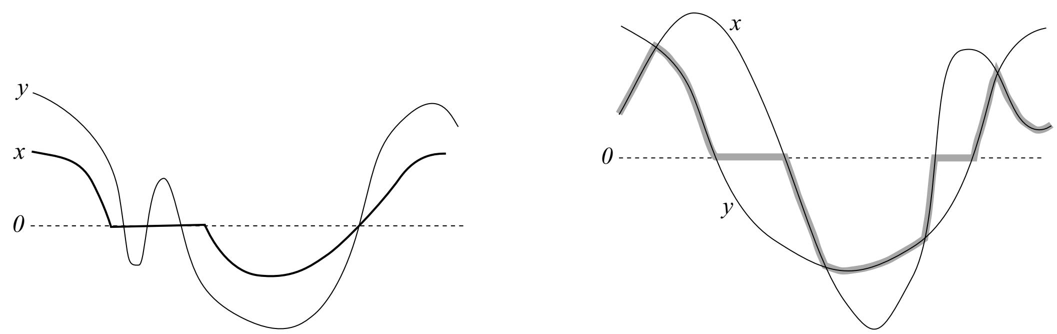


Figure 3: Left: $x \preceq y$ in the cisl $\mathcal{L}$ of functions from $\mathbb{R}$ to $\mathbb{R}_0$. Right: the infimum of two signals $x, y \in \mathcal{L}$ (fat grey line).

is straightforward, says that a bijection between a cisl $\mathcal{L}$ and another set $\mathcal{M}$ induces a cisl-structure on $\mathcal{M}$.

3.3. Proposition. *Assume that $(\mathcal{L}, \preceq)$ is a cisl, that $\mathcal{M}$ is some nonempty set, and that $\theta : \mathcal{L} \to \mathcal{M}$ is a bijection. Define the relation $\preceq_\theta$ on $\mathcal{M} \times \mathcal{M}$ by*

$$y_1 \preceq_\theta y_2 \iff \theta^{-1}(y_1) \preceq \theta^{-1}(y_2).$$

Then $(\mathcal{M}, \preceq_\theta)$ is a cisl with infimum given by

$$\curlywedge_\theta y_i = \theta(\curlywedge \theta^{-1}(y_i)).$$

If $(\mathcal{L}, \preceq)$ and $(\mathcal{M}, \preceq)$ are cisl's, then a bijective mapping $\theta : \mathcal{L} \to \mathcal{M}$ is called an *cisl-isomorphism* if

$$\theta(\curlywedge x_i) = \curlywedge \theta(x_i)$$

for every collection $\{x_i\} \subseteq \mathcal{L}$.

3.2 Reference cisl's

A class of cisl's that is important for our purposes, are the so-called *reference cisl's*. Before giving a formal definition we recall the concept of 'infinite distributivity' on a complete lattice. Given a complete lattice $(\mathcal{L}, \leq)$, we say that $\mathcal{L}$ satisfies the infinite distributive laws if

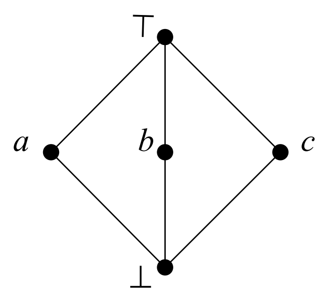
$$y \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (y \wedge x_i) \tag{3.3}$$

$$y \vee \bigwedge_{i \in I} x_i = \bigwedge_{i \in I} (y \vee x_i) \tag{3.4}$$

for an arbitrary family $\{x_i \mid i \in I\} \subseteq \mathcal{L}$ and $y \in \mathcal{L}$. We call (3.3) and (3.4) the infinite supremum distributive law and the infinite infimum distributive law, respectively. It is evident that every complete lattice in which these laws hold is distributive; the converse is not true, however.

3.4. Definition. Let $(\mathcal{L}, \leq)$ be a lattice. An element $r \in \mathcal{L}$ is called *reference element* if for every two elements $x, y \in \mathcal{L}$ we have $x \wedge r = y \wedge r$ and $x \vee r = y \vee r$ if and only if $x = y$.

Obviously, the least and greatest element in a lattice, if they exist, are automatically reference elements, but it is easy to find lattices which do not contain any other reference elements. This is e.g. the case for the lattice represented by the following Hasse diagram: We have $a \wedge c = b \wedge c = \perp$ and $a \vee c = b \vee c = \top$, hence c is not a reference



element. The same is true for a and b.

Let $\mathcal{L}$ be a lattice and $r \in \mathcal{L}$ a fixed element. Define the binary relation $\preceq_r$ on $\mathcal{L} \times \mathcal{L}$ by

$$x \preceq_r y \text{ if } \begin{cases} r \wedge y \leq r \wedge x \\ r \vee y \geq r \vee x \end{cases}$$

If we choose for r the least element of $\mathcal{L}$ (presumed that it exists), then $\preceq_r$ coincides with the partial ordering $\leq$. If, on the other hand, we choose for r the greatest element of $\mathcal{L}$

(again, supposed that it exists), then $\preceq_r$ is the dual ordering $\geq$ on $\mathcal{L}$, also sometimes denoted by $\leq'$.

3.5. Proposition. *Let $\mathcal{L}$ be a complete lattice for which the infinite distributive laws hold. If r is a reference element of $\mathcal{L}$, then $(\mathcal{L}, \preceq_r)$ is a cisl with least element r and with infimum given by*

$$\curlywedge_r x_i = (r \wedge \bigvee_{i\in I} x_i) \vee \bigwedge_{i\in I} x_i = (r \vee \bigwedge_{i\in I} x_i) \wedge \bigvee_{i\in I} x_i\,. \tag{3.5}$$

Proof. The second equality in (3.5) is a straightforward consequence of the distributivity of $\mathcal{L}$. We show that $\preceq_r$ defines a partial ordering on $\mathcal{L}$. It is evident that $x \preceq_r x$ for $x \in \mathcal{L}$. Assume that $x \preceq_r y$ and $y \preceq_r x$. We get that $x \wedge r = y \wedge r$ and $x \vee r = y \vee r$. From the fact that r is a reference element we conclude that $x = y$. The transitivity of $\preceq_r$ (i.e. $x \preceq_r y$ and $y \preceq_r z$ implies $x \preceq_r z$) is trivial and we conclude that $(\mathcal{L}, \preceq_r)$ is a poset.

It remains to be shown that the expressions in (3.5) define the infimum of a family $x_i \in \mathcal{L}$, $i \in I$. Let us denote the element defined by (3.5) by a. We must show that

(*i*) $a \preceq_r x_i$ for $i \in I$;

(*ii*) $a' \preceq_r a$ for every a' with property (*i*).

Using the infinite distributivity laws we get

$$\begin{aligned} r \wedge a &= r \wedge \bigvee_{i\in I} x_i = \bigvee_{i\in I}(r \wedge x_i) \geq r \wedge x_i\,, i \in I \\ r \vee a &= r \vee \bigwedge_{i\in I} x_i = \bigwedge_{i\in I}(r \vee x_i) \leq r \vee x_i\,, i \in I\,. \end{aligned}$$

But this yields that $a \preceq_r x_i$, hence (*i*) is proved. Now if $a' \preceq_r x_i$ for $i \in I$, then $a' \wedge r \geq x_i \wedge r$, hence

$$a' \wedge r \geq \bigvee_{i\in I}(x_i \wedge r) = r \wedge \bigvee_{i\in I} x_i\,,$$

where we have used the infinite supremum distributivity law. This yields $a' \wedge r \geq a \wedge r$. Similarly, we deduce $a' \vee r \leq a \vee r$ and we conclude that $a' \preceq_r a$, which was to be shown. Finally, it is easy to see that r is the least element of $(\mathcal{L}, \preceq_r)$. □

We mention some special cases of lattices $\mathcal{L}$ where the infinite distributive laws hold and every element is a reference element.

3.6. Proposition. *For every complete chain, the infinite distributive laws hold, and every element is a reference element.*

The proof of this result is straightforward and therefore omitted. Thus, the conclusions of Proposition 3.5 are valid if $\mathcal{L}$ is a complete chain. In fact, it is easy to see that, for $r \in \mathcal{L}$, the cisl $(\mathcal{L}, \preceq_r)$ is a concatenation of two chains, namely $((\leftarrow, r], \geq)$ and $([r, \rightarrow), \leq)$, where $(\leftarrow, r] = \{x \in \mathcal{L} \mid x \leq r\}$ and $[r, \rightarrow) = \{x \in \mathcal{L} \mid x \geq r\}$. Note that $\mathbb{R}_0$ is an example of a cisl that possesses this structure, apart from the fact that the least and greatest element $-\infty$ and $+\infty$ are not included.

Every complete Boolean lattice satisfies the infinite distributive laws. Furthermore, every element $x \in \mathcal{L}$ is a reference element. Thus, Proposition 3.5 yields that $(\mathcal{L}, \preceq_r)$ is a cisl for every $r \in \mathcal{L}$. Actually, we can prove a stronger result in this case. Recall that we denote the complement of an element x of a Boolean lattice by x^{c}.

3.7. Proposition. *If $\mathcal{L}$ is a complete Boolean lattice, then*

$$x \preceq_r y \quad \text{iff} \quad y \preceq_{r^{\mathrm{c}}} x, \qquad r, x, y \in \mathcal{L}\,.$$

In particular, $(\mathcal{L}, \preceq_r)$ is a complete lattice with least and greatest element r and r^{c}, respectively, with infimum given by (3.5) *and supremum given by*

$$\curlyvee_r x_i = (r^{\mathrm{c}} \wedge \bigvee_{i \in I} x_i) \vee \bigwedge_{i \in I} x_i \tag{3.6}$$

for $\{x_i \mid i \in I\} \subseteq \mathcal{L}$.

Proof. It suffices to prove the first equivalence relation as the other results are easy consequences of this fact. Now $x \preceq_r y$ means

$$x \wedge r \geq y \wedge r \quad \text{and} \quad x \vee r \leq y \vee r\,.$$

In the first equality we take at both sides the supremum with r^{c}, and in the second equality we take at both sides the infimum with r^{c}. Thus we get

$$(x \wedge r) \vee r^{\mathrm{c}} \geq (y \wedge r) \vee r^{\mathrm{c}} \quad \text{and} \quad (x \vee r) \wedge r^{\mathrm{c}} \leq (y \vee r) \wedge r^{\mathrm{c}}\,,$$

which, by using distributivity, can be rewritten as

$$x \vee r^{\mathrm{c}} \geq y \vee r^{\mathrm{c}} \quad \text{and} \quad x \wedge r^{\mathrm{c}} \leq y \wedge r^{\mathrm{c}}\,.$$

But this means $y \preceq_{r^{\mathrm{c}}} x$, as we wanted to show. ☐

Later, in Section 5, we will discuss another family of lattices, the so-called *lattice-ordered groups*, for which the assumptions in Proposition 3.5 are valid.

We conclude this section with an example. Let E be a nonempty set and let $\mathcal{T} = \mathbb{Z}$ or $\mathbb{R}$ with the usual ordering. Consider the complete lattice $(\mathcal{T}^E, \leq)$, where $\leq$ denotes the pointwise ordering of functions. It is easy to show that this complete lattice satisfies the infinite distributive laws and that each of its elements is a reference element. Thus, following Proposition 3.5, we conclude that $(\mathcal{T}^E, \preceq_r)$ is a cisl for every reference function $r \in \mathcal{T}^E$. We will denote this cisl by $\mathcal{F}_r$. The mapping $x \mapsto x^\circ$ on $\mathcal{F}_r$ given by

$$x^\circ(p) = 2r(p) - x(p) \tag{3.7}$$

defines a $\circ$-negation. Observe that $\mathcal{F}_r$ can be regarded as a special case of Proposition 3.2, where $\mathcal{T}_p = \mathcal{T}$ for all $p \in E$ and $\preceq_p$ on $\mathcal{T}$ is the partial ordering $\preceq_{r(p)}$. An illustration is given in Fig. 4. The operator λ_r given by $\lambda_r(x) = x - r$ defines a cisl-

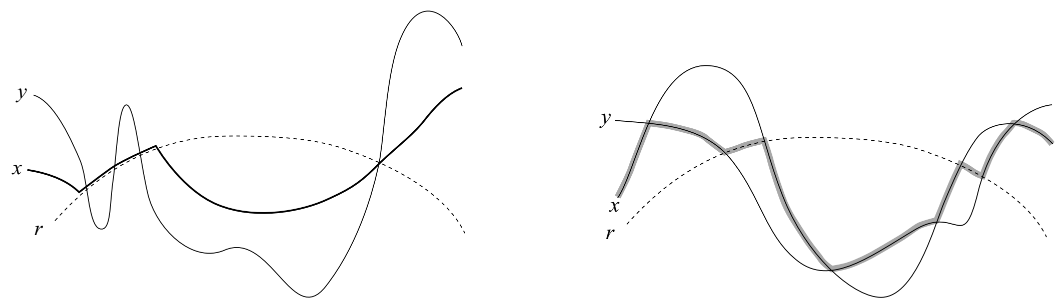


Figure 4: *Left: $x \preceq y$ in the cisl $\mathcal{F}_r$. Right: the infimum (in grey) of two signals $x, y \in \mathcal{F}_r$.*

isomorphism between the cisl's $\mathcal{F}_r$ and $\mathcal{F}_0$, and more generally, between $\mathcal{F}_{r+s}$ and $\mathcal{F}_s$. This leads to the following intertwining diagram for operators on $\mathcal{F}_r$ and operators on $\mathcal{F}_0$.

$$\begin{array}{ccc} \mathcal{F}_r & \xrightarrow{\psi} & \mathcal{F}_r \\ {\scriptstyle x\mapsto x-r}\downarrow & & \uparrow{\scriptstyle x\mapsto x+r} \\ \mathcal{F}_0 & \xrightarrow{\psi_0} & \mathcal{F}_0 \end{array}$$

Intertwining diagram: $\psi(x) = \psi_0(x - r) + r$.

The inverse λ_r^{-1} is given by $\lambda_r^{-1}(x) = \lambda_{-r}(x) = x + r$, and it is a cisl-isomorphism between $\mathcal{F}_0$ and $\mathcal{F}_r$, and more generally, between $\mathcal{F}_s$ and $\mathcal{F}_{s+r}$. The operators in the diagram above are related by

$$\psi = \lambda_r^{-1} \psi_0 \lambda_r$$

It is easy to verify that ψ is increasing on $\mathcal{F}_r$ iff ψ_0 is increasing on $\mathcal{F}_0$. Later we will use this intertwining diagram to define erosions on $\mathcal{F}_r$.

3.3 Adjunctions on cisl's

In Section 2 we have defined erosions, dilations and adjunctions on general posets. As we shall see below, the various expressions become simpler in the case of cisl's. First of all, an operator $\varepsilon : \mathcal{L} \to \mathcal{M}$, where both $\mathcal{L}$ and $\mathcal{M}$ are cisl's, is an erosion if

$$\varepsilon(\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I} \varepsilon(x_i),$$

for every collection $\{x_i\} \subseteq \mathcal{L}$. The set $\mathcal{M}[\varepsilon]$ defined in (2.2) is now given by

$$\mathcal{M}[\varepsilon] = \{y \in \mathcal{M} \mid \exists x \in \mathcal{L} : \; y \leq \varepsilon(x)\},$$

and the dilation $\delta = \Delta(\varepsilon)$ is the same as in (2.4), i.e.,

$$\delta(y) = \bigwedge\{x \in \mathcal{L} \mid y \leq \varepsilon(x)\}, \; y \in \mathcal{M}[\varepsilon].$$

Note that the infimum exists since the set over which the infimum is taken is nonempty. It is evident that $\bot \in \mathcal{M}[\varepsilon]$ and that $\delta(\bot) = \bot$.

The following proposition is concerned with composition of adjunctions.

3.8. Proposition. *Let $\mathcal{L}, \mathcal{M}$ be cisl's and $\mathcal{N}$ a poset. Assume that $\varepsilon_1 : \mathcal{L} \to \mathcal{M}$ and $\varepsilon_2 : \mathcal{M} \to \mathcal{N}$ are erosions, and that $\varepsilon = \varepsilon_2\varepsilon_1$. Then ε is an erosion from $\mathcal{L}$ into $\mathcal{N}$ and*

(i) $\mathcal{N}[\varepsilon] \subseteq \mathcal{N}[\varepsilon_2]$;

(ii) $\Delta(\varepsilon_2)$ *maps* $\mathcal{N}[\varepsilon]$ *into* $\mathcal{M}[\varepsilon_1]$;

(iii) $\Delta(\varepsilon_1)\Delta(\varepsilon_2) = \Delta(\varepsilon)$ *on* $\mathcal{N}[\varepsilon]$.

Proof. We write $\delta_i = \Delta(\varepsilon_i)$ for $i = 1, 2$ and $\delta = \Delta(\varepsilon)$.

(i) $z \in \mathcal{N}[\varepsilon]$ means that $z \leq \varepsilon_2\varepsilon_1(x)$ for some $x \in \mathcal{L}$. But this implies $z \leq \varepsilon_2(\varepsilon_1(x))$, and therefore $z \in \mathcal{N}[\varepsilon_2]$.

(ii) We show that $\delta_2(z) \in \mathcal{M}[\varepsilon_1]$ for $z \in \mathcal{N}[\varepsilon]$. Now $z \in \mathcal{N}[\varepsilon]$ means $z \leq \varepsilon_2\varepsilon_1(x)$ for some $x \in \mathcal{L}$. Furthermore,

$$\delta_2(z) = \bigwedge\{y \in \mathcal{M} \mid z \leq \varepsilon_2(y)\},$$

and since $\varepsilon_1(x)$ is an element of the set at the right hand-side we derive that $\delta_2(z) \leq \varepsilon_1(x)$, which yields that $\delta_2(z) \in \mathcal{M}[\varepsilon_1]$.

(iii) For $x \in \mathcal{L}$ and $z \in \mathcal{N}[\varepsilon]$ we have

$$\begin{aligned} z \leq \varepsilon_2\varepsilon_1(x) \quad &\iff \quad \delta_2(z) \leq \varepsilon_1(x) \quad [\text{since } z \in \mathcal{N}[\varepsilon_1] \text{ by } (i)] \\ &\iff \quad \delta_1\delta_2(z) \leq x \quad [\text{since } \delta_2(z) \in \mathcal{M}[\varepsilon_1] \text{ by } (ii)] \end{aligned}$$

where we have respectively used that $(\varepsilon_2, \delta_2)$ forms an adjunction between $\mathcal{M}$ and $\mathcal{N}[\varepsilon_2]$, and that $(\varepsilon_1, \delta_1)$ is an adjunction between $\mathcal{L}$ and $\mathcal{M}[\varepsilon_1]$. On the other hand,

$$z \leq \varepsilon_2\varepsilon_1(x) = \varepsilon(x) \iff \delta(z) \leq x\,.$$

This yields that $\delta = \delta_1\delta_2$ on $\mathcal{N}[\varepsilon]$. □

We now give a simple example.

3.9. Example. Let $\mathcal{L} = \mathcal{M} = \mathcal{N} = [-3, 3]$ and define $\varepsilon_1 = \varepsilon_2$ as in Fig. 5 below. We have $M[\varepsilon_1] = [-2, 2]$ and $\mathcal{N}[\varepsilon] = [-1, 1]$. Note that the dilations $\Delta(\varepsilon_1), \Delta(\varepsilon_2)$ cannot be extended beyond $[-2, 2]$.

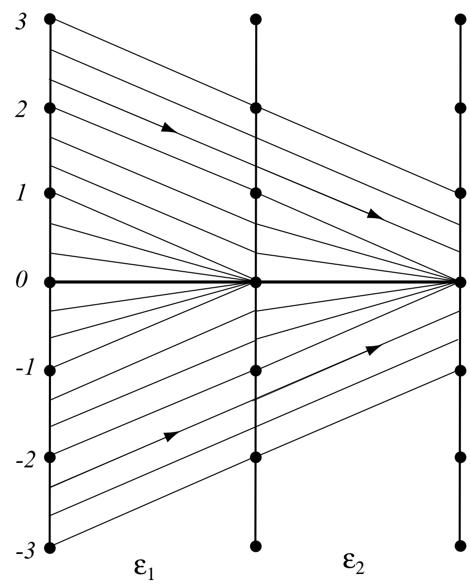


Figure 5: *Composition of two erosions.*

The next result is concerned with the ∘-negative of an erosion; see (2.1) for the corresponding definition.

3.10. Proposition. *Assume that $\mathcal{L}$ a cisl, that $\mathcal{M}$ is a poset, and that both sets have an ∘-negation. If ε is an erosion between $\mathcal{L}$ and $\mathcal{M}$ then ε° is an erosion between $\mathcal{L}$ and $\mathcal{M}$ too and we have*

$$M[\varepsilon^\circ] = (\mathcal{M}[\varepsilon])^\circ = \{y^\circ \mid y \in \mathcal{M}[\varepsilon]\}$$
$$\Delta(\varepsilon^\circ) = (\Delta(\varepsilon))^\circ$$

Proof. That ε° is an erosion follows immediately from the fact that $(\bigwedge_{i\in I} x_i)^\circ = \bigwedge_{i\in I} x_i{}^\circ$ for every family $\{x_i \mid i \in I\} \subseteq \mathcal{L}$. Furthermore,

$$y \le \varepsilon(x) \iff y^\circ \le \varepsilon(x)^\circ = \varepsilon^\circ(x^\circ)\,,$$

which yields that $M[\varepsilon^\circ] = (\mathcal{M}[\varepsilon])^\circ$. Finally, for $y \in M[\varepsilon^\circ]$ we have

$$\begin{aligned}
\Delta(\varepsilon^\circ)(y) &= \bigwedge_{i\in I}\{x \in \mathcal{L} \mid y \le \varepsilon^\circ(x)\} \\
&= \bigwedge_{i\in I}\{x \in \mathcal{L} \mid y \le (\varepsilon(x^\circ))^\circ\} \\
&= \bigwedge_{i\in I}\{x \in \mathcal{L} \mid y^\circ \le \varepsilon(x^\circ)\} \\
&= \bigwedge_{i\in I}\{x^\circ \mid x \in \mathcal{L} \ \text{ and } \ y^\circ \le \varepsilon(x)\} \\
&= [\bigwedge_{i\in I}\{x \mid x \in \mathcal{L} \ \text{ and } \ y^\circ \le \varepsilon(x)\}]^\circ \\
&= (\Delta(\varepsilon)(y^\circ))^\circ\,,
\end{aligned}$$

which proves that $\Delta(\varepsilon^\circ) = (\Delta(\varepsilon))^\circ$. □

The space of operators mapping a set $\mathcal{L}$ into a cisl $\mathcal{M}$ can be regarded as a power cisl $\mathcal{M}^{\mathcal{L}}$ (see Section 3). Thus the infimum of an arbitrary collection of operators between $\mathcal{L}$ and $\mathcal{M}$ exists. The following result is concerned with the infimum of erosions.

3.11. Proposition. *Let ε_i, $i \in I$, be erosions between the poset $\mathcal{L}$ and the cisl $\mathcal{M}$, and define $\varepsilon = \curlywedge_{i\in I} \varepsilon_i$. Then ε is an erosion between $\mathcal{L}$ and $\mathcal{M}$ with*

$$\mathcal{M}[\varepsilon] \subseteq \bigcap_{i\in I} M[\varepsilon_i] \tag{3.8}$$

$$\Delta(\varepsilon) = \curlyvee_{i\in I} \Delta(\varepsilon_i) \text{ on } \mathcal{M}[\varepsilon]\,. \tag{3.9}$$

Proof. It is evident that ε is an erosion and that (3.8) holds. To prove (3.9), observe that $\Delta(\varepsilon)(y) = \curlywedge\{x \in \mathcal{L} \mid y \preceq \varepsilon(x)\}$. If $y \preceq \varepsilon(x)$, then $y \preceq \varepsilon_i(x)$ for all $i \in I$, and therefore

$$\Delta(\varepsilon)(y) \succeq \curlywedge\{x \in \mathcal{L} \mid \ y \preceq \varepsilon_i(x)\} = \Delta(\varepsilon_i)(y)\,.$$

Thus $\Delta(\varepsilon)(y)$ is an upper bound of $\{\Delta(\varepsilon_i)(y) \mid i \in I\}$. Assume that $\Delta(\varepsilon_i)(y) \preceq a$ for $i \in I$. Then $y \preceq \varepsilon_i(a)$ for $i \in I$, which yields that $y \le \curlywedge_{i\in I} \varepsilon_i(a) = \varepsilon(a)$. But this means that $\Delta(\varepsilon)(y) \preceq a$, and we conclude that $\Delta(\varepsilon)(y)$ is the least upper bound of $\{\Delta(\varepsilon_i)(y) \mid i \in I\}$. This proves (3.9). □

The inclusion in (3.8) may be a strict inclusion as we show by means of an example.

3.12. Example. Let $\mathcal{L} = \{0, 1, 2, \dots\}$ with the following partial ordering: $n \preceq m$ if $n = 0$ or if $n+m$ is even and $n \leq m$. Thus $\mathcal{L}$ consists of two chains which are connected at the origin: $0 \preceq 1 \preceq 3 \preceq 5 \preceq \cdots$ and $0 \preceq 2 \preceq 4 \preceq 6 \preceq \cdots$. Consider the erosions $\varepsilon_1 = \mathsf{id}$ and $\varepsilon_2(n) = n - 1$ (with $\varepsilon_2(0) = 0$) from $\mathcal{L}$ into $\mathcal{L}$. Then $\mathcal{L}[\varepsilon_1] = \mathcal{L}[\varepsilon_2] = \mathcal{L}$ but $(\varepsilon_1 \,\mathrm{f}\, \varepsilon_2)(n) = 0$ for every n, thus in particular $\mathcal{L}[\varepsilon_1 \,\mathrm{f}\, \varepsilon_2] = \{0\}$. For the sake of completeness we mention that $\delta_1 = \mathsf{id}$ and that δ_2 is given by $\delta_2(n) = n + 1$ for $n > 0$ and $\delta_2(0) = 0$.Let $\mathcal{L} = \{0, 1, 2, \dots\}$ with the following partial ordering: $n \preceq m$ if $n = 0$ or if $n + m$ is even and $n \leq m$. Thus $\mathcal{L}$ consists of two chains which are connected at the origin: $0 \preceq 1 \preceq 3 \preceq 5 \preceq \cdots$ and $0 \preceq 2 \preceq 4 \preceq 6 \preceq \cdots$. Consider the erosions $\varepsilon_1 = \mathsf{id}$ and $\varepsilon_2(n) = n - 1$ (with $\varepsilon_2(0) = 0$) from $\mathcal{L}$ into $\mathcal{L}$. Then $\mathcal{L}[\varepsilon_1] = \mathcal{L}[\varepsilon_2] = \mathcal{L}$ but $(\varepsilon_1 \,\mathrm{f}\, \varepsilon_2)(n) = 0$ for every n, thus in particular $\mathcal{L}[\varepsilon_1 \,\mathrm{f}\, \varepsilon_2] = \{0\}$. For the sake of completeness we mention that $\delta_1 = \mathsf{id}$ and that δ_2 is given by $\delta_2(n) = n + 1$ for $n > 0$ and $\delta_2(0) = 0$.

3.4 Invariance properties

Adjunctions on Consider the cisl $\mathcal{T} = \mathbb{R}_0$ with the partial ordering $\preceq$ as defined in the previous section. Define the family of mappings ρ_v, $v \in \mathbb{R}$, on $\mathbb{R}_0$ by

$$\rho_v(t) = \begin{cases} t + v & \text{if } t, t + v > 0 \\ t - v & \text{if } t, t - v < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Note that the erosion ε_1 in Fig. 5 coincides with ρ_{-1} (restricted to the interval $[-3, 3]$).

We can show the following properties.

3.13. Proposition. *The family ρ_v satisfies the following properties:*

- *(a)* $\rho_0 = \mathsf{id}$;
- *(b)* $\rho_w \rho_v = \rho_{v+w}$ *if* $v, w \geq 0$;
- *(c)* $\rho_{-w} \rho_{-v} = \rho_{-v-w}$ *if* $v, w \geq 0$;
- *(d)* $\rho_{-w} \rho_v = \rho_{v-w}$ *if* $v \geq w \geq 0$;
- *(e)* $(\rho_v)^\circ = \rho_v$ *for all* v.

The proof of this result is not very difficult and we leave it as an exercise for the reader.

3.14. Proposition. *For every $v \geq 0$, the pair (ρ_{-v}, ρ_v) defines an adjunction on $\mathbb{R}_0$.*

Proof. We must show that

$$\rho_v(t) \preceq s \iff t \preceq \rho_{-v}(s)\,,$$

for $s, t \in I\!R_0$. Assume first that $\rho_v(t) \preceq s$. Without loss of generality we may assume that $t \geq 0$. If $t = 0$ the result follows immediately. If $t > 0$ then $t + v \leq s$ hence $t \leq s - v = s + (-v) = \rho_{-v}(s)$. This yields that $t \preceq \rho_{-v}(s)$.

Assume on the other hand that $t \preceq \rho_{-v}(s)$. Without loss of generality we may assume that $s \geq 0$. If $0 \leq s \leq v$ then $\rho_{-v}(s) = 0$, hence $t = 0$ as well and the result follows. If $s > v$, then $\rho_{-v}(s) = s - v$ and $t \preceq \rho_{-v}(s)$ means that $0 \leq t \leq s - v$, hence $t + v \leq s$. This implies that $\rho_v(t) \preceq s$. □

For $v > 0$ we write:

$$t \dot{+} v = \rho_v(t) \text{ and } t \dot{-} v = \rho_{-v}(t), \quad \text{for } t \in I\!R_0. \tag{3.11}$$

It is easy to see that all the previous result remain valid on $\mathbb{Z}_0$ (with also $v \in \mathbb{Z}_0$).

In many practical cases our interest goes towards adjunctions with additional properties. Here we consider adjunctions which are invariant under a given automorphism group. Let $\mathcal{L}$ be a poset and T an Abelian automorphism group on $\mathcal{L}$. An operator ψ on $\mathcal{L}$ is said to be T-*invariant* if

$$\psi\tau = \tau\psi, \quad \tau \in T\,.$$

The proof of the following result is easy.

3.15. Proposition. *Assume that the erosion $\varepsilon : \mathcal{L} \to \mathcal{L}$ is T-invariant, then $\mathcal{L}[\varepsilon]$ is T-invariant, i.e., $y \in \mathcal{L}[\varepsilon]$ implies $\tau(y) \in \mathcal{L}[\varepsilon]$ for every $\tau \in T$, and $\Delta(\varepsilon)$ is T-invariant.*

Below we consider a specific example in more detail. In the forthcoming sections we shall be concerned with adjunctions on $\mathcal{F}_0$ and $\mathcal{F}_r$ that are translation invariant in a sense to be specified later.

3.16. Example. Consider the cisl $\mathbb{C}$ provided with the partial ordering defined in (3.2). The mapping $\varepsilon : \mathbb{C} \to \mathbb{C}$ given by

$$\varepsilon(z) = E(|z|) \cdot \exp\bigl(iA(\arg z)\bigr) \tag{3.12}$$

defines an erosion if and only if E is an erosion on the cisl $(I\!R_+, \leq)$ with $E(0) = 0$ and $A : [0, 2\pi) \to [0, 2\pi)$ is an injective mapping. Let $R_A \subseteq [0, 2\pi)$ denote the range of A. It is easy to verify that

$$\mathbb{C}[\varepsilon] = \{w \in \mathbb{C} \mid \arg w \in R_A \text{ and } |w| \in I\!R_+[E]\,\}.$$

Note that $\mathbb{R}_+[E]$ is of the form $[0, W]$ with $W < \infty$ or $[0, W)$ with $W \leq \infty$.

The adjoint dilation $\delta = \Delta(\varepsilon)$ is given by

$$\delta(w) = D(|w|) \cdot \exp\bigl(iA^{-1}(\arg w)\bigr) ,$$

where D is the dilation on $\mathbb{R}_+[E]$ adjoint to erosion E. A simple example is given by $E(r) = cr$, where $c \geq 0$, and $A(\varphi) = \varphi + \alpha \,(\mathrm{mod}\, 2\pi)$. This corresponds to the erosion $\varepsilon(z) = ce^{i\alpha}z$. The adjoint dilation is $\delta(w) = c^{-1}e^{-i\alpha}w$ if $c > 0$ and $\delta(w) = 0$ for all w if $c = 0$.

The erosion in (3.12) can be generalized to the cisl $\mathcal{L} = \mathbb{C}^S$, where $S = \mathbb{R}^d$ or $\mathbb{Z}^d$, as follows:

$$\varepsilon(x)(s) = E_s(|x(s)|) \cdot \exp\bigl(iA_s(\arg x(s))\bigr) \tag{3.13}$$

where, for every $s \in S$, E_s and A_s satisfy the properties given above. Now consider the family of operators $M = \{\mu_{q,a} \mid q > 0, a \in S\}$ on $\mathcal{L}$ given by

$$\mu_{q,a}(x)(s) = qe^{i\langle a\,,\,s\rangle}x(s), \quad s \in S\,.$$

Here $\langle \cdot\,,\,\cdot \rangle$ is the vector product on $S \times S$. It is easy to see that every $\mu_{q,a}$ is a cisl-automorphism on $\mathcal{L}$ and that

$$\mu_{q,a}\mu_{r,b} = \mu_{qr,a+b},\ q, r > 0,\ a, b \in S\,,$$

whence it follows that M is an Abelian automorphism group on $\mathcal{L} = \mathbb{C}^S$. If ε given by (3.13) is required to be M-invariant, we find that for every $s \in S$, the mappings E_s and A_s satisfy

$$\begin{aligned} E_s(qr) &= qE_s(r),\ q, r > 0\,, \\ A_s(\varphi + \langle a\,,\,s\rangle) &= A_s(\varphi) + \langle a\,,\,s\rangle,\ \varphi \in [0, 2\pi),\ a \in S. \end{aligned}$$

Thus we get that E_s and A_s are of the form

$$E_s(r) = c(s)r, \quad A_s(\varphi) = \varphi + \alpha(s)\,,$$

where $c : S \to \mathbb{R}_+$ and $\alpha : S \to [0, 2\pi)$. Writing $e(s) = c(s)e^{i\alpha(s)}$, we obtain that

$$\varepsilon(x)(s) = e(s)x(s),\ s \in S.$$

Before concluding this example, we point out the relation with linear filtering. Taking the Fourier transform $\hat{f}$ of a signal $f : \mathbb{R} \to \mathbb{R}$ (integrable or square integrable), we end up in the cisl $\mathcal{L}$ (where $S = \mathbb{R}$). The cisl ordering on $\mathcal{L}$ thus induces a cisl ordering on the original space; see e.g. Proposition 3.3. Furthermore, the Fourier transform maps translation invariance of an operator on the original space onto invariance under modulations (the mappings $\mu_{1,a}$) and grey-scale invariance onto grey-scale invariance. The erosion $\varepsilon(x) = ex$ on the Fourier transformed domain corresponds (via the inverse Fourier transform) to a linear convolution on the original domain. These observations suggest that linear convolution operators can be considered as erosions with respect to a very specific partial ordering on the underlying space. We shall not pursue this matter further here.

4 Translation Invariance

4.1 Standard translations

Let $\mathcal{T} = \mathbb{R}_0$ or $\mathbb{Z}_0$ provided with the cisl ordering $\preceq$. As before, we denote by $\mathcal{F}_r(E, \mathcal{T})$ the functions $x : E \to \mathcal{T}$ provided with cisl ordering $\preceq_r$; here $r : E \to \mathcal{T}$ is a given reference function. When no confusion about E or $\mathcal{T}$ is possible, we write $\mathcal{F}_r$.

The operators ρ_v defined in (3.10) can be extended to the cisl $\mathcal{F}_0$ by pointwise application: $\rho_v(x)(p) = \rho_v(x(p))$. Using the intertwining diagram in Section 3, these operators can also be extended to $\mathcal{F}_r$ for any reference function r. The properties in Propositions 3.13 and 3.14 remain valid. An illustration is given in Fig. 6. Define the

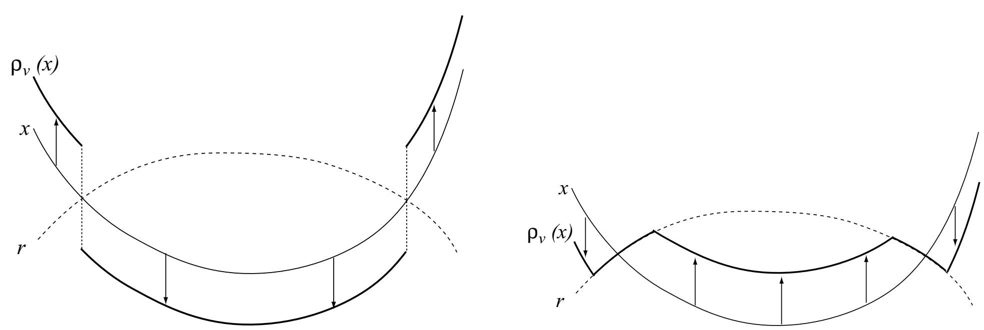


Figure 6: *Vertical translation for $v > 0$ and $v < 0$.*

translation operator τ_h, $h \in E$, on $\mathcal{F}_0$ as follows:

$$\tau_h(x)(p) = x(p - h), \; x \in \mathcal{F}_0, \; p \in E.$$

The following properties are straightforward.

4.1. Proposition. *The family τ_h, $h \in E$, of operators on $\mathcal{F}_0$ has the following properties:*

(a) *every τ_h is a cisl-automorphism;*

(b) $\tau_h \tau_k = \tau_{h+k}$, *for* $h, k \in E$;

(c) $(\tau_h)^\circ = \tau_h$, *for* $h \in E$.

Furthermore, it is easy to verify the following commutation relation:

$$\tau_h \rho_v = \rho_v \tau_h, \quad h \in E,\ v \in \mathcal{T}. \tag{4.1}$$

In other words, the operators ρ_v are T-invariant, T being the family of translations τ_h.

Let A be a subset of E and assume that e_h is an erosion on $\mathcal{T}$ for every $h \in A$. It is not difficult to see that the operator $\varepsilon : \mathcal{F}_0 \to \mathcal{F}_0$ given by

$$\varepsilon = \bigwedge_{h \in A} e_h \tau_h \,,$$

or alternatively

$$\varepsilon(x)(p) = \bigwedge_{h \in A} e_h(x(p-h)) \,, \tag{4.2}$$

defines an erosion. The set $\mathcal{F}_0[\varepsilon]$ comprises all functions $y \in \mathcal{F}_0$ for which

$$y(p) \preceq e_h(x(p-h)), \quad \text{for all} \quad h \in A,\ p \in E,$$

or alternatively,

$$y(p+h) \in \mathcal{T}[e_h],\ h \in A,\ p \in E.$$

Let d_h be the dilation on $\mathcal{T}[e_h]$ that forms an adjunction with e_h, then

$$d_h(y(p+h)) \preceq x(p),\ h \in A,\ p \in E\,.$$

We conclude that the supremum of $d_h(y(p+h))$ over $h \in A$ exists in this case, and

$$\bigvee_{h \in A} d_h(y(p+h)) \preceq x(p),\ p \in E\,.$$

The expression at the left is the dilation $\delta = \Delta(\varepsilon)$ adjoint to ε:

$$\delta(y)(p) = \bigvee_{h \in A} d_h(y(p+h)) \preceq x(p),\ p \in E\,, \tag{4.3}$$

or alternatively,

$$\delta = \bigvee d_h \tau_{-h}\,.$$

If we choose e_h on $\mathcal{T}$ as (see (3.11))

$$e_h(t) = t \dot{-} g(h)\,,$$

where $g(h) > 0$ for $h \in A$, then we find

$$\varepsilon(x)(p) = \bigwedge_{h \in A} \bigl(x(p-h) \dot{-} g(h)\bigr) . \tag{4.4}$$

The adjoint dilation is given by

$$\delta(y)(p) = \bigvee_{h \in A} \bigl(y(p+h) \dot{+} g(h)\bigr) , \tag{4.5}$$

presumed that $y \in \mathcal{F}_0[\varepsilon]$. Besides being translation invariant, ε and δ have the following invariance property:

$$\varepsilon \rho_{-v} = \rho_{-v} \varepsilon \quad \text{and} \quad \delta \rho_v = \rho_v \delta ,$$

for every $v > 0$. Note that ρ_v maps $\mathcal{F}_0[\varepsilon]$ into $\mathcal{F}_0[\varepsilon]$.

4.2. Remark. In the classical case where $\mathcal{L}$ comprises the functions from E to $\mathcal{T} = \overline{\mathbb{R}}$ or $\overline{\mathbb{Z}}$ provided with the usual complete lattice ordering, it is true that every translation invariant erosion ε is of the form (4.2). In the cisl case discussed here this is no longer true. For example, the operator ε given by

$$\varepsilon(x)(p) = \begin{cases} x(p), & \text{if } x(p)x(p-1) > 0 , \\ 0, & \text{otherwise} , \end{cases}$$

is a translation invariant erosion on $\mathcal{F}_0$ which is not of the form (4.2).

An important subclass of erosions, as defined by (4.2), is obtained if one chooses for e_h the identity mapping, for every h in the structuring element A. Such erosions are given by

$$\varepsilon(x)(p) = \bigwedge_{h \in A} x(p-h) . \tag{4.6}$$

In Fig. 7 we depict the erosion ε and the correspond opening $\delta\varepsilon$ on $\mathcal{F}_0$ for the case where $A = \{-a, -a+1, \dots, -1, 0, 1, \dots, a\}$ for $a = 3$. In Fig. 8 we show the same operators for the cisl $\mathcal{F}_r$; here we have used the intertwining construction given in Section 3. In this case the expression for ε is

$$\varepsilon(x)(p) = r(p) + \bigwedge_{h \in A} \bigl[x(p-h) - r(p-h)\bigr] . \tag{4.7}$$

In Fig. 9 we show the 2-dimensional erosion of a given input image with respect to a given reference image. In Fig. 10 we show how to use the cisl opening for noise filtering. Fig. 11, which is partially given for the sake of curiosity, but also since it illustrates the mechanism behind the cisl reference erosion, shows the transition from an input image to another reference image.

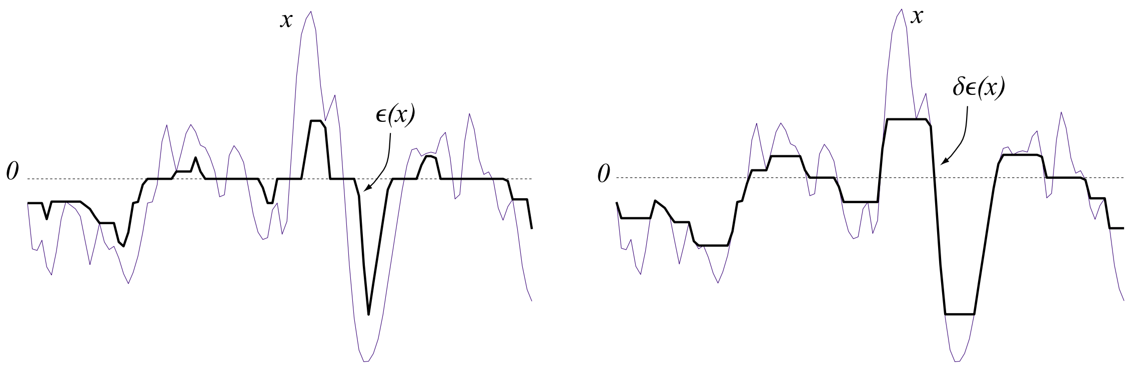


Figure 7: *Erosion (left) and opening (right) of a signal in* $\mathcal{F}_0$.

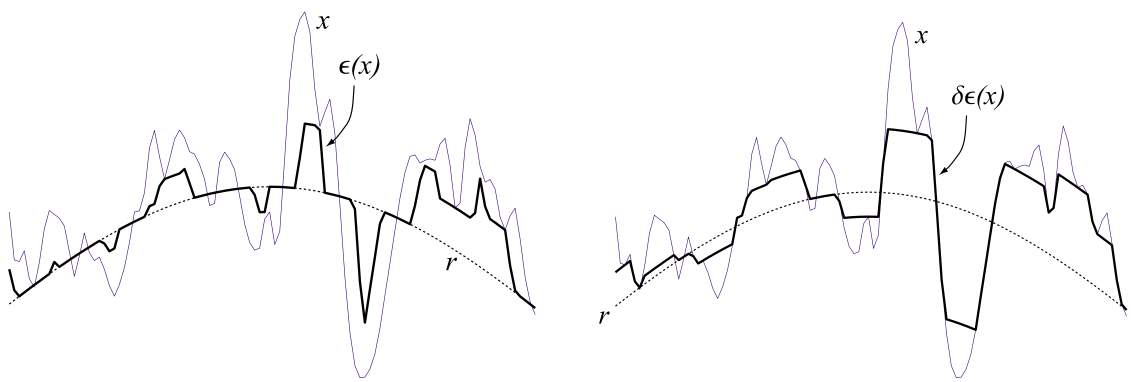


Figure 8: *Erosion (left) and opening (right) of a signal in* $\mathcal{F}_r$.

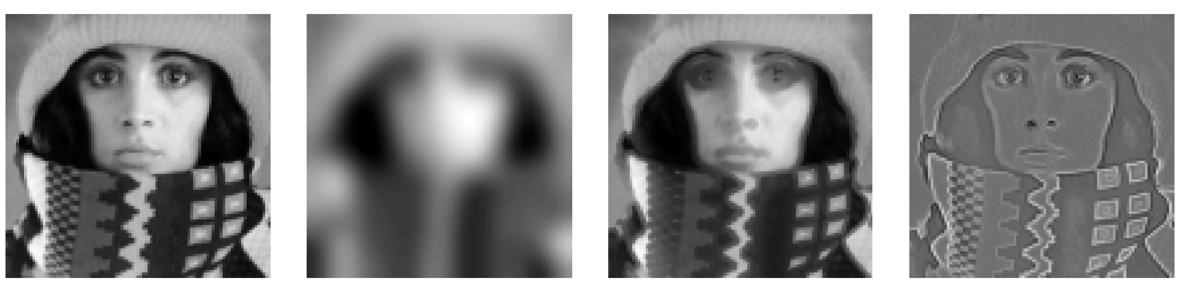


Figure 9: *Erosion as defined by* (4.7) *of an input image (left) with respect to a given reference image (second). The third image is the eroded image and the right-most image shows the difference between the input and the output (with enhanced contrast).*

4.2 Signed translations

In this section we introduce an alternative class of translations for discrete signals. The key difference with the translations defined in the previous subsection is that the

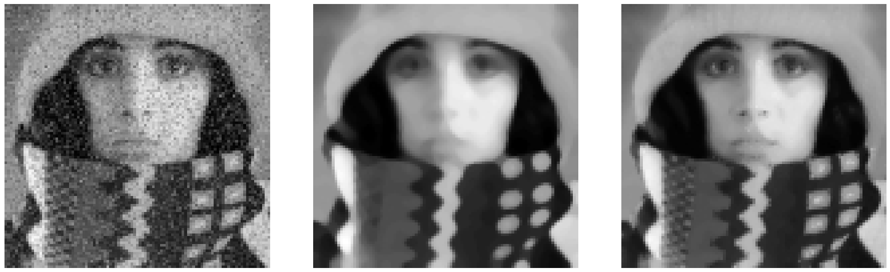


Figure 10: *The cisl reference opening obtained by composing ε in* (4.7) *with its adjoint dilation can be used to remove noise. From left to right: the input noisy image, the result after iterative filtering, and the image obtained by applying the cisl reference opening to the input image, with the median-filtered image as reference image.*

Figure 11: *The two images that we start with are the input image x at the top left and the reference image r at the bottom right. The sequence $\varepsilon^n(x)$, where ε is given by* (4.7)*, converges to r when n increases.*

translations defined below do not change the sign of a function at a given point. Thus, if the function is nonnegative at a given location, then it cannot become negative due to translation. In other words, the sign of the function is preserved. For that reason we call them *signed translations.*

Consider the cisl $\mathcal{F}_0$ of functions mapping $\mathbb{Z}$ into $\mathbb{Z}_0$ provided with the partial ordering

$\preceq$. Define the operator ε on $\mathcal{F}_0$ by

$$\varepsilon(x)(n) = \begin{cases} x(n-1) \vee 0, & \text{if } x(n) > 0 \\ x(n-1) \wedge 0, & \text{if } x(n) < 0 \\ 0, & \text{if } x(n) = 0 \end{cases}$$

At a given location n this operator describes a shift towards the right as long as $x(n-1)$ and $x(n)$ are both positive or both negative, i.e., $x(n-1)x(n) > 0$. If $x(n-1)x(n) \leq 0$, then $\varepsilon(x)(n) = 0$. Thus we can also write

$$\varepsilon(x)(n) = \begin{cases} x(n-1), & \text{if } x(n-1)x(n) > 0 \\ 0, & \text{if } x(n-1)x(n) \leq 0\,. \end{cases}$$

Furthermore, we define a second operator $\delta : \mathcal{F}_0 \to \mathcal{F}_0$ describing a leftward shift:

$$\delta(y)(n) = \begin{cases} y(n+1), & \text{if } y(n+1) \neq 0 \\ \operatorname{sign} y(n), & \text{if } y(n+1) = 0\,. \end{cases}$$

Here $\operatorname{sign} t$ denotes the sign of t, which is defined to be 0 if $t = 0$. Define

$$\mathcal{F}_0^{(1)} = \{x \in \mathcal{F}_0 \mid x(n)x(n+1) \geq 0 \text{ for } n \in \mathbb{Z}\}\,.$$

4.3. Proposition. *The operator ε defines an erosion on $\mathcal{F}_0$ with*

$$\mathcal{F}_0[\varepsilon] = \mathcal{F}_0^{(1)}$$

and $\Delta(\varepsilon) = \delta$.

Proof. First we show that (ε, δ) is an adjunction between $\mathcal{F}_0$ and $\mathcal{F}_0^{(1)}$, that is $\delta(y) \preceq x$ iff $y \preceq \varepsilon(x)$ for $x \in \mathcal{F}_0$ and $y \in \mathcal{F}_0^{(1)}$.

Assume that $\delta(y) \preceq x$; we must show that $y \preceq \varepsilon(x)$, i.e., that $y(n) \preceq \varepsilon(x)(n)$, for $n \in \mathbb{Z}$. If $y(n) = 0$ then this is obvious. We consider the case where $y(n) > 0$; evidently the case $y(n) < 0$ is treated analogously. Using that $\delta(y) \preceq x$ at $n-1$ we get $y(n) \preceq x(n-1)$. Suppose $x(n) = 0$, then $\delta(y)(n) = 0$, hence $y(n) = 0$, which contradicts our assumption that $y(n) > 0$. Suppose $x(n) < 0$, then $\delta(y)(n) < 0$. Obviously, $y(n+1) \neq 0$, for otherwise $\delta(y)(n) = \operatorname{sign} y(n) = 1$. We get $\delta(y)(n) = y(n+1)$. From $\delta(y)(n) \preceq x(n)$ we find that $y(n+1) \preceq x(n) < 0$. But then $y(n)y(n+1) < 0$ which contradicts the fact that $y \in \mathcal{F}_0^{(1)}$. We conclude that $x(n) > 0$. We have seen above that $0 < y(n) \preceq x(n-1)$, hence $\varepsilon(x)(n) = x(n-1)$, and indeed we have shown that $y(n) \preceq \varepsilon(x)(n)$.

Assume that $y \preceq \varepsilon(x)$; we must show that $\delta(y) \preceq x$, i.e., that $\delta(y)(n) \preceq x(n)$ for $n \in \mathbb{Z}$. First assume that $x(n) = 0$. Then $\varepsilon(x)(n) = \varepsilon(x)(n+1) = 0$ and therefore $y(n) = y(n+1) = 0$. This yields that $\delta(y)(n) = 0$, and thus $\delta(y)(n) \preceq x(n)$ in this case. Thus it remains to consider the case where $x(n) > 0$; the case $x(n) < 0$ is treated analogously. We distinguish two cases: $y(n+1) = 0$ and $y(n+1) > 0$. (Again, the case $y(n+1) < 0$ is treated analogously to the case where $y(n+1) > 0$.)

(i) $y(n+1) = 0$. Then $\delta(y)(n) = \operatorname{sign} y(n)$. If $y(n) = 0$ then $\delta(y)(n) = 0$ and the inequality $\delta(y)(n) \preceq x(n)$ is trivially satisfied. If $y(n) > 0$ then $\delta(y)(n) = 1$. Since $\varepsilon(x)(n) \succeq y(n) > 0$ we get that $x(n) > 0$, hence $\delta(y)(n) \preceq x(n)$. If $y(n) < 0$ then $\delta(y)(n) = -1$ and from $\varepsilon(x)(n) \succeq y(n)$ we get that $x(n) < 0$, which then yields that $\delta(y)(n) \preceq x(n)$.

(ii) $y(n+1) > 0$. Then $\delta(y)(n) = y(n+1) > 0$. Since $\varepsilon(x)(n+1) \succeq y(n+1)$ we conclude that $x(n+1) > 0$ and $x(n) \geq y(n+1) > 0$, i.e., $\delta(y)(n) \preceq x(n)$.

Thus we have shown that (ε, δ) is an adjunction between $\mathcal{F}_0$ and $\mathcal{F}_0^{(1)}$. Thus we are left with the task to show that

$$\mathcal{F}_0[\varepsilon] = \mathcal{F}_0^{(1)} .$$

Suppose first that $y \in \mathcal{F}_0^{(1)}$. Then $y \preceq \varepsilon\delta(y)$ hence $y \in \mathcal{F}_0[\varepsilon]$. On the other hand, let $y \in \mathcal{F}_0[\varepsilon]$; we must demonstrate that $y(n)y(n+1) \geq 0$ for every $n \in \mathbb{Z}$. Assume that $y(n) > 0$ and $y(n+1) < 0$. Then $\varepsilon(x)(n) > 0$, which requires that $x(n-1) > 0$ and $x(n) > 0$. But in this case also $\varepsilon(x)(n+1) \geq 0$ which contradicts $y(n+1) \preceq \varepsilon(x)(n+1)$. This concludes the proof. □

The adjunction (ε, δ) forms the basis ingredient for a new class of translations. Define, for every integer $k \geq 1$ the set $\mathcal{F}_0^{(k)} \subseteq \mathcal{F}_0$ by

$$\mathcal{F}_0^{(k)} = \{x \in \mathcal{F}_0 \mid x(n)x(n+j) \geq 0 \text{ for } n \in \mathbb{Z} \text{ and } j = 1, 2, \ldots, k\} .$$

One can easily show that

$$\mathcal{F}_0^{(k)} \subseteq \mathcal{F}_0^{(k-1)}, \text{ for } k \geq 1 ,$$

where $\mathcal{F}_0^{(0)} = \mathcal{F}_0$. Furthermore, ε maps $\mathcal{F}_0^{(k)}$ into $\mathcal{F}_0^{(k+1)}$ and δ maps $\mathcal{F}_0^{(k+1)}$ into $\mathcal{F}_0^{(k)}$. In fact, we have the following extension of the previous proposition.

4.4. Proposition. *For every $n \geq 0$ and $k \geq 1$, the operator ε^k defines an erosion on $\mathcal{F}_0^{(n)}$ with*

$$\mathcal{F}_0^{(n)}[\varepsilon^k] = \mathcal{F}_0^{(n+k)}$$

and $\Delta(\varepsilon^k) = \delta^k$.

In the sequel we use the following notation:

$$\sigma_k = \varepsilon^k \text{ and } \sigma_k^{\leftarrow} = \delta^k, \; k \geq 0 .$$

For $k \leq 0$ we define σ_k and $\sigma_k^{\leftarrow}$ by using the erosion ε' governing translation to the left as starting point:

$$\varepsilon'(x)(n) = \begin{cases} x(n+1) \vee 0, & \text{if } x(n) > 0 \\ x(n+1) \wedge 0, & \text{if } x(n) < 0 \\ 0, & \text{if } x(n) = 0 \end{cases}$$

It is evident that the corresponding set $\mathcal{F}_0[\varepsilon']$ equals $\mathcal{F}_0^{(1)}$ and more generally, that $\mathcal{F}_0[\varepsilon'^k]$ equals $\mathcal{F}_0^{(k)}$.

In Fig. 12 we depict a signal x, its signed translate $\sigma_k(x)$, and the inverse $\sigma_k^{\leftarrow}\sigma_k(x)$, which, being a composition of an erosion and a dilation, is an opening. Note that the

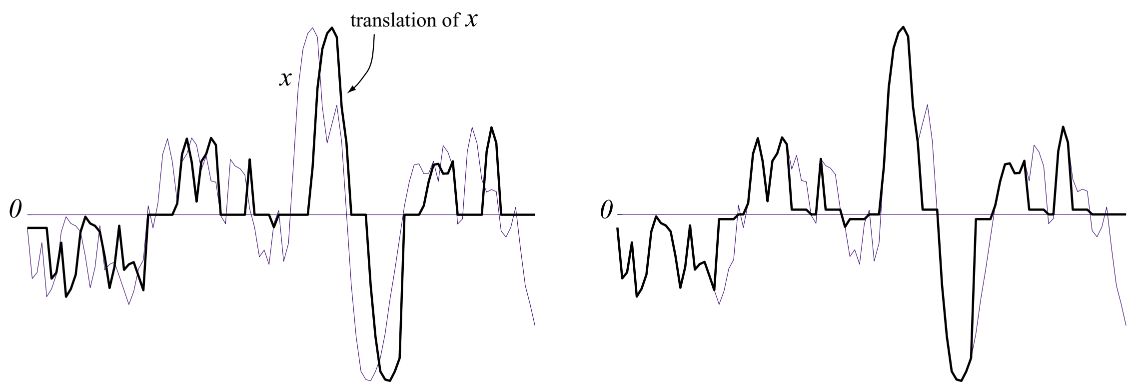


Figure 12: *A signal x (thin line), it's signed translate $\sigma_k(x)$ with $k = 4$ (left), and the opening $\sigma_k^{\leftarrow}\sigma_k(x)$ (right).*

translation family σ_k does not have the same nice properties as the family τ_k introduced in the previous section. In particular, it is not a group: we only have

$$\sigma_k \sigma_l = \sigma_{k+l} \quad \text{if} \quad kl \geq 0\,.$$

If, however, k, l have opposite signs, then this relation fails to be true. In particular, it does not hold that $\sigma_k \sigma_{-k} = \mathsf{id}$: *signed translations are not invertible.*

It is easy to verify that

$$\sigma_k \rho_v = \rho_v \sigma_k\,, \quad k \in \mathbb{Z},\ v \in \mathbb{Z}\,;$$

(c.f. relation (4.1)). Furthermore, we can easily establish the following relationships:

$$\sigma_k \tau_l = \tau_l \sigma_k,\ \ k, l \in \mathbb{Z}$$
$$\sigma_k^{\leftarrow} \tau_l = \tau_l \sigma_k^{\leftarrow},\ \ k, l \in \mathbb{Z}\,.$$

Furthermore, it holds that

$$\mathsf{id}\,\mathrm{f}\,\sigma_1 = \mathsf{id}\,\mathrm{f}\,\tau_1\,, \tag{4.8}$$

and as we argue below, this relation has some important consequences.

Using the signed translation operators σ_k, we can define a new family of erosions in the following way: let $A \subseteq \mathbb{Z}$ be a finite structuring element and define $K = \max\{|k| \mid k \in A\}$. Then

$$\varepsilon_A = \bigcurlywedge_{k \in A} \sigma_k \tag{4.9}$$

defines an erosion that maps $\mathcal{F}_0$ into $\mathcal{F}_0^{(K)}$. The proof that the range of ε_A is contained in $\mathcal{F}_0^{(K)}$ is based on the observation that $x \in \mathcal{F}_0^{(k)}$ and $y \in \mathcal{F}_0^{(l)}$ implies $x \,\mathrm{f}\, y \in \mathcal{F}_0^{(m)}$, where $m = \max\{k, l\}$. The dilation adjoint to ε_A is

$$\delta_A = \bigcurlyvee_{k \in A} \overleftarrow{\sigma_k}, \tag{4.10}$$

which is well-defined on $\mathcal{F}_0^{(K)}$. In Fig. 13 we compare the adjunction corresponding

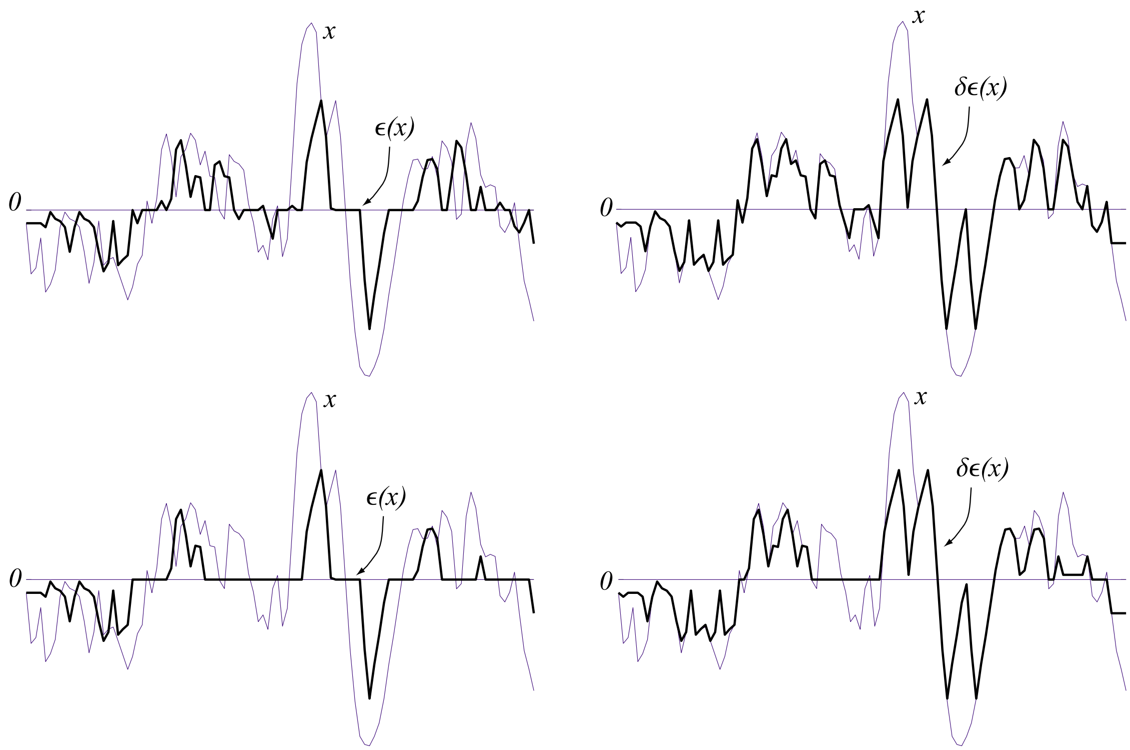


Figure 13: *Comparison between operators deriving from the standard translation (top row) and the signed translation (bottom row). The first column shows the signal (thin line) and its erosion (fat line), the second column the signal and its opening. In both cases we use structuring element* $\{-3, 3\}$.

to the signed translation and the adjunction deriving from the standard translation as discussed in the previous section. We used a structuring element of the form $A = \{-a, a\}$. If A is of the form $[-a, a]$, both erosions yield the same ouput. This is a straightforward consequence of the identity given in (4.8).

We can extend the signed translations to two dimensions by decomposition into a horizontal and vertical component. Translation of a 2-dimensional signal over the vector (k, l) only yields a positive (resp. negative) value at the point with coordinates (m, n) if and only if $x(i, j)$ is positive (resp. negative) at the entire rectangle $[m - k, m] \times [n - l, n]$.

5 Lattice Ordered Groups and Inf-Semilattices

5.1 Lattice ordered groups

An interesting construction method for cisl's uses so called *lattice ordered groups* and will be discussed below.

5.1. Definition. A nonempty set $\mathcal{L}$ with an addition $+$ and a a partial ordering relation $\leq$ is called a *lattice ordered group* if

(i) $(\mathcal{L}, +)$ is a group;

(ii) $(\mathcal{L}, \leq)$ is a lattice;

(iii) the addition is isotone, i.e.,

$$x \leq y \text{ implies } x + a \leq y + a \quad \text{and} \quad a + x \leq a + y \tag{5.1}$$

for $a, x, y \in \mathcal{L}$.

(iv) the addition $+$ is distributive over the supremum and infimum, i.e.,

$$(x \wedge y) + a = (x + a) \wedge (y + a) \quad \text{and} \quad a + (x \wedge y) = (a + x) \wedge (a + y) \tag{5.2}$$

$$(x \vee y) + a = (x + a) \vee (y + a) \quad \text{and} \quad a + (x \vee y) = (a + x) \vee (a + y) \tag{5.3}$$

for $x, y, a \in \mathcal{L}$.

Some background on lattice ordered groups can be found in [1, 2, 3].

Denoting the inverse of an element x with respect to the group operation $+$ by $-x$ we have

$$x \leq y \iff -y \leq -x.$$

In fact, it follows that $x \mapsto -x$ is a dual automorphism on the lattice $(\mathcal{L}, \leq)$. In particular, we have

$$-(x \wedge y) = -x \vee -y \quad \text{and} \quad -(x \vee y) = -x \wedge -y\,. \tag{5.4}$$

In [2] the following result has been proved; see also [1].

5.2. Proposition. *If $(\mathcal{L}, +, \leq)$ is a partially ordered group such that $(\mathcal{L}, \leq)$ is an inf-semilattice, then $(\mathcal{L}, +, \leq)$ is a lattice ordered group and, moreover, the lattice $(\mathcal{L}, \leq)$ is distributive.*

Throughout the remainder of this section we assume that $(\mathcal{L}, +, \leq)$ is a lattice ordered group. Let 0 be the unit element with respect to the group operation $+$. Define the *cone* $\mathcal{L}^+$ as

$$\mathcal{L}^+ = \{x \in \mathcal{L} \mid x \geq 0\}.$$

Elements in $\mathcal{L}_+$ are said to be *positive*. It is obvious that $x, y \in \mathcal{L}^+$ implies that $x + y \in \mathcal{L}^+$. Furthermore, $\mathcal{L}^+ \cap -\mathcal{L}^+ = \{0\}$. Define, for an element $x \in \mathcal{L}$ the elements $x^+, x^- \in \mathcal{L}^+$ by[2]:

$$x^+ = x \vee 0 \quad \text{and} \quad x^- = -(x \wedge 0)\,. \tag{5.5}$$

It is easy to see that

$$(-x)^+ = x^- \quad \text{and} \quad (-x)^- = x^+\,. \tag{5.6}$$

5.3. Proposition. *For every $x \in \mathcal{L}$ we have*

$$x = x^+ - x^-\,. \tag{5.7}$$

Proof. Let $x \in \mathcal{L}$, then

$$\begin{aligned} x^+ - x^- &= (x \vee 0) + (x \wedge 0) = (x + (x \wedge 0)) \vee (0 + (x \wedge 0)) \\ &= (x + (x \wedge 0)) \vee (x + (0 \wedge -x)) = x + ((x \wedge 0) \vee (0 \wedge -x)) \\ &\leq x + 0 = x\,, \end{aligned}$$

where we have used that $(x \wedge 0) \vee (0 \wedge -x) \leq 0$. Hence $x^+ - x^- \leq x$. Substituting $-x$ for x this yields $(-x)^+ - (-x)^- \leq -x$. Using (5.6) we get $x^- - x^+ \leq -x$ which implies $x^+ - x^- \geq x$, whence the assertion follows. □

We define the *absolute value* of $x \in \mathcal{L}$ by

$$|x| = x^+ + x^-. \tag{5.8}$$

For the proof of the following result we refer to Birkhoff [1]; see also [3, Chapter V].

[2] Observe that our definition of x^- is different from that often found in the literature. There one usually defines $x^- = x \wedge 0$

5.4. Proposition. *For $x, y \in \mathcal{L}$ the following is true:*

(a) $|x| > 0$ *if* $x \neq 0$*;*

(b) $x^+ \wedge x^- = 0$*;*

(c) $|x| = x \vee -x$*;*

(d) $|x - y| = (x \vee y) - (x \wedge y)$*.*

5.5. Definition. Two positive elements $x, y \in \mathcal{L}$ are said to be *disjoint* (or *orthogonal*) if $x \wedge y = 0$.

It is easy to see that

$$x, y \text{ are disjoint} \iff x \vee y = x + y\,. \tag{5.9}$$

Namely,

$$x \wedge y = 0 \iff -x \vee -y = 0 \iff 0 \vee (x - y) = x \iff y \vee x = x + y.$$

In particular this means that $x + y = y + x$ if x, y are disjoint.

Property (b) in Proposition 5.4 above says that x^+ and x^- are disjoint, for every element $x \in \mathcal{L}$.

We make the following additional assumption on $\mathcal{L}$.

5.6. Assumption. Every nonempty subset of $\mathcal{L}$ which possesses a lower bound has an infimum in $\mathcal{L}$.

If this assumption holds, it is automatically true that every nonempty subset which has an upper bound has a supremum. Furthermore, the relations in (5.2)-(5.4) carry over to infinite infima and suprema. For example, the first relation in (5.2) generalizes to

$$(\bigwedge_{i \in I} x_i) + a = \bigwedge_{i \in I} (x_i + a)\,.$$

This means that $\{x_i \mid i \in I\}$ has an infimum iff $\{x_i + a \mid i \in I\}$ has an infimum and the previous relation holds.

5.2 Defining a new partial ordering

Define a binary relation $\preceq$ on $\mathcal{L}$ as follows:

$$x \preceq y \quad \text{if} \quad x^+ \leq y^+ \quad \text{and} \quad x^- \leq y^- \,. \tag{5.10}$$

The following important result holds.

5.7. Proposition. *$(\mathcal{L}, \preceq)$ is a cisl with the infimum of a collection $\{x_i\}$ given by*

$$\curlywedge_{i\in I} x_i = \bigwedge_{i\in I} x_i^+ - \bigwedge_{i\in I} x_i^- \,. \tag{5.11}$$

This infimum satisfies

$$(\curlywedge_{i\in I} x_i)^+ = \bigwedge_{i\in I} x_i^+ \quad \text{and} \quad (\curlywedge_{i\in I} x_i)^- = \bigwedge_{i\in I} x_i^- \tag{5.12}$$

The least element of $(\mathcal{L}, \preceq)$ is 0.

Proof. First we show that '$\preceq$' defines a partial ordering on $\mathcal{L}$. Reflexivity and transitivity are evident. We show that '$\preceq$' is anti-symmetric, i.e., that $x \preceq y$ and $y \preceq x$ implies $x = y$. Obviously, $x \preceq y$ and $y \preceq x$ yield that $x^+ = y^+$ and $x^- = y^-$. But then $x = y$. It is also clear that $0 \preceq x$ for $x \in \mathcal{L}$.

Now consider a family $\{x_i\} \subseteq \mathcal{L}$ and define $a = \bigwedge_{i\in I} x_i^+$ and $b = \bigwedge_{i\in I} x_i^-$. Evidently, $a, b \in \mathcal{L}^+$ and

$$a \wedge b = \bigwedge_{i\in I} (x_i^+ \wedge x_i^-) = 0\,,$$

since $x_i^+ \wedge x_i^- = 0$ for every $i \in I$. Subtracting b at both sides and using the distributivity in (5.2) we get that $(a - b) \wedge 0 = -b$, that is

$$(a-b)^- = b = \bigwedge_{i\in I} x_i^- \,,$$

hence $(a - b)^- \leq x_i^-$ for every $i \in I$. Furthermore, from the disjointness of a and b and (5.9) we derive that

$$a \vee b = a + b\,.$$

Subtracting b from both sides and using the distributivity in (5.3) we get

$$(a-b) \vee 0 = a = \bigwedge_{i\in I} x_i^+ \,,$$

hence $(a - b)^+ \leq x_i^+$ for $i \in I$. We have shown that $(a - b)^- \leq x_i^-$ and $(a - b)^+ \leq x_i^+$ for $i \in I$, and therefore $a - b \preceq x_i$ for $i \in I$. This means that $a - b$ is a lower bound of $\{x_i\}$ with respect to '$\preceq$'. Suppose that c is another lower bound; we show that $c \preceq a - b$. In fact, if c is a lower bound of $\{x_i\}$ then $c^+ \preceq \bigwedge_{i\in I} x_i^+ = (a - b)^+$. Analogously we get $c^- \leq (a - b)^-$ and thus $c \preceq a - b$. We conclude that $a - b$ is the greatest lower bound of $\{x_i\}$. The equalities in (5.12) follow from the arguments above. □

Note that

$$x \preceq y \;\Rightarrow\; |x| \leq |y|. \tag{5.13}$$

The converse is not true in general, however.

5.8. Example. (a) Consider once again the set $\mathcal{L} = \mathcal{T}^E$ where $\mathcal{T} = \mathbb{R}$ or $\mathbb{Z}$ provided with the standard ordering $x \leq y$ if $x(p) \leq y(p)$ for $p \in E$, and the addition $(x + y)(p) = x(p) + y(p)$. It is evident that $(\mathcal{L}, +, \leq)$ is a lattice ordered group for which Assumption 5.6 holds. The corresponding cisl $(\mathcal{L}, \preceq)$ coincides with $\mathcal{F}_0$ in this case.

(b) We define another addition $+_r$, where $r \in \mathcal{L}$ is a given element, by

$$(x +_r y)(p) = x(p) + y(p) - r(p)\,.$$

Again, it is not difficult to verify that $(\mathcal{L}, +_r, \leq)$ is a lattice ordered group for which Assumption 5.6 holds. The unit element of the group $(\mathcal{L}, +_r)$ is r. The cisl $(\mathcal{L}, \preceq_r)$ that we obtain in this case is the reference cisl $\mathcal{F}_r$.

If ψ is an automorphism on $(\mathcal{L}, \leq)$ with $\psi(0) = 0$, then

$$\psi(x^+) = \psi(x \vee 0) = \psi(x) \vee \psi(0) = \psi(x) \vee 0 = \psi(x)^+\,.$$

Analogously, we get that $\psi(x^-) = \psi(x)^-$, and the following result holds.

5.9. Proposition. *If ψ is an automorphism on $(\mathcal{L}, \leq)$ with $\psi(0) = 0$, then ψ is also an automorphism on $(\mathcal{L}, \preceq)$.*

But also $*$-negations on $(\mathcal{L}, \leq)$ yield automorphisms on $(\mathcal{L}, \preceq)$, as the following result shows.

5.10. Proposition. *Every $*$-negation on $(\mathcal{L}, \leq)$ which maps 0 onto 0 defines an automorphism on $(\mathcal{L}, \preceq)$.*

Proof. Let ν be a $*$-negation on $(\mathcal{L}, \leq)$ with $\nu(0) = 0$. We must show that ν defines an increasing mapping on $(\mathcal{L}, \preceq)$. Assume that $x \preceq y$, that is, $x^+ \leq y^+$ and $x^- \leq y^-$. From the fact that ν is a $*$-negation on $(\mathcal{L}, \leq)$, we derive that

$$\nu(x)^+ = \nu(x) \vee 0 = \nu(x) \vee \nu(0) = \nu(x \wedge 0) = \nu(-x^-)\,.$$

Similarly we derive that $\nu(x)^- = \nu(-x^+)$. Thus

$$\nu(x)^+ = \nu(-x^-) \leq \nu(-y^-) = \nu(y)^+\,,$$

as well as

$$\nu(x)^- = \nu(-x^+) \leq \nu(-y^+) = \nu(y)^-\,,$$

and we conclude that $\nu(x) \preceq \nu(y)$, which finishes the proof. $\square$

This last result holds in particular for the $*$-negation $\nu(x) = -x$. A combination of the two previous results leads to the following corollary.

5.11. Corollary. *If ψ is an automorphism on $(\mathcal{L}, \leq)$ with $\psi(0) = 0$, then ψ and $-\psi$ define automorphisms on $(\mathcal{L}, \preceq)$.*

5.3 Operator constructions

In what follows we shall define operators ψ on $\mathcal{L}$ starting from two operators ψ^+, ψ^- on $\mathcal{L}^+$.

5.12. Definition. A pair ψ^+, ψ^- of operators on $\mathcal{L}^+$ is called *disjointness-preserving* if

$$x \wedge y = 0 \;\Rightarrow\; \psi^+(x) \wedge \psi^-(y) = 0,\; x, y \in \mathcal{L}^+.$$

The operator ψ^+ is called *disjointness-preserving* if the pair ψ^+, ψ^+ is disjointness-preserving.

It is easy to see that the pair ψ^+, ψ^- is disjointness-preserving if both operators are anti-extensive. The converse is not true, however.

Given two arbitrary operators ψ^+, ψ^- on $\mathcal{L}^+$, we define the operator ψ on $\mathcal{L}$ by

$$\psi(x) = \psi^+(x^+) - \psi^-(x^-),\; x \in \mathcal{L}.$$

5.13. Proposition. *Assume that ψ^+, ψ^- are disjointness-preserving and that ψ is of the form given above.*

- *(a) If ψ^+, ψ^- are increasing on $(\mathcal{L}^+, \leq)$ then ψ is increasing on $(\mathcal{L}, \preceq)$.*
- *(b) If $\psi^+ = \psi^-$ then ψ is self-dual, i.e., $\psi(-x) = -\psi(x)$ for $x \in \mathcal{L}$.*
- *(c) If ψ^+, ψ^- are anti-extensive on $(\mathcal{L}^+, \leq)$ then ψ is anti-extensive on $(\mathcal{L}, \preceq)$.*
- *(d) If ψ^+, ψ^- are idempotent then ψ is idempotent.*

Proof. First we show that

$$[\psi(x)]^+ = \psi^+(x^+) \quad \text{and} \quad [\psi(x)]^- = \psi^-(x^-) \tag{5.14}$$

for every $x \in \mathcal{L}$. Using the distributivity relation in (5.3) we get

$$\begin{aligned} [\psi(x)]^+ &= [\psi^+(x^+) - \psi^-(x^-)]^+ \\ &= (\psi^+(x^+) - \psi^-(x^-)) \vee 0 \\ &= (\psi^+(x^+) \vee \psi^-(x^-)) - \psi^-(x^-)\,. \end{aligned}$$

The fact that $\psi^+(x^+)$ and $\psi^-(x^-)$ are disjoint in combination with (5.9) yields that this latter expression reduces to

$$(\psi^+(x^+) + \psi^-(x^-)) - \psi^-(x^-) = \psi^+(x^+)\,.$$

This proves the first relation in (5.14). The second one follows by an analogous argument.

(a) Now assume that $x \preceq y$, then

$$\begin{aligned}[\psi(x)]^+ = \psi^+(x^+) &\leq \psi^+(y^+) = [\psi(y)]^+ \\ [\psi(x)]^- = \psi^-(x^-) &\leq \psi^-(y^-) = [\psi(y)]^-\end{aligned}$$

which yields that $\psi(x) \preceq \psi(y)$.

(b) Using that $(-x)^+ = x^-$ and $(-x)^- = x^+$ we get

$$\begin{aligned}\psi(-x) &= \psi^+((-x)^+) - \psi^+((-x)^-) \\ &= \psi^+(x^-) - \psi^+(x^+) \\ &= -\psi(x)\,.\end{aligned}$$

(c) For $x \in \mathcal{L}$ we have $(\psi(x))^+ = \psi^+(x^+) \leq x^+$ and $(\psi(x))^- = \psi^-(x^-) \leq x^-$ which yields that $\psi(x) \preceq x$.

(d) For $x \in \mathcal{L}$ we have

$$\begin{aligned}\psi^2(x) &= \psi(\psi(x)) = \psi^+((\psi(x))^+) - \psi^-((\psi(x))^-) \\ &= (\psi^+)^2(x^+) - (\psi^-)^2(x^-) \\ &= \psi^+(x^+) - \psi^-(x^-) = \psi(x)\,,\end{aligned}$$

where we have used the identities in (5.14). □

Combination of the results in $(a), (c), (d)$ yields the following interesting fact.

5.14. Corollary. *If ψ^+, ψ^- are openings on $(\mathcal{L}^+, \leq)$, then ψ is an opening on $(\mathcal{L}, \preceq)$. In particular, if ψ^+ is an opening on $(\mathcal{L}^+, \leq)$ then the operator ψ given by $\psi(x) = \psi^+(x^+) - \psi^+(x^-)$ is a self-dual opening on $(\mathcal{L}, \preceq)$.*

We now show how to construct adjunctions on $(\mathcal{L}, \preceq)$ given an adjunction on $\mathcal{L}^+$.

5.15. Proposition. *Let ε^+ be an erosion on $(\mathcal{L}^+, \leq)$ which is disjointness-preserving, and let ε be the extension to $\mathcal{L}$ given by*

$$\varepsilon(x) = \varepsilon^+(x^+) - \varepsilon^+(x^-)\,, \quad x \in \mathcal{L}\,.$$

Then ε defines an erosion on $(\mathcal{L}, \preceq)$. For every $y \in \mathcal{L}[\varepsilon]$ we have $y^+, y^- \in \mathcal{L}^+[\varepsilon^+]$, and the adjoint dilation $\delta : \mathcal{L}[\varepsilon] \to \mathcal{L}$ is given by

$$\delta(y) = \delta^+(y^+) - \delta^+(y^-)\,,$$

where $\delta^+ : \mathcal{L}^+[\varepsilon^+] \to \mathcal{L}^+$ is the adjoint dilation of ε^+.

Proof. First we show that $y \in \mathcal{L}[\varepsilon]$ implies that $y^+, y^- \in \mathcal{L}^+[\varepsilon^+]$. Assume that $y \preceq \varepsilon(x)$ for some $x \in \mathcal{L}$, that is, $y \preceq \varepsilon^+(x^+) - \varepsilon^+(x^-)$. This means in particular that

$$y \vee 0 \leq (\varepsilon^+(x^+) - \varepsilon^+(x^-)) \vee 0 = (\varepsilon^+(x^+) \vee \varepsilon^+(x^-)) - \varepsilon^+(x^-)\,.$$

Since $\varepsilon^+(x^+)$ and $\varepsilon^+(x^-)$ are disjoint, we can replace the supremum at the right hand-side by a summation, and we find that $y^+ = y \vee 0 \leq \varepsilon^+(x^+)$, that is $y^+ \in \mathcal{L}^+[\varepsilon^+]$. Similarly, we find that $y^- \in \mathcal{L}^+[\varepsilon^+]$.

It remains to be shown that $\delta(y) \preceq x \iff y \preceq \varepsilon(x)$ for $x \in \mathcal{L}$ and $y \in \mathcal{L}[\varepsilon]$. Assume first that $y \preceq \varepsilon(x)$. Then $y^+ \leq (\varepsilon(x))^+ = \varepsilon^+(x^+)$ and $y^- \leq (\varepsilon(x))^- = \varepsilon^+(x^-)$. This yields that $y^+, y^- \in \mathcal{L}^+[\varepsilon^+]$, and since $(\varepsilon^+, \delta^+)$ is an adjunction, we get that $\delta^+(y^+) \leq x^+$ and $\delta^+(y^-) \leq x^-$. But this implies that $\delta(y) \preceq x$.

Now assume that $\delta(y) \preceq x$, i.e., $(\delta(y))^+ \leq x^+$ and $(\delta(y))^- \leq x^-$. Since $y \in \mathcal{L}[\varepsilon]$, we have $y \preceq \varepsilon(x_0)$ for some $x_0 \in \mathcal{L}$, that is

$$\begin{aligned} y^+ &\leq (\varepsilon(x_0))^+ = (\varepsilon^+(x_0^+) - \varepsilon^+(x_0^-))^+ = \varepsilon^+(x_0^+) \\ y^- &\leq (\varepsilon(x_0))^- = (\varepsilon^+(x_0^+) - \varepsilon^+(x_0^-))^- = \varepsilon^+(x_0^-) \end{aligned}$$

This yields that $\delta^+(y^+) \leq x_0^+$ and $\delta^+(y^-) \leq x_0^-$, which means in particular that $\delta^+(y^+)$ and $\delta^+(y^-)$ are disjoint. Therefore $(\delta(y))^+ = \delta^+(y^+) \leq x^+$, which yields that $y^+ \leq \varepsilon^+(x^+)$. Similarly we get that $y^- \leq \varepsilon^+(x^-)$, and we find that $y \preceq \varepsilon^+(x^+) - \varepsilon^+(x^-) = \varepsilon(x)$. □

Note from Proposition 5.13(b) that the erosion ε given by Proposition 5.15 is self-dual in the sense that $\varepsilon(-x) = -\varepsilon(x)$ for every $x \in \mathcal{L}$. Furthermore, $\mathcal{L}[\varepsilon]$ is invariant under the $\circ$-negation $y \mapsto -y$, i.e., $y \in \mathcal{L}[\varepsilon]$ iff $-y \in \mathcal{L}[\varepsilon]$ and $\delta(-y) = -\delta(y)$ for such y.

Note also that the previous result can easily be extended to the case where we start with two different erosions $\varepsilon^+, \varepsilon^-$ on $\mathcal{L}^+$. In that case we define an erosion on $(\mathcal{L}, \preceq)$ by $\varepsilon(x) = \varepsilon^+(x^+) - \varepsilon^-(x^-)$ for $x \in \mathcal{L}$.

5.16. Remark. We briefly discuss an alternative approach for the construction of a cisl which possesses a $\circ$-negation. The starting point for this construction is a given inf-semilattice $(\mathcal{L}, \leq)$. For example, if we are interested in the functions $\mathbb{R}^E$, then $\mathcal{L}$ represents the positive part, i.e. $\mathcal{L} = \mathbb{R}_+^E$. Denote the least element of $\mathcal{L}$ by 0. Define

$$\mathcal{M} = \{x = (x^+, x^-) \in \mathcal{L} \times \mathcal{L} \mid x^+ \wedge x^- = 0\}\,,$$

and define a partial ordering $\preceq$ on $\mathcal{M}$ by

$$(x^+, x^-) \preceq (y^+, y^-) \text{ if } x^+ \leq y^+ \text{ and } x^- \leq y^-.$$

Now $(\mathcal{M}, \preceq)$ is an inf-semilattice with least element $(0, 0)$.

The mapping γ given by

$$\gamma(x^+, x^-) = (x^-, x^+)$$

defines a $\circ$-negation on $\mathcal{M}$. Given an operator ψ^+ on $\mathcal{L}$ that has the property

$$x \wedge y = 0 \;\Rightarrow\; \psi^+(x) \wedge \psi^+(y) = 0,$$

then the extension ψ to $\mathcal{M}$ given by

$$\psi(x^+, x^-) = (\psi^+(x^+), \psi^+(x^-))$$

has the property

$$\gamma\psi\gamma = \psi,$$

expressing the self-duality with respect to γ.

6 Conclusions

A theory for self-dual morphology was developed, in which the concept of self-duality is defined in posets, by means of $\circ$-negations.

Two different approaches for operator derivation were proposed on inf-semilattices. The reference semilattice approach uses a given fixed function as the least element of the inf-semilattice, and as a reference for self-duality. One practical way of using reference semilattices is having the input signal be the reference element itself, and use a transformation of this signal as starting point for self-dual processing. If this processing is an opening, then the result is a filtered version of the original signal. Special attention is given to translation invariance, and two cases are considered: The traditional translation invariance, and invariance to a new type of translation, called signed translation.

The second approach is through lattice ordered groups. Through this approach, it becomes simple to modify traditional lattice operators, so they become self-dual within a lattice ordered group.

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