

## Existence of Eigenvectors for Monotone Homogeneous Functions

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Collatz-Wielandt  
property,  
Hilbert  
projective  
metric,  
nonexpansive  
function,  
nonlinear  
eigenvalue,  
Perron-  
Frobenius  
theorem,  
strongly  
connected graph,  
sub-eigenspace

We consider function  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  which are additively homogeneous and monotone in the product ordering on  $\mathbb{R}^n$  (topical functions). We show that if some non-empty sub-eigenspace of  $\mathbf{f}$  is bounded in the Hilbert semi-norm then  $\mathbf{f}$  has an additive eigenvector and we give a Collatz-Wielandt characterisation of the corresponding eigenvalue. The boundedness condition is satisfied if a certain directed graph associated to  $\mathbf{f}$  is strongly connected. The Perron-Frobenius theorem for non-negative matrices, its analogue for the max-plus semiring, a version of the mean ergodic theorem for Markov chains and theorems of Birkhoff and Zijm all follow as immediate corollaries.

# EXISTENCE OF EIGENVECTORS FOR MONOTONE HOMOGENEOUS FUNCTIONS

STÉPHANE GAUBERT AND JEREMY GUNAWARDENA

ABSTRACT. We consider functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  which are additively homogeneous and monotone in the product ordering on  $\mathbb{R}^n$  (topical functions). We show that if some non-empty sub-eigenspace of  $f$  is bounded in the Hilbert semi-norm then  $f$  has an additive eigenvector and we give a Collatz-Wielandt characterisation of the corresponding eigenvalue. The boundedness condition is satisfied if a certain directed graph associated to  $f$  is strongly connected. The Perron-Frobenius theorem for non-negative matrices, its analogue for the max-plus semiring, a version of the mean ergodic theorem for Markov chains and theorems of Bather and Zijm all follow as immediate corollaries.

## 1. INTRODUCTION

1.1. **Notation.** The partial order on  $\mathbb{R}$  will be extended pointwise to functions  $f, g : X \rightarrow \mathbb{R}$  so that  $f \leq g$  if, and only if,  $f(x) \leq g(x)$  for all  $x \in X$ . The least upper bound and greatest lower bound with respect to this ordering, will be denoted in infix form by  $\vee$  and  $\wedge$ , respectively:  $(f \vee g)(x) = \max(f(x), g(x))$  and  $(f \wedge g)(x) = \min(f(x), g(x))$ . In particular, taking  $X = \{1, \dots, n\}$ , this gives the product ordering on  $\mathbb{R}^n$  with its usual structure as a distributive lattice.

It will also be convenient to use the following *vector-scalar* convention: if, in an operation or a relation, a vector and a scalar appear together, then the operation is applied to, or the relation is taken to hold for, each component of the vector. For example, if  $\lambda \in \mathbb{R}$  and  $x \in \mathbb{R}^n$ , then  $\lambda + x = (\lambda + x_1, \dots, \lambda + x_n)$  and  $x \leq \lambda$  if, and only if,  $x_i \leq \lambda$  for  $1 \leq i \leq n$ .

1.2. **Topical functions.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is (additively) *homogeneous* if

$$(1) \quad \forall \lambda \in \mathbb{R} \text{ and } \forall x \in \mathbb{R}^n, \quad f(\lambda + x) = \lambda + f(x) ,$$

and *monotone* if

$$(2) \quad \forall x, y \in \mathbb{R}^n, \quad x \leq y \implies f(x) \leq f(y) .$$

Functions which are monotone and homogeneous have been called *topical functions* in [12] and we adopt this terminology here. If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a topical function, we say that  $x \in \mathbb{R}^n$  is an (additive) *eigenvector* for the *eigenvalue*  $\lambda \in \mathbb{R}$  if  $f(x) = \lambda + x$ . The main results of this paper are existence theorems for eigenvectors, Theorems 1 and 2, and a Collatz-Wielandt characterisation of the eigenvalue, Proposition 1. Our methods are elementary.

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Topical functions include many examples that have been extensively studied: non-negative matrices (see below); max-plus matrices, (see §4 and [2]), and other models of discrete event systems, [3, 9, 10, 22]; operators arising in Markov decision theory and the theory of stochastic games, [6, 13]; problems in fixed point theory, [17, 19]; matrix scaling problems and related problems of entropy minimisation, [15, 20]. This paper shows the emergence of elementary general results of wide applicability: we recover some well-known theorems as immediate corollaries of our main results.

**1.3. The multiplicative context.** Non-negative matrices are familiar in a multiplicative form so it will be helpful to note first that the additive and multiplicative contexts are interchangeable.

Let  $\mathbb{R}^+$  denote the positive reals:  $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ . The whole space,  $\mathbb{R}^n$ , can be placed in bijective correspondence with the positive cone,  $(\mathbb{R}^+)^n$ , via the mutually inverse functions  $\exp : \mathbb{R}^n \rightarrow (\mathbb{R}^+)^n$  and  $\log : (\mathbb{R}^+)^n \rightarrow \mathbb{R}^n$ , where  $\exp(x)_i = \exp(x_i)$ , for  $x \in \mathbb{R}^n$ , and  $\log(x)_i = \log(x_i)$ , for  $x \in (\mathbb{R}^+)^n$ . If  $A : (\mathbb{R}^+)^n \rightarrow (\mathbb{R}^+)^n$  is any self-map of the positive cone then  $\mathcal{E}(A) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  will denote the function defined by  $\mathcal{E}(A)(x) = \log(A(\exp(x)))$ . This induces a bijective functional between self-maps of  $(\mathbb{R}^+)^n$  and self-maps of  $\mathbb{R}^n$ . Clearly,  $\mathcal{E}(AB) = \mathcal{E}(A)\mathcal{E}(B)$ , so that the dynamics of  $A$  on  $(\mathbb{R}^+)^n$  and  $\mathcal{E}(A)$  on  $\mathbb{R}^n$  are equivalent.

If  $A : (\mathbb{R}^+)^n \rightarrow (\mathbb{R}^+)^n$  is represented by a non-negative matrix in the standard basis (for which the same notation,  $A$ , will be used) then it is easy to see that  $\mathcal{E}(A)$  is a topical function. Furthermore,  $x \in \mathbb{R}^n$  is an (additive) eigenvector of  $\mathcal{E}(A)$ , with eigenvalue  $\lambda \in \mathbb{R}$ , if, and only if,  $\exp(x) \in (\mathbb{R}^+)^n$  is an eigenvector of  $A$  in the usual sense, with eigenvalue  $\exp(\lambda)$ :  $A \exp(x) = \exp(\lambda) \exp(x)$ .

Note that (additive) eigenvectors of  $\mathcal{E}(A)$  correspond bijectively to the (multiplicative) eigenvectors of  $A$  *all of whose components are positive*. The word eigenvector will be used in both contexts; the reader should have no difficulty inferring the right meaning. Note further that a non-negative matrix  $A$  corresponds to a topical function under  $\mathcal{E}$  if, and only if, no row of  $A$  is the zero vector:

$$(3) \quad \forall i, \exists j \text{ such that } A_{ij} \neq 0 .$$

**1.4. Nonexpansiveness.** A key property of topical functions is their nonexpansiveness with respect to certain norms. Let  $\mathbf{t}, \mathbf{b} : \mathbb{R}^n \rightarrow \mathbb{R}$  be defined as (“top”)  $\mathbf{t}(x) = x_1 \vee \cdots \vee x_n$ , and (“bottom”)  $\mathbf{b}(x) = -\mathbf{t}(-x) = x_1 \wedge \cdots \wedge x_n$ , both of which are topical functions. The supremum, or  $\ell^\infty$ , norm on  $\mathbb{R}^n$  can then be defined as  $\|x\|_\infty = \mathbf{t}(x) \vee -\mathbf{b}(x)$ . We shall also need the Hilbert semi-norm,  $\|x\|_H = \mathbf{t}(x) - \mathbf{b}(x)$ , which defines a metric on the space of lines parallel to the main diagonal in  $\mathbb{R}^n$ . This metric is the additive version of the Hilbert projective metric while  $\|x\|_\infty$  gives rise to the additive version of Thompson’s “part” metric on  $(\mathbb{R}^+)^n$ , [17].

An elementary application of (1) and (2), [12, Proposition 1.1], shows that a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is topical if, and only if,

$$\forall x, y \in \mathbb{R}^n, \mathbf{t}(f(x) - f(y)) \leq \mathbf{t}(x - y) .$$

(This provides some justification for the term *topical*.) We see immediately that a topical function is nonexpansive with respect to both the supremum norm and the Hilbert semi-norm:  $\forall x, y \in \mathbb{R}^n$ ,

$$(4) \quad \|f(x) - f(y)\|_\infty \leq \|x - y\|_\infty$$

$$(5) \quad \|f(x) - f(y)\|_H \leq \|x - y\|_H .$$

In fact, as first observed by Crandall and Tartar [7], if  $f$  is homogeneous, then it is monotone if, and only if, it is nonexpansive in the supremum norm, [12, Proposition 1.1].

## 2. SUB-EIGENSPACES AND THE COLLATZ-WIELANDT PROPERTY

The results of the present paper originate in the study of sub-eigenspaces. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . For any  $\lambda \in \mathbb{R}$ , define the sub-eigenspace of  $f$  associated to  $\lambda$ ,  $S_\lambda(f) \subseteq \mathbb{R}^n$ , by

$$S_\lambda(f) = \{x \in \mathbb{R}^n \mid f(x) \leq \lambda + x\} .$$

If  $S_\lambda(f) \neq \emptyset$  then  $\lambda$  is said to be a *sub-eigenvalue*, and any  $x \in S_\lambda(f)$  is a *sub-eigenvector*. Let  $\Lambda(f) \subseteq \mathbb{R}$  denote the set of sub-eigenvalues:  $\Lambda(f) = \{\lambda \in \mathbb{R} \mid S_\lambda(f) \neq \emptyset\}$ . For any functions  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and any  $\lambda, \mu \in \mathbb{R}$ , the following are easily seen to hold.

$$(6a) \quad f \leq g \quad \Rightarrow \quad S_\lambda(f) \supset S_\lambda(g) ,$$

$$(6b) \quad \lambda \leq \mu \quad \Rightarrow \quad S_\lambda(f) \subset S_\mu(f) ,$$

$$(6c) \quad S_{(\lambda+\mu)}(f) = S_\lambda(f - \mu) .$$

It follows immediately from (6b) that for any function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\Lambda(f)$  must be an interval of the form  $(-\infty, \infty)$ ,  $(a, \infty)$  or  $[a, \infty)$  and it is easy to see that all three forms can appear. For a topical function the first form can be ruled out. To see this, it is helpful to recall first some well-understood facts about the asymptotic dynamics of a topical function,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

First, (4) implies that all trajectories of  $f$  are asymptotically the same:

$$(7) \quad f^k(x) = f^k(y) + O(1) \text{ as } k \rightarrow \infty .$$

Second, an elementary argument using (1) and (2) shows that the sequence  $\mathbf{t}(f^k(0))$  is sub-additive,

$$\mathbf{t}(f^{k+l}(0)) \leq \mathbf{t}(f^k(0)) + \mathbf{t}(f^l(0)) .$$

It follows from (7) that the sequence  $\mathbf{t}(f^k(x)/k)$  converges as  $k \rightarrow \infty$  and that the limit is independent of  $x$ , [12, 22]. The *upper cycle-time* of  $f$ ,  $\overline{\chi}(f) \in \mathbb{R}$ , is defined as

$$\overline{\chi}(f) = \lim_{k \rightarrow \infty} \mathbf{t}(f^k(x)/k) .$$

Dually, the *lower cycle-time* is  $\underline{\chi}(f) = \lim_{k \rightarrow \infty} \mathbf{b}(f^k(x)/k)$ . The existence of the *cycle-time vector* of  $f$ ,  $\chi(f) = \lim_{k \rightarrow \infty} f^k(x)/k$ , is another matter altogether. It does not always exist, [12, Theorem 3.1], and one of the central problems in the field is to characterise those topical functions for which it does.

Now suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a topical function and that  $f(x) \leq \lambda + x$  for some  $x \in \mathbb{R}^n$  and some  $\lambda \in \mathbb{R}$ . Using (1) and (2),  $f^k(x) \leq k\lambda + x$ . Hence,

$$\mathbf{t}(f^k(x)/k) \leq \lambda + \mathbf{t}(x/k) ,$$

from which the following lemma immediately follows.

**Lemma 1.** *If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a topical function then either  $\Lambda(f) = (a, \infty)$  or  $\Lambda(f) = [a, \infty)$ , where  $\overline{\chi}(f) \leq a$ .*

Both possibilities can occur. It follows from Proposition 1 and (10a) below that if  $f$  has an eigenvector,  $f(x) = \lambda + x$ , then  $\Lambda(f) = [\lambda, \infty]$ . If  $f = \mathcal{E}(A)$  where  $A$  is the non-negative matrix below

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then it is easy to see that  $\Lambda(f) = (0, \infty)$ .

We shall now show that  $\overline{\chi}(f) = a$ . This requires the following simple but crucial observation.

**Lemma 2.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a topical function and let  $k$  be any positive integer. If  $S_\lambda(f^k) \neq \emptyset$ , then  $S_{\lambda/k}(f) \neq \emptyset$ .*

*Proof.* If  $S_\lambda(f^k) \neq \emptyset$ , then  $f^k(x) \leq \lambda + x$  for some  $x \in \mathbb{R}^n$ . Let

$$y = x \wedge (f(x) - \lambda/k) \wedge \cdots \wedge (f^{k-1}(x) - (k-1)\lambda/k) .$$

Using (1) and (2) we see that

$$\begin{aligned} f(y) &\leq f(x) \wedge (f^2(x) - \lambda/k) \wedge \cdots \wedge (f^k(x) - (k-1)\lambda/k) \\ &\leq f(x) \wedge (f^2(x) - \lambda/k) \wedge \cdots \wedge (x + \lambda/k) \\ &= y + \lambda/k . \end{aligned}$$

Thus,  $y \in S_{\lambda/k}(f) \neq \emptyset$ . □

Lemma 2 allows us to give the following characterisation of  $\overline{\chi}(f)$  which may be thought of as a generalised Collatz-Wielandt formula, [16, §1.3].

**Proposition 1.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a topical function. Then,*

$$(8) \quad \inf \Lambda(f) = \inf_{x \in \mathbb{R}^n} \mathbf{t}(f(x) - x) = \overline{\chi}(f) .$$

*Proof.* Let  $a = \inf \Lambda(f)$ . Since  $f(x) \leq x + \lambda$  if, and only if,  $\mathbf{t}(f(x) - x) \leq \lambda$  the first equality in (8) follows easily. Lemma 1 has already shown that  $\overline{\chi}(f) \leq a$ . Now choose  $\epsilon > 0$ . For sufficiently large  $k$ ,  $f^k(0) \leq (\overline{\chi}(f) + \epsilon)k$ . Hence,  $S_{(\overline{\chi}(f) + \epsilon)k}(f^k) \neq \emptyset$ . By Lemma 2,  $S_{\overline{\chi}(f) + \epsilon}(f) \neq \emptyset$ . Hence,  $a \leq \overline{\chi}(f) + \epsilon$ . Since  $\epsilon$  was chosen arbitrarily,  $a \leq \overline{\chi}(f)$  and so  $a = \overline{\chi}(f)$ . □

A result on topical functions can be dualised by applying it to the topical function  $-f(-x)$ . Using this method on the Collatz-Wielandt formula, we deduce that

$$(9) \quad \underline{\chi}(f) = \sup_{x \in \mathbb{R}^n} \mathbf{b}(f(x) - x) .$$

If  $f$  has an eigenvector, so that  $f(x) = \lambda + x$  then it follows from (1) that

$$(10a) \quad \underline{\chi}(f) = \lambda = \overline{\chi}(f) \text{ and}$$

$$(10b) \quad \chi(f) = (\lambda, \dots, \lambda) .$$

### 3. EXISTENCE OF EIGENVECTORS

**3.1. The main result.** It is convenient for the proofs that follow to make use of the normalised sub-eigenspace,  $S'_\lambda(f) \subseteq \mathbb{R}^n$ , defined by

$$S'_\lambda(f) = \{x \in \mathbb{R}^n \mid f(x) \leq \lambda + x \text{ and } \mathbf{b}(x) = 0\} .$$

If  $b(x) = 0$  then  $\|x\|_{\mathbb{H}} = \|x\|_{\infty}$ . It follows that if  $f$  is homogeneous then  $S_{\lambda}(f)$  is non-empty and bounded in the Hilbert semi-norm if, and only if,  $S'_{\lambda}(f)$  is non-empty and bounded in the supremum norm.

**Theorem 1.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a topical function for which some sub-eigenspace is non-empty and bounded in the Hilbert semi-norm. Then  $f$  has an eigenvector.*

*Proof.* Assume that  $S_{\mu}(f)$  is non-empty and bounded in the Hilbert semi-norm. Let  $a = \inf \Lambda(f)$ . Evidently  $a \leq \mu$ . We may assume, without loss of generality, that  $a = \mu$ . To see why, suppose that  $a < \mu$ . Since  $S_{\mu}(f)$  is bounded in the Hilbert semi-norm, the sets  $S'_b(f)$ , for  $a < b \leq \mu$  are compact in the supremum norm. It follows easily from (6b) that  $S'_a(f) = \bigcap_{a < b \leq \mu} S'_b(f)$ . The right hand side is a decreasing intersection of non-empty compact sets and so  $S'_a(f)$  is also non-empty and compact. Hence  $S_a(f)$  is non-empty and bounded in the Hilbert semi-norm, as claimed.

Let  $g = -a + f$ . By (6c),  $\Lambda(g) = [0, \infty)$ , so that we can find  $x \in \mathbb{R}^n$  such that  $g(x) \leq x$ . Hence  $g^{k+1}(x) \leq g^k(x)$  and  $g^k(x) \in S_0(g) = S_a(f)$  for all  $k \in \mathbb{N}$ . If  $\lim_{k \rightarrow \infty} t(g^k(x)) = -\infty$ , then  $g^k(x) \leq -1 + x$ , for some sufficiently large  $k$  and Lemma 2 shows that  $S_{-1/k}(g) \neq \emptyset$ , contradicting  $\Lambda(g) = [0, \infty)$ . Hence  $t(g^k(x))$  is bounded from below as  $k \rightarrow \infty$ . By hypothesis,  $\|g^k(x)\|_{\mathbb{H}}$  remains bounded and this can only happen if  $g^k(x)$  itself remains bounded. Let  $y = \lim_{k \rightarrow \infty} g^k(x)$ . Then by continuity of  $g$ ,  $g(y) = y$ , so that  $f(y) = a + y$ .  $\square$

The following examples are instructive in the light of this result. Consider the topical functions  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$\begin{aligned} f_1(x) &= ((x_1 - 1) \vee x_2) \wedge (x_1 + 1) & \text{and} & & g_1(x) &= x_1 \wedge x_2 \\ f_2(x) &= x_1 \vee x_2 & & & g_2(x) &= x_1 \vee x_2 . \end{aligned}$$

We leave it to the reader to show that  $\Lambda(f) = [0, \infty)$  and

$$S_{\lambda}(f) = \begin{cases} \{x \in \mathbb{R}^2 \mid -\lambda + x_1 \leq x_2 \leq \lambda + x_1\} & \text{for } 0 \leq \lambda < 1 \\ \{x \in \mathbb{R}^2 \mid -\lambda + x_1 \leq x_2\} & \text{for } \lambda \geq 1 . \end{cases}$$

It follows that  $S_{\lambda}(f)$  is bounded for  $0 \leq \lambda < 1$  and unbounded for  $1 \leq \lambda$ . As for  $g$ , it has the eigenvector  $(0, 0)$  and  $\Lambda(g) = [0, \infty)$  but  $S_{\lambda}(g) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq \lambda + x_2\}$  is unbounded for all  $\lambda \geq 0$ . (The dual super-eigenspaces are also unbounded.)

**3.2. Graphs associated to topical functions.** If  $A$  is a  $n \times n$  non-negative matrix, its associated graph,  $G(A)$ , is the directed graph with vertices  $\{1, \dots, n\}$  and an edge from  $i$  to  $j$  if, and only if,  $A_{ij} \neq 0$ , [5, Chapter 2]. The matrix  $A$  is irreducible if, and only if,  $G(A)$  is *strongly connected*: if there is a directed path between any two vertices. The Perron-Frobenius theorem (see Corollary 1 below) asserts that an irreducible non-negative matrix has an eigenvector all of whose components are positive. We now generalise this to topical functions.

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Define the associated graph of  $f$ ,  $G(f)$ , to be the directed graph with vertices  $\{1, \dots, n\}$  and an edge from  $i$  to  $j$  if, and only if,  $\lim_{\nu \rightarrow \infty} f_i(\nu e_j) = \infty$ , where  $e_j$  is the  $j$ -th vector of the canonical basis of  $\mathbb{R}^n$ .

**Theorem 2.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a topical function whose associated graph is strongly connected. Then all non-empty sub-eigenspaces of  $f$  are bounded. In particular,  $f$  has an eigenvector.*

*Proof.* For each edge from  $i$  to  $j$  of  $G(f)$  define  $h_{ji} : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$  by

$$h_{ji}(x) = \sup\{\nu \in \mathbb{R} \mid f_i(\nu e_j) \leq x\} ,$$

where we use the convention that  $\sup \emptyset = -\infty$ . For any  $\lambda \in \mathbb{R}$ , let  $h_{ji}^\lambda(x) = h_{ji}(\lambda + x)$ . Let  $S_\lambda(f)$  be any non-empty sub-eigenspace of  $f$  and choose  $x \in S_\lambda(f)$ , which we may assume to satisfy  $\mathbf{b}(x) = 0$ . Let  $i \in \{1, \dots, n\}$  be the component for which  $x_i = 0$ . Choose any other component  $j \in \{1, \dots, n\}$ . By hypothesis there exists a directed path from  $i$  to  $j$  in  $G(f)$ . Suppose that the nodes on this are  $i = i_1, \dots, i_k = j$ , where there is an edge from  $i_{p-1}$  to  $i_p$  for  $1 < p \leq k$ . Since  $\mathbf{b}(x) = 0$ , we must have  $x_{i_p} e_{i_p} \leq x$ . Hence

$$f_{i_{p-1}}(x_{i_p} e_{i_p}) \leq f_{i_{p-1}}(x) \leq \lambda + x_{i_{p-1}}$$

and so  $x_{i_p} \leq h_{i_p i_{p-1}}^\lambda(x_{i_{p-1}})$ . Putting these together we find that

$$x_j \leq h_{i_k i_{k-1}}^\lambda \circ \dots \circ h_{i_2 i_1}^\lambda(0) .$$

It follows that  $S_\lambda(f)$  is bounded in the Hilbert semi-norm. By Theorem 1,  $f$  has an eigenvector.  $\square$

Amghibech and Dellacherie state a similar but weaker result in [1]. They use a different graph which is, in general, not strongly connected for the examples studied in the next section, with the exception of that in Corollary 3. However, the proof technique of [1], based on an approximation procedure, could be used to obtain an independent proof of Theorem 2.

Consider the topical function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by

$$\begin{aligned} f_1(x) &= x_1 \vee (x_2 \wedge x_3) \\ f_2(x) &= x_1 \vee x_2 \vee x_3 \\ f_3(x) &= x_1 \vee x_2 \vee x_3 . \end{aligned}$$

$G(f)$  is not strongly connected since there are no edges from 1 to 2 and from 1 to 3. Nevertheless it is easy to check that  $f$  has bounded sub-eigenspaces. Is there a combinatorial object associated to a topical function which determines when the function has bounded sub-eigenspaces? This is an interesting problem which we hope to address elsewhere.

For convex topical functions, Theorem 2 has a converse. Recall that a function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex if, for all  $x, y \in \mathbb{R}^n$ ,

$$(11) \quad h(\lambda x + \mu y) \leq \lambda h(x) + \mu h(y) ,$$

where  $0 \leq \lambda, \mu \leq 1$  and  $\lambda + \mu = 1$ . A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is convex if each component function  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex. A simple deduction from (11), which is left to the reader, captures the intuition that the derivative of  $h$  is increasing. With the same notation as above, let  $x' = \lambda x + \mu y = x + \mu(y - x) = y - \lambda(y - x)$ . Then,

$$(12) \quad \frac{h(x') - h(x)}{\mu} \leq \frac{h(y) - h(x')}{\lambda} .$$

For any function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  define its *syntactic* graph,  $G^s(f)$ , to be the directed graph with vertices  $1, \dots, n$  and an edge from  $i$  to  $j$  if, and only if,  $f_i$  depends on  $x_j$  in the following sense: there is no map  $h : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  such that  $f_i(x) = h(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ .

**Proposition 2.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a convex topical function. Then  $G(f) = G^s(f)$ . Moreover,  $G^s(f)$  is strongly connected if, and only if, all sub-eigenspaces of  $f$  are bounded in the Hilbert semi-norm.*

*Proof.* Clearly, an edge of  $G(f)$  is an edge of  $G^s(f)$ . Conversely, if there is an edge from  $i$  to  $j$  in  $G^s(f)$ , then we can find  $x, x' \in \mathbb{R}^n$  such that  $x_k = x'_k$  for all  $k \neq j$ ,  $x_j \neq x'_j$ , and  $f_i(x) \neq f_i(x')$ . Without loss of generality, assume that  $x'_j > x_j$ . Choose  $\nu > 0$  and let  $y = x' + \nu e_j$ . Let  $\alpha = x'_j - x_j + \nu$ ,  $\lambda = \nu/\alpha$  and  $\mu = (x'_j - x_j)/\alpha$ . Evidently,  $0 \leq \lambda, \mu \leq 1$  and  $\lambda + \mu = 1$  and it is easy to check that  $x' = \lambda x + \mu y$ , in accordance with the notation used in (12). Using this inequality we see that

$$\frac{f_i(y) - f_i(x')}{\lambda} \geq \frac{f_i(x') - f_i(x)}{\mu},$$

which can be rewritten as

$$f_i(x' + \nu e_j) \geq \frac{\nu}{x'_j - x_j} (f_i(x') - f_i(x)) + f_i(x').$$

Since this holds for any  $\nu > 0$ , it follows that  $\lim_{\nu \rightarrow \infty} f_i(x' + \nu e_j) = \infty$ . But,  $x' + \nu e_j \leq \mathbf{t}(x') + \nu e_j$ . Using (2), we see that  $f_i(\nu e_j) \geq f_i(x' + \nu e_j) - \mathbf{t}(x')$  and so  $\lim_{\nu \rightarrow \infty} f_i(\nu e_j) = \infty$ . It follows that there is an edge from  $i$  to  $j$  in  $G(f)$  and so  $G^s(f)$  is identical to  $G(f)$ .

If  $G^s(f) = G(f)$  is strongly connected then Theorem 2 shows that all the sub-eigenspaces of  $f$  are bounded. Conversely, suppose that  $G^s(f)$  is not strongly connected. Then, by standard arguments, [5, Chapter 2], we can, after possibly reordering, partition the variables so that  $x = (y, z)$ , where  $y \in \mathbb{R}^p$ ,  $z \in \mathbb{R}^q$ ,  $n = p + q$  and  $f(x) = (g(y, z), h(z))$ , for some topical functions  $g : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^p$ , and  $h : \mathbb{R}^q \rightarrow \mathbb{R}^q$ . Suppose that  $S_\lambda(f)$  is non-empty and choose  $x \in S_\lambda(f)$ . Since  $f(x) \leq \lambda + x$ , we must have  $g(y, z) \leq \lambda + y$  and  $h(z) \leq \lambda + z$ . Now choose  $\mu \geq 0$ . Using (1) and (2) it follows that  $g(\mu + y, z) \leq g(\mu + y, \mu + z) \leq \lambda + \mu + y$ . Hence  $(\mu + y, z) \in S_\lambda(f)$  for all  $\mu > 0$  and so all non-empty sub-eigenspaces of  $f$  are unbounded in the Hilbert semi-norm.  $\square$

#### 4. APPLICATIONS

We now show that several well-known theorems are immediate corollaries of the elementary results above.

**Corollary 1.** (Perron-Frobenius theorem, [5]) *Let  $A$  be a  $n \times n$  non-negative matrix. If  $A$  is irreducible then its spectral radius is an eigenvalue, for which  $A$  has an eigenvector all of whose components are positive.*

*Proof.* Since  $A$  is irreducible the nondegeneracy condition (3) holds. Let  $f = \mathcal{E}(A)$ . It is easy to see that  $G(f)$ , the graph associated to  $f$ , is identical to  $G(A)$ , the graph associated to  $A$ . Since  $A$  is irreducible,  $G(A)$  is strongly connected and so, by Theorem 2,  $f$  has an eigenvector:  $f(x) = r + x$ . Evidently,  $A \exp(x) = \exp(r) \exp(x)$ , where  $\exp(x)$  has all its components positive. It remains to show that  $\exp(r)$  is the spectral radius of  $A$ . For completeness, we reproduce the standard argument using the Collatz-Wielandt property. Suppose that  $z \in \mathbb{C}^n$  is a (multiplicative) eigenvector of  $A$  with eigenvalue  $\lambda \in \mathbb{C}$ :  $Az = \lambda z$ . Let  $|z| \in \mathbb{R}^n$  be the vector of absolute values:  $|z| = (|z_1|, \dots, |z_n|)$  and let  $x = \log(|z|)$ . A simple application



of the triangle inequality shows that  $A|z| \geq |\lambda||z|$ . Hence  $f(x) \geq \log(|\lambda|) + x$ . It follows, using (9) and (10a), that

$$\log(|\lambda|) \leq \mathbf{b}(f(x) - x) \leq \underline{\chi}(f) = r$$

and so  $\exp(r)$  is the spectral radius of  $A$ .  $\square$

The function  $\mathbb{R}^2 \rightarrow \mathbb{R}$  which takes  $x \mapsto \log(\exp(x_1) + \exp(x_2))$  is convex, from which it follows that  $\mathcal{E}(A)$  is a convex topical function. Hence we could have used Proposition 2, together with Theorem 1 in the proof of Corollary 1. This applies also to the topical functions in the corollaries below, which are all convex.

Property (1) applied to the function  $\mathbf{t} : \mathbb{R}^2 \rightarrow \mathbb{R}$  illustrates that addition distributes over maximum. It follows that the set  $\mathbb{R} \cup \{-\infty\}$  equipped with the operations of maximum (as addition) and addition (as multiplication) forms an idempotent semiring (a semiring whose addition satisfies  $a + a = a$ ), called the *max-plus semiring* and denoted  $\mathbb{R}_{\max}$ , [11]. Suppose that  $A$  is a  $n \times n$  matrix over  $\mathbb{R}_{\max}$  which satisfies a similar nondegeneracy condition to (3):

$$\forall i, \exists j \text{ such that } A_{ij} \neq -\infty .$$

If  $x \in \mathbb{R}^n$  then it is easy to see that  $x \mapsto Ax$  defines a topical function. For instance, the matrix on the left below gives rise to the function on the right.

$$\begin{pmatrix} 2 & -1 \\ -\infty & 4 \end{pmatrix} \quad \begin{array}{l} f_1(x) = (x_1 + 2) \vee (x_2 - 1) \\ f_2(x) = x_2 + 4 . \end{array}$$

If  $A$  is a  $n \times n$  matrix over  $\mathbb{R}_{\max}$ , its associated graph,  $G(A)$ , is the directed graph with vertices  $\{1, \dots, n\}$  and an edge from  $i$  to  $j$  if, and only if,  $A_{ij} \neq -\infty$ . It is customary, in max-plus theory, to adjoin labels (“weights”) to the edges in  $G(A)$ , [2]. This unlabelled version will be sufficient for our purposes.  $A$  is said to be irreducible if  $G(A)$  is strongly connected. It is easy to see that if  $f$  is the topical function corresponding to  $A$  then the graphs  $G(A)$  and  $G(f)$  coincide. The following result follows immediately. The cited reference is to a standard source but the result has been proved independently many times.

**Corollary 2.** (Perron-Frobenius for max-plus, [2, Theorem 3.28]) *An irreducible max-plus matrix has an eigenvector.*

In max-plus theory, the eigenvectors of a matrix lie in  $(\mathbb{R} \cup \{-\infty\})^n$ . The point of Corollary 2 is that such an eigenvector can be found in  $\mathbb{R}^n$ . The formula for the eigenvalue, based on the structure of the circuits of  $G(A)$ , lies outside the scope of the present paper, [2].

For the next result, assume that  $P$  is the transition matrix of a Markov chain (so that  $P$  is row-stochastic) and let  $f(x) = c + Px$ , for some  $c \in \mathbb{R}^n$ . Evidently,  $f$  is a topical function. By (10b), if  $f$  has an eigenvector with eigenvalue  $\lambda$ , then

$$\frac{(1 + P + \dots + P^{k-1})c}{k} = \frac{f^k(0)}{k}$$

converges to  $(\lambda, \dots, \lambda)$ . The next result can hence be thought of as a version of the mean ergodic theorem for Markov chains, [23, Chapter XIII, §1, Theorem 2].

**Corollary 3.** *Let  $c \in \mathbb{R}^n$  and let  $P$  denote a  $n \times n$  irreducible row-stochastic matrix. The function  $f(x) = c + Px$  has an eigenvector.*

*Proof.*  $f$  is a topical function and  $G(f)$  is strongly connected so the result follows from Theorem 2.  $\square$

A family of non-negative matrices  $\{P^u\}_{u \in U}$  is said to be *communicating* if the matrix  $\sup_{u \in U} P^u$ , obtained by taking entrywise suprema, has finite entries and is irreducible. The following result is due to Bather.

**Corollary 4.** (Bather's theorem, [4, Theorem 2.4]) *Let  $\{P^u\}_{u \in U}$  be a communicating family of row-stochastic matrices, and let  $\{c^u\}_{u \in U}$  be a family of vectors  $c^u \in \mathbb{R}^n$  that is bounded above. Then the function  $f(x) = \sup_{u \in U} (c^u + P^u x)$  has an eigenvector.*

*Proof.* It is easy to check that  $G(f) = G(\sup_{u \in U} P^u)$ . Since the latter is strongly connected by hypothesis, the result follows immediately from Theorem 2.  $\square$

The next result was proved by Zijm in the special case of a finite communicating family. It follows by combining the argument of Corollary 4 with that of Corollary 1.

**Corollary 5.** (Zijm's theorem, [24, Theorem 3.4]) *Let  $\{A^u\}_{u \in U}$  be a communicating family of non-negative matrices. Then the function  $f(x) = \sup_{u \in U} A^u x$  has a (multiplicative) eigenvector.*

As a last illustration of the ideas developed here, consider the topical function  $\mathcal{E}(f)$  where  $f : (\mathbb{R}^+)^3 \rightarrow (\mathbb{R}^+)^3$  is defined by:

$$\begin{aligned} f_1(x) &= 2x_1 \vee 3x_2 \\ f_2(x) &= \sqrt{x_1(4x_2 + 15x_3)} \\ f_3(x) &= x_2 \end{aligned}$$

None of the above corollaries can be applied to  $f$ . However,  $G(\mathcal{E}(f))$  is strongly connected. By Theorem 2,  $f$  has a (multiplicative) eigenvector. In fact,  $f(3, 3, 1) = 3(3, 3, 1)$ .

## 5. CONCLUSIONS

An alternative approach to the eigenvector problem stems from the observation in (7) that all trajectories of a topical function  $f$  are asymptotically equivalent. This suggests that the asymptotics of  $f^k(x)$  contain information on the existence of fixed points, an idea confirmed in recent work, [8, 10].

Topical functions can be defined and studied on cones in Banach spaces, as Krein and Rutman have done for Perron-Frobenius theory. Some attractive examples have emerged here, [21], but with the exception of Nussbaum's work, [17, 18], little general progress has been made.

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