On the Asymptotics of the Eigenvalue Counting Function for Random Recursive Sierpinski Gaskets

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We consider natural Laplace operators on random recursive affine nested fractals based on the Sierpinski gasket and prove an analogue of Weyl's classical result on their eigenvalue asymptotics. The eigenvalue counting function $\boldsymbol{N}(\lambda)$ is shown to be of order $\lambda^{ds/2}$ as $\lambda \to \infty$ where we can explicitly compute the spectral dimension d_s. Moreover the limit $\boldsymbol{N}(\lambda)\lambda^{-ds/2}$ will typically exist and can be expressed as a deterministic constant multiplied by a random variable. This random variable is a power of the limiting random variable in a suitable general branching process and has an interpretation as the volume of the fractal.

On the asymptotics of the eigenvalue counting function for random recursive Sierpinski gaskets

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Abstract

We consider natural Laplace operators on random recursive affine nested fractals based on the Sierpinski gasket and prove an analogue of Weyl's classical result on their eigenvalue asymptotics. The eigenvalue counting function $N(\lambda)$ is shown to be of order $\lambda^{d_s/2}$ as $\lambda \to \infty$ where we can explicitly compute the spectral dimension d_s. Moreover the limit $N(\lambda)\lambda^{-d_s/2}$ will typically exist and can be expressed as a deterministic constant multiplied by a random variable. This random variable is a power of the limiting random variable in a suitable general branching process and has an interpretation as the volume of the fractal.

1 Introduction

The eigenvalue counting function for the Laplacian on a bounded domain has asymptotics that depend on geometric information about the domain. Let $D \subset \mathbb{R}^d$ be a bounded open subset. The Laplacian is a compact operator on D and hence has a discrete spectrum consisting of eigenvalues. If we let $N(\lambda)$ denote the eigenvalue counting function for either the Dirichlet or Neumann Laplacian, then the classical result of Weyl states that

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{B_d|D|}{(2\pi)^d},$$

where $|D|$ denotes the d-dimensional volume of the set D and B_d the volume of the unit ball in $\mathbb{R}^d$.

We will be concerned with the behaviour of the function $N(\lambda)$, when the bounded domain is a fractal subset of $\mathbb{R}^d$. In the case of the compact Sierpinski gasket, it is shown in [7] how to use an exact description of the eigenvalues, as the backwards orbits of a renormalization map, to obtain the following analogue of Weyl's result,

$$0 < \liminf_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d_s/2}} < \limsup_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d_s/2}} < \infty, \tag{1.1}$$

where the exponent $d_s = 2\log 3/\log 5$ is called the spectral dimension of the fractal. The non-existence of this limit is directly related to localization phenomena for the eigenfunctions of the Laplacian on the Sierpinski gasket, [4]. Indeed it is possible to show that for the Sierpinski gasket it is the eigenvalues corresponding to localized eigenfunctions which grow at the rate determined by the spectral dimension. The non-localized eigenfunctions have eigenvalues which grow at a slower rate [14].

For a large class of fractals, called p.c.f. self-similar sets, it has been shown, [15], that for the Laplacian with respect to any Bernoulli measure, μ, the existence of the limit in (1.1), for a suitably defined exponent $d_s(\mu)$, is the generic case. The spectral dimension is then defined to be the maximal exponent over these measures, $d_s = \max_\mu d_s(\mu)$. However, whenever there is a lot of symmetry in the fractal this limit will not necessarily exist as there can be many eigenfunctions with the same

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eigenvalue, leading to large jumps in the eigenvalue counting function. For conditions on p.c.f. self-similar sets for which this can occur see [21]. The constant which appears when the limit does exist has no simple interpretation as a volume.

The question we will address here is the asymptotics of the eigenvalue counting function for a bounded random fractal subset of $\mathbb{R}^d$. The random fractals we consider are affine nested fractals [6] based on the Sierpinski gasket in arbitrary dimension. They are built from a finite family of possible configurations but with a possibly uncountable set of scale factors. We will be able to construct a natural Laplacian on such random fractals in the same way as [11] and obtain results which provide analogues of those of [15] in this setting. The spectral dimension of the Laplacian can be computed as the solution to a suitable expectation equation. The lack of symmetry suggests that the limit in (1.1) will exist and indeed, this is the case. Only if there are a finite number of constituent fractals is it possible to have the non-existence of the limit in (1.1) and as yet there are no known non-trivial examples of non-existence.

As the fractals are random there is an underlying probability space and we will see that the constant appearing in the limit is random. It can be expressed as a deterministic constant multiplied by a random variable. The constant is an extension of that obtained in [15] and is obtained from a renewal equation for the mean behaviour of the eigenvalue counting function. The random variable is related to the limiting random variable for the normalized population size of a general branching process and is a measure of the volume of the fractal.

There is an alternative randomization for fractals which has been explored in more detail in [10, 3, 12]. The randomness appears in an environment sequence and, in this case, the spatial symmetry is preserved but the fractal is not exactly self-similar. They are scale irregular in that there is no scaling factor which leaves the set invariant. The asymptotics of the eigenvalue counting function have not been tackled directly in this case and information about $N(\lambda)$ has come from the trace of the heat kernel. This has shown that there will be oscillation in the eigenvalue counting function asymptotics if there is sufficient irregularity in the environment sequence. In particular, in the case where the environment sequence is generated by a sequence of independent and identically distributed random variables, the limit in (1.1) does not exist and

$$0 < \limsup_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d_s/2}\phi(\lambda)} < \infty,$$

where $\phi(\lambda) = \exp(\sqrt{\log\lambda \log\log\log\lambda})$.

For the Laplacian on a bounded domain $D \subset \mathbb{R}^d$, there has also been extensive investigation of the effect of the boundary on the second term in the asymptotic expansion of $N(\lambda)$. In the case of a smooth boundary the second order term is determined by the $(d-1)$-dimensional volume of the boundary. In the case where the boundary is fractal, then under some mild assumptions, it can be shown that

$$N(\lambda) = \frac{B_d|D|}{(2\pi)^d}\lambda^{d/2} + O(\lambda^{d_m/2}),$$

where d_m is the Minkowski dimension of the boundary. For a discussion of this result and various conjectures about the behaviour of the eigenvalue counting function for fractals and domains with fractal boundary, see [16].

In order to demonstrate the main result we consider the following two random fractals constructed from three affine nested fractals. We take the original Sierpinski gasket, SG(2), the fractal SG(3), as defined initially in [10], and a modified version MSG(3). These are illustrated in Figure 1. As can be seen SG(2) is constructed from 3, 2-similitudes, SG(3) from 6, 3-similitudes and MSG(3) from 3, l-similitudes and 3, $2l/(l-1)$-similitudes, where we choose l, the scale factor of the small triangle, according to a distribution on $(3,\infty)$. We can compute the Hausdorff dimensions and spectral dimensions of each fractal, using standard approaches, [15].

We now construct two examples of random recursive Sierpinski gaskets from these three possible types. In Section 2 we will define the full class of fractals in which we work. Firstly we construct a random fractal from SG(2) and SG(3) by choosing independently for each triangle which set of similitudes to apply within it. With probability p we use the family of similitudes corresponding

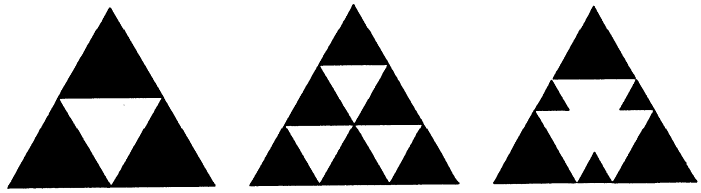

Figure 1: The first level of SG(2), SG(3) and MSG(3)

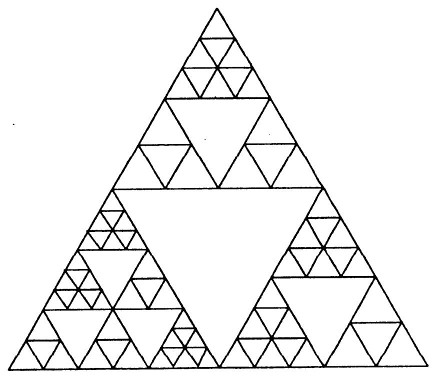

Figure 2: The graph approximation to the random recursive fractal built from SG(2) and SG(3)

to SG(2) and with probability $(1-p)$ the similitudes of SG(3). This fractal, shown in Figure 2, is an example of the fractals discussed in [11]. For our second random fractal we replace SG(3) with the modified version MSG(3), where, once we have chosen to apply the set of maps corresponding to MSG(3), we choose the scale factor of the triangles on the middle of each side according to a distribution function ϕ for a measure with support on a bounded interval in $(3,\infty)$. Thus, in this example, there can be an uncountable family of fractals used to construct the random fractal.

These sets are realizations of a random process and hence they are elements in a space of random recursive fractals $(\Omega,\mathcal{F},\mathbb{P})$. For both examples the probability measure $\mathbb{P}$ is the measure on the path space of a Galton-Watson branching process $\{Z_n; n\geq 0\}$ in which each individual has either 3 offspring or 6 offspring with probability p and $1-p$ respectively, at each generation. Each branch of the resulting Galton-Watson tree determines a subset of the fractal as we associate the similitude $\psi^{a_1,\ldots,a_n}_{i_1,\ldots,i_n}$ with the branch $i_1,\ldots,i_n$ where a_j represents the type of the individual (i.e. the type of

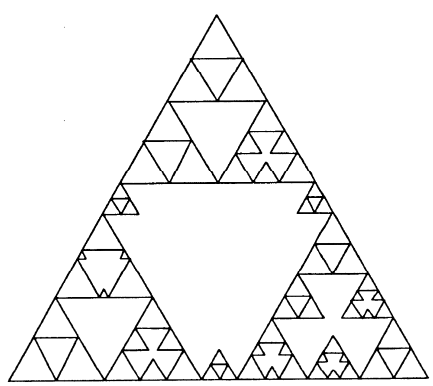

Figure 3: The graph approximation to the random recursive fractal built from SG(2) and MSG(3)

map) at generation j. The random recursive fractal can then be constructed as

$$G = \bigcap_{n=0}^{\infty} \bigcup_{(i_1,\ldots,i_n)\in Z_n} \psi_{i_1,\ldots,i_n}^{a_1,\ldots,a_n}(\Delta).$$

As the fractals are different we will extend the branching process to a general branching process in order to incorporate all the information about the fractal. This will be done in Section 3, where we define and discuss the properties of general branching processes.

The Laplace operator on this set is constructed in Section 4 and we give a brief discussion here. Firstly we observe that there is a scale parameter ρ_a which renormalizes the resistance in a fractal of type a. We then take the graph formed by the images of the edges of the initial triangle after n iterations and define a resistor network by setting a conductance $\rho_{i_1} \ldots \rho_{i_n}$ on each edge in the triangle at $i_1, \ldots, i_n$. This allows us to define a Laplace operator as the limit of the difference operators on these networks. From the construction and choice of ρ_a it is easy to define the Dirichlet form for the fractal and hence a diffusion process. The eigenvalue problem can be expressed in terms of the Dirichlet form and, using a natural decomposition of the form, we may express the eigenfunctions associated with one random fractal in terms of eigenfunctions for other random fractals. This will lead to an expression for the eigenvalue counting function in terms of a process closely associated with a general branching process, which we will be able to describe in enough detail to prove the following result.

Theorem 1.1 *For either of the two random recursive Sierpinski gaskets there exists a constant* $0 < c < \infty$ *and a strictly positive mean one random variable* W*, such that*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d_s/2}} = CW^{1-d_s/2}, \quad \mathbb{P}-a.s.$$

In the case of our first random fractal $d_s/2 = \alpha/(\alpha+1)$*, where* α *satisfies*

$$3p(\frac{5}{3})^{-\alpha} + 6(1-p)(\frac{15}{7})^{-\alpha} = 1.$$

For the second fractal we have $d_s/2 = \alpha/(\alpha+1)$*, where*

$$3p(\frac{5}{3})^{-\alpha} + (1-p)\left(\int_3^{\infty}\left(3(\frac{5}{3}+\frac{4}{3(l-1)})^{-\alpha} + 3(\frac{2}{3}+\frac{5(l-1)}{6})^{-\alpha}\right)d\Phi(l)\right) = 1.$$

We will give an explicit expression for the constant C and the full statement of our main result in Theorem 5.5. Note that the spectral dimension $d_s < 2$ and hence $1 - d_s/2 > 0$ for all the fractals in this class.

2 Random recursive Sierpinski gaskets

As the building blocks for our scale irregular Sierpinski gaskets will all be affine nested fractals, we begin by recalling from [17], [6] the definition of an affine nested fractal.

For $l > 1$, an *l-similitude* is a map $\psi : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\psi(x) = l^{-1}U(x) + x_0, \tag{2.1}$$

where U is a unitary, linear map and $x_0 \in \mathbb{R}^d$. Let $\psi = \{\psi_1, \ldots, \psi_m\}$ be a finite family of maps where ψ_i is an l_i-similitude. For $B \subset \mathbb{R}^d$, define

$$\Psi(B) = \cup_{i=1}^m \psi_i(B),$$

and let

$$\Psi_n(B) = \Psi \circ \ldots \circ \Psi(B).$$

The map Ψ on the set of compact subsets of $\mathbb{R}^d$ has a unique fixed point F, which is a self-similar set satisfying $F = \Psi(F)$.

As each ψ_i is a contraction, it has a unique fixed point. Let F_0' be the set of fixed points of the mappings ψ_i, $1 \leq i \leq m$. A point $x \in F_0'$ is called an *essential fixed point* if there exist $i, j \in \{1, \ldots, m\}$, $i \neq j$ and $y \in F_0'$ such that $\psi_i(x) = \psi_j(y)$. We write F_0 for the set of essential fixed points. Now define

$$\psi_{i_1,\ldots,i_n}(B) = \psi_{i_1} \circ \ldots \circ \psi_{i_n}(B), \quad B \subset \mathbb{R}^D.$$

The set $F_{i_1,\ldots,i_n} = \psi_{i_1,\ldots,i_n}(F_0)$ is called an *n-cell* and the set $E_{i_1,\ldots,i_n} = \psi_{i_1,\ldots,i_n}(F)$ an *n-complex*. The lattice of fixed points F_n is defined by

$$F_n = \Psi_n(F_0), \tag{2.2}$$

and the set F can be recovered from the essential fixed points by setting

$$F = cl(\cup_{n=0}^{\infty} F_n).$$

We can now define an affine nested fractal as follows.

Definition 2.1 The set F is an affine nested fractal if $\{\psi_1, \ldots, \psi_m\}$ satisfy:
(A1) *(Connectivity)* For any 1-cells C and C', there is a sequence $\{C_i : i = 0, \ldots, n\}$ of 1-cells such that $C_0 = C, C_n = C'$ and $C_{i-1} \cap C_i \neq \emptyset$, $i = 1, \ldots, n$.
(A2) *(Symmetry)* If $x, y \in F_0$, then reflection in the hyperplane $H_{xy} = \{z : |z - x| = |z - y|\}$ maps F_n to itself.
(A3) *(Nesting)* If $\{i_1, \ldots, i_n\}, \{j_1, \ldots, j_n\}$ are distinct sequences, then

$$\psi_{i_1,\ldots,i_n}(F) \bigcap \psi_{j_1,\ldots,j_n}(F) = \psi_{i_1,\ldots,i_n}(F_0) \bigcap \psi_{j_1,\ldots,j_n}(F_0).$$

(A4) *(Open set condition)* There is a non-empty, bounded, open set V such that the $\psi_i(V)$ are disjoint and $\cup_{i=1}^m \psi_i(V) \subset V$.

Note that the difference between nested and affine nested fractals is that affine nested fractals can have similitudes with different scale factors. There is a finite number k of length scale factors which correspond to the k possibly different sizes of components of an affine nested fractal.

Our affine nested fractals are based on tetrahedra and hence in $\mathbb{R}^d$ there are $d + 1$ essential fixed points. We will assume that there are a finite number of families of similitudes but allow an uncountable set of scale factors for the similitudes, provided they respect the symmetry of the fractal and are suitably bounded (see section 4). Let A be a finite set. For each $a \in A$, let

$$\psi^a = \{\psi_i^a; i = 1, \ldots, m_a\},$$

denote a set of m_a similitudes in $\mathbb{R}^d$ for some fixed $d > 1$. We can divide these into k_a classes which are each invariant under the symmetry group of the fractal and for $j \in \{1, \ldots, k_a\}$ write $m_a(j)$ or sometimes $m(a, j)$, for the number of similitudes in class j. We can then choose the scale factors $l_a(j)$ for the similitudes according to a distribution function Φ_a on $\mathbb{R}_+^{k_a}$. As this must respect the structure of the affine nested fractal, in that the fractal is connected and there are no overlaps, we see that the distribution is in fact on $\mathbb{R}_+^{k_a - 1}$ and we assume that it lies in a bounded subset. As above there is a unique compact subset K_a of $\mathbb{R}^d$ which satisfies

$$K_a = \bigcup_{i=1}^{m_a} \psi_i^a(K_a).$$

Under the open set condition (A4), this set will have Hausdorff dimension

$$d_f(K_a) = \{\alpha : \sum_{j=1}^{k_a} m_a(j) l_a(j)^{-\alpha} = 1\}.$$

In order to construct random fractals we require an address space. Let $I_n = \cup_{k=0}^{n}\mathbb{N}^k$ and let $I = \cup_k I_k$ be the space of arbitrary length sequences. We will write $\mathbf{i},\mathbf{j}$ for concatenation of sequences. For a point $\mathbf{i} \in I\backslash I_n$ denote by $[\mathbf{i}]_n$ the sequence of length n such that $\mathbf{i} = [\mathbf{i}]_n, \mathbf{k}$ for a sequence $\mathbf{k}$. We write $\mathbf{j} \leq \mathbf{i}$, if $\mathbf{i} = \mathbf{j}, \mathbf{k}$ for some $\mathbf{k}$, which provides a natural ordering on branches. Also denote by $|\mathbf{i}|$ the length of the sequence $\mathbf{i}$.

The infinite random tree, T, is a subset of the space I, defined as the sample path of a Galton-Watson process. Let the root be $T_0 = I_0$, the empty sequence. Let $U_{\mathbf{i}}, \mathbf{i} \in T$ be independent and identically distributed A-valued random variables, indicating the type of nested fractal to be used, with probability distribution

$$P(U = a) = p_a, \quad a \in A.$$

Then $\mathbf{i} \in T$ if $[\mathbf{i}]_n \in T_n \subset I_n$ for each $n \leq |\mathbf{i}|$, where $[\mathbf{i}]_n \in T_n$ if

1. $[\mathbf{i}]_{n-1} \in T_{n-1}$,
2. there is a $j : 1 \leq j \leq m(U_{[\mathbf{i}]_{n-1}})$ such that $[\mathbf{i}]_{n-1}, j = [\mathbf{i}]_n$.

There is a natural probability space associated with these trees given by $(\Omega, \mathcal{B}, \mathbb{P})$. We will now denote a random tree T as a sample point $\omega \in \Omega$. The σ-algebras are defined as

$$\mathcal{B}_n = \sigma(U_{\mathbf{i}}; \mathbf{i} \in T_{n-1}(\omega)), \quad \mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n,$$

and the probability measure, $\mathbb{P}$, is determined by the Galton-Watson process in which an individual has m_a offspring with probability p_a for $a \in A$. For random recursive fractals which are connected, the branching process will be supercritical with no possibility of extinction.

In the case of either of the examples discussed in the introduction and shown in Figures 2 and 3, we have generating function for the offspring distribution $f(u) = pu^3 + (1-p)u^6$. As these fractals are different we will need to incorporate more information into the branching process to provide an adequate description of the fractal.

The address of each branch in the tree is now used to specify a set in our random fractal through the application of the maps determined by the address. We need another independent random variable $V(a, \mathbf{i}) \in \mathbb{R}_+^{k_a}$, chosen according to Φ_a, which specifies the length scale factor. Thus $l(U_{\mathbf{i}}, V_{\mathbf{i}}, i) = V_i(U_{\mathbf{i}}, \mathbf{i})$. Let $E = E_0$ be the unit equilateral tetrahedron, and let F_0 denote the complete graph on the vertices of E_0. Then set $E_{\mathbf{i}}$, $\mathbf{i} \in T_n$, geometrically similar to E, to be

$$E_{\mathbf{i}} = \psi_{\mathbf{i}}(E) = \psi_{[\mathbf{i}]_1}^{U_{[\mathbf{i}]_1}, V_{[\mathbf{i}]_1}}(\cdots(\psi_{[\mathbf{i}]_n}^{U_{[\mathbf{i}]_n}, V_{[\mathbf{i}]_n}}(E))).$$

We regard $\mathbf{i}$ as the address of the set $E_{\mathbf{i}}$. A random gasket can then be defined by

$$F^{\omega} = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i} \in T_n(\omega)} E_{\mathbf{i}}.$$

The Hausdorff dimension of the set F^{ω} can be found by applying the results of [5], [18], [8] and is given by,

$$d_f(F^{\omega}) = \inf\{\alpha : \mathbb{E}\left(\sum_{i=1}^{m(U_0)} l(U_0, V_0, i)^{-\alpha}\right) = 1\}, \quad \text{for a.e. } \omega \in \Omega. \tag{2.3}$$

3 General branching processes

The main tool for proving our result is to describe the fractal with a general branching process. We can then use a limit theorem related to that derived by Nerman [19] which describes the growth of the general branching counted with random characteristic. The Galton-Watson process, introduced in the above construction of the random fractal, contains information about the number of sets in the fractal. However there is no information about the size of the sets after a certain number of generations. We will enlarge the probability space to include more information about the set and the probabilistic setting for this is that of general or C-M-J branching processes.

In the general branching process a typical individual in the population has a reproduction point process, ξ which describes the birth events, as well as a life-length L, and a function ϕ, on $[0,\infty)$, called a random characteristic of the process. We make no assumptions about the joint distributions of these quantities. We write $\xi(t)$ for the ξ-measure of $[0,t]$ and $\nu(t) = E\xi(t)$ for the mean reproduction measure. The basic probability space is now

$$(\Omega, \mathcal{B}, \mathbb{P}) = \prod_{\mathbf{i} \in I} (\Omega_{\mathbf{i}}, \mathcal{B}_{\mathbf{i}}, \mathbb{P}_{\mathbf{i}}),$$

where the spaces $(\Omega_{\mathbf{i}}, \mathcal{B}_{\mathbf{i}}, \mathbb{P}_{\mathbf{i}})$ are identical and contain independent copies of (ξ, L). We now denote a random tree by $\omega \in \Omega$ and we will write $\theta_{\mathbf{i}}(\omega)$ for the subtree of ω rooted at individual $\mathbf{i}$.

The individuals in the population are ordered according to their birth times σ_n. As we can have multiple births this will not be a strictly increasing sequence. We denote the attributes of the n-th individual by (ξ_n, L_n, ϕ_n). At time 0 we have an initial ancestor so that $\sigma_1 = 0$. The process that we wish to consider is written as

$$Z^{\phi}(t) = \sum_{n:\sigma_n \leq t} \phi_n(t - \sigma_n).$$

That is the individuals in the population are counted according to the random characteristic ϕ. By considering the offspring of the initial individual we have a decomposition of the process as

$$Z^{\phi}(t) = \phi_1(t) + \sum_{i=1}^{\xi_1(t)} Z_i^{\phi}(t - \sigma_i), \tag{3.1}$$

where Z_i is an independent copy of the general branching process.

An example of a random characteristic is

$$\phi(t) = I_{\{L>t\}},$$

so that $Z^{\phi}(t)$ is the total number of individuals alive at time t. If the characteristic is $\varphi(t) = 1$ for all t, then the process $Z^{\varphi}(t)$ counts the total number of individuals born up to time t. Later we will choose a characteristic which counts eigenvalues.

We will assume that $\nu(0) = 0$ and there exists a Malthusian parameter $\alpha > 0$, such that

$$\int_0^{\infty} e^{-\alpha t} \nu(dt) = 1.$$

Let $\xi_\alpha(t) = \int_0^t e^{-\alpha s} \xi(ds)$, and define the probability measure $\nu_\alpha(dt) = E(\xi_\alpha(dt))$. We also assume that each individual has at least two offspring so there is no possibility of extinction and the process will be strictly supercritical. We will write

$$\nu_\alpha^{\phi}(t) = E(e^{-\alpha t} Z^{\phi}(t)),$$

for the discounted mean of the process with random characteristic ϕ. We now introduce a martingale, analogous to the standard branching process martingale, which will enable us to discuss the asymptotic growth of this process.

We define the σ-algebra determined by the first n individuals and their characteristics as

$$\mathcal{A}_n = \sigma((\xi_k, L_k, \phi_k^{(1)}, \ldots) : 1 \leq k \leq n).$$

Observe that the birth time of an individual is determined by their parent's reproduction process, so that the birth times σ_k are $\mathcal{A}_{k-1}$ measurable. Now define

$$R_n = \sum_{l=n+1}^{\infty} e^{-\alpha \sigma_l} I_{\{l \text{ is a child of } 1 \ldots n\}}.$$

Then we have the following theorem.

Theorem 3.1 *([1] Chapter VI, Theorem 4.1) The quantity $\{R_n\}_{n=1}^{\infty}$ is a non-negative martingale with respect to $\mathcal{A}_n$ and*

$$W = \lim_{n\to\infty} R_n \text{ exists.}$$

Also $W > 0$ if and only if

$$E\left(\xi_\alpha(\infty)\log^+\xi_\alpha(\infty)\right) < \infty,$$

otherwise $W = 0$, a.s..

There is also a continuous time martingale obtained by setting

$$Y_t = R_{Z^\varphi(t)}.$$

In [19] it is shown that this is a martingale and will converge to the same limit W. We note that for all the general branching processes that we will consider here $\xi_\alpha(\infty)$ is bounded and hence $W > 0$ almost surely.

We will extend a result obtained by Nerman which shows that even when the characteristic depends on the entire line of descent there is still an almost sure limit. We state the extension of [19] Theorem 5.4 as discussed in [19] Section 7. We also give the lattice version of the theorem.

Theorem 3.2 *Let $D[0,\infty)$ denote the set of $\mathbb{R}_+$-valued cadlag paths and let ϕ be a $D[0,\infty)$-valued characteristic satisfying;*
(1) There exists a non-increasing, bounded positive integrable function g, such that

$$E\sup_{t\geq 0}\left(\frac{\xi_\alpha(\infty)-\xi_\alpha(t)}{g(t)}\right) < \infty.$$

(2) There exists a non-increasing, bounded positive integrable function h, such that

$$E\sup_{t>0}\left(\frac{e^{-\alpha t}\phi(t)}{h(t)}\right) < \infty.$$

Then, if the lifelength distribution is non-lattice,

$$\lim_{t\to\infty} e^{-\alpha t}Z^\phi(t) = W\nu_\alpha^\phi(\infty), \quad a.s. \tag{3.2}$$

If the lifelength distribution is lattice, then there exists a periodic function G_α^ϕ, such that for large t,

$$Z^\phi(t) = We^{\alpha t}(G_\alpha^\phi(t) + o(1)), \quad a.s. \tag{3.3}$$

At this stage it should be clear that there is an intimate connection between these processes and the random recursive fractals. We assume that for each fractal $a \in A$ the scale factors for the fractal are chosen according to a distribution function Φ_a on a bounded subset $S_a \subset \mathbb{R}_+^{k_a}$. Now take the general branching process with reproduction and lifelength given by

$$(\xi(ds), L) = (\sum_{i=1}^{k_a} m_a(i)I_{\{\log x_i\in ds\}}, \max_i \log x_i) \text{ with probability } p_a d\Phi_a(x_1,\ldots,x_k),$$

then the process z_n counts the number of sets in the fractal of radius roughly e^{-n}. To be precise, if we let ϕ denote the characteristic

$$\phi_i(t) = \xi_i(\infty) - \xi_i(t), \tag{3.4}$$

which counts the individuals born after time t to mothers born at or before time t, then the process z_n^ϕ is the number of sets in a e^{-n}-cover for the fractal. From this we easily obtain the upper box counting dimension of the fractal as the Malthusian parameter of this general branching process and it is not difficult to establish that it is also the Hausdorff dimension.

4 Laplacians on random recursive Sierpinski gaskets

We now define a Laplace operator on each possible random fractal $\omega \in \Omega$ and give some properties. There is a question as to what is a natural Laplacian on this fractal, as there are no symmetries. We use the idea that the movement of Brownian motion through a medium is determined by the resistance of the medium.

Firstly we note that for affine nested fractals based upon the Sierpinski gasket there is no difficulty in solving the fixed point problem of [17]. Recall that there are k_a classes of set in the affine nested fractal (some of these could be the same size). Let $s(\mathbf{i}) \in \{1, \dots, k_a\}$ denote the class of the set with address $\mathbf{i}$. We can allocate a fixed resistance $r_a(j), j = 1, \dots, k_a$ to all cells in a given class in the fractal K_a. Let

$$\mathcal{E}_0^{(a)}(f,g) = \frac{1}{2} \sum_{x,y \in F_0} (f(x) - f(y))(g(x) - g(y)),$$

for $f, g \in C(F_0^a)$. If we let

$$\tilde{\mathcal{E}}_1^{(a)}(f,f) = \sum_{i=1}^{m_a} r_a(s(i))^{-1} \mathcal{E}_0^{(a)}(f \circ \psi_i, f \circ \psi_i) = \sum_{j=1}^{k_a} \sum_{i=1}^{m(a,j)} r_a(j)^{-1} \mathcal{E}_0^{(a)}(f \circ \psi_i, f \circ \psi_i),$$

for $f \in C(F_1^a)$, then there is a constant λ_a such that

$$\mathcal{E}_0^{(a)}(f,f) = \lambda_a \inf\{\tilde{\mathcal{E}}_1^{(a)}(g,g) : g = f|_{F_0}\}.$$

This allows us to define the Dirichlet form for each fractal in our family A, for details see [2]. We will let $\rho_a(j) = \rho(a,j) = \lambda_a / r_a(j)$ denote the conductance of a cell of class j in the fractal.

Our aim is to construct a Dirichlet form $\mathcal{E}$ on an appropriate $L^2(F, \mu)$ for the random fractal for each $\omega \in \Omega$. As usual we build this up from a sequence of approximating forms on the lattice approximations to the fractal. Let

$$\mathcal{E}_0(f,g) = \frac{1}{2} \sum_{x,y \in F_0} (f(x) - f(y))(g(x) - g(y)),$$

for $f, g \in C(F_0)$. Now define the resistance of a triangle by

$$R(\mathbf{i})^{-1} = \prod_{i=1}^{|\mathbf{i}|} \rho(U_{[\mathbf{i}]_i}, s([\mathbf{i}]_i)).$$

We can then write

$$\mathcal{E}_n^\omega(f,g) = \sum_{\mathbf{i} \in \omega_n} R(\mathbf{i})^{-1} \mathcal{E}_0(f \circ \psi_{\mathbf{i}}, f \circ \psi_{\mathbf{i}}).$$

By the construction of the conductances we see that the sequence of Dirichlet forms is monotone increasing as, for $f : F \to \mathbb{R}$, we can define

$$\mathcal{E}_n^\omega(f|_{F_n}, f|_{F_n}) = \inf\{\mathcal{E}_{n+1}^\omega(g,g) : g \in C(F_{n+1}), g = f|_{F_n}\}.$$

Once we have such a sequence we can clearly define the limiting Dirichlet form as the limit of the sequence. However, in order to define the associated Laplace operator, we need to put this Dirichlet form on an appropriate L^2 space and hence need to define a measure. As in [11] we will choose the measure defined by the invariant measure of the Markov chain on the resistor network, which is equivalent to the Hausdorff measure of the fractal in the resistance metric. In the case of p.c.f. self-similar sets this measure is the one which maximizes the spectral exponent, [15]. To do this we define a new set of approximations to the fractal determined by keeping the resistance of each edge in the graph of approximately the same resistance.

We can modify the general branching process description of the fractal, introduced at the end of Section 3, to describe this new approximation to the fractal. It is now the resistance of a set rather than its length that is crucial and hence we assume that the conductances are now chosen according to the random variable $V(a, \mathbf{i})$ with distribution function Φ_a. We introduce an assumption.

Assumption 4.1 *For each $a \in A$, the support S_a of the distribution function Φ_a for the distribution of conductances on the cells in the fractal K_a is bounded away from 0 and ∞.*

This assumption ensures that conductance and resistance can be controlled uniformly. We choose the conductances $\rho_a(i)$ according to Φ_a satisfying this assumption. Note that, as in Section 2 where the length scale factor of the similitude was chosen and one degree of freedom was lost as the side length must be one, here the equation for λ_a fixes a coordinate. Let

$$(\xi(ds), L) = (\sum_{i=1}^{k_a} m_a(i) I_{\{\log x_i \in ds\}}, \max_i \log x_i) \text{ with probability } p_a d\Phi_a(x_1, \ldots, x_{k_a}),$$

so that the offspring of an individual are born at times given by $\log \rho_a(i)$. Let ϕ denote the characteristic, defined in (3.4), which counts the number of individuals in the population born after time t to mothers born before or at time t, and denote the corresponding general branching process by z_n^{ϕ}.

Let

$$\Lambda_n = \{\mathbf{i} \in z_n^{\phi}\},$$

where we abuse notation by identifying an individual with their line of descent, and then define

$$\tilde{F}_n = \bigcup_{\mathbf{i} \in \Lambda_n} \psi_{\mathbf{i}}(F_0).$$

The graph based on $\tilde{F}_n$ has approximately the same resistance for the edge of each triangle, in that, by our assumption, there exists a constant $c_1 > 0$ such that $c_1 e^{-n} \leq R(\mathbf{i}) \leq e^{-n}$. We will refer to the sets $E_{\mathbf{i}}$ for $\mathbf{i} \in \Lambda_n$ as n-cells.

We use the conductivity to define the measure μ, as this is the invariant measure for the associated Markov chain. Firstly, for an m-cell $E_{\mathbf{i}} \subset F^{\omega}$, define

$$\mu_n^{\omega}(E_{\mathbf{i}}) = \frac{\sum_{\mathbf{j} \in \Lambda_{n-m}} R(\mathbf{i}, \mathbf{j})^{-1}}{\sum_{\mathbf{j} \in \Lambda_n} R(\mathbf{j})^{-1}}. \tag{4.1}$$

As the fractal F^{ω} is compact, by tightness there is a subsequence of measures μ_n^{ω} which converges weakly to a limit measure μ^{ω} on the fractal F^{ω}. We can then define the Dirichlet form $(\mathcal{E}^{\omega}, \mathcal{F}^{\omega})$ on $L^2(F^{\omega}, \mu^{\omega})$ for each $\omega \in \Omega$.

However from now on we will work with a subset $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ where the general branching process converges. On this set we can describe the limit measure using the general branching process. By Theorem 3.2 we have that there exists Ω' with $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$,

$$e^{-\alpha t} z_t(\omega) \to \nu_{\alpha}^{\phi}(\infty) W(\omega),$$

where α satisfies the equation

$$E\left(\sum_{i=1}^{\xi_1(\infty)} \rho_1(s(i))^{-\alpha}\right) = \sum_{a \in A} \int_{S_a} \sum_{j=1}^{k_a} m_a(j) x_j^{-\alpha} d\Phi_a(x_1, \ldots, x_{k_a}) p_a = 1. \tag{4.2}$$

Under our assumption the branching process counted with random characteristic ϕ can be written for a fixed m, by taking t large enough, as

$$z_t^{\phi} = \sum_{\mathbf{i} \in \Lambda_m} z_{t-\sigma_{\mathbf{i}}}^{\phi}(\mathbf{i}),$$

where $z^{\phi}(\mathbf{i})$ are iid copies of z^{ϕ}. Substituting the convergence result into the above, and using the definition of Λ_m we see that

$$W = \sum_{\mathbf{i} \in \Lambda_m} R(\mathbf{i})^{\alpha} W_{\mathbf{i}},$$

where

$$W_{\mathbf{i}} = W(\theta_{\mathbf{i}}(\omega)) = \lim_{s\to\infty} e^{-\alpha s} z_s^{\phi}(\mathbf{i})/\nu_{\alpha}^{\phi}(\infty).$$

Hence, for an m-cell $E_{\mathbf{i}}$ in conductivity coordinates, we have

$$\mu(E_{\mathbf{i}}) = \frac{R(\mathbf{i})^{\alpha} W(\theta_{\mathbf{i}}(\omega))}{W(\omega)}. \tag{4.3}$$

By taking the characteristic $\phi_{\mathbf{i}}(t) = R(\mathbf{i})^{-1}$ and using Theorem 3.2 we can see that this is the behaviour of the limit of the sequence of measures defined by (4.1).

For the rest of the section we omit reference to the sample point $\omega \in \Omega'$ when it is not required. We define the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on the space $L^2(F, \mu)$ as

$$\mathcal{F} = \{f : \sup_n \mathcal{E}_n(f,f) < \infty\},$$

and

$$\mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}_n(f,f), \quad \forall f \in \mathcal{F}.$$

We can define the effective resistance between two points in the random fractal F, by

$$r(x,y) = \left(\inf\{\mathcal{E}(f,f) : f(x) = 0, f(y) = 1\}\right)^{-1}.$$

As in [11] we have the following estimate on the effective resistance.

Lemma 4.2 *There exist constants c_2, c_3 such that for each edge $(x,y) \in \tilde{F}_n$,*

$$c_2 e^{-n} \le r(x,y) \le c_3 e^{-n}.$$

The measure μ is equivalent to the α-dimensional Hausdorff measure in the effective resistance metric. This ensures that the spectral exponent associated with the Laplacian with respect to this metric will be maximal over Bernoulli measures (see the appendix to [15]).

We note that using our conductivity coordinates, and the definition of effective resistance, we can prove the following estimate on the continuity of functions in the domain $\mathcal{F}$.

Lemma 4.3 *There exists a constant c_4 such that*

$$\sup_{x,y\in E_{\mathbf{i}}} |f(x) - f(y)| \le c_4 R(\mathbf{i}) \mathcal{E}(f,f), \quad \forall f \in \mathcal{F}, \quad \forall \mathbf{i} \in \Lambda_m.$$

By construction we have $c_1 e^{-m} \le R(\mathbf{i}) \le e^{-m}$ for $\mathbf{i} \in \Lambda_m$ and this shows that the domain $\mathcal{F} \subset C(F)$. The following theorem can be proved in our setting, in the same way as [11].

Theorem 4.4 *The bilinear form $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on $L^2(F, \mu)$ and has the property that there exists a constant c_4 such that*

$$\sup_{x,y\in F} |f(x) - f(y)| \le c_4 \mathcal{E}(f,f), \quad \textit{for all } f \in \mathcal{F} \tag{4.4}$$

We can also observe a scaling property of this Dirichlet form. We write $\rho_1(j)$ for the conductance of the sets of class j in the first division of the fractal. This corresponds to the fact that the first individual has $m(U_0, j)$ offspring at times $\log \rho_1(j)$.

Lemma 4.5 *We can write for all $f, g \in \mathcal{F}^{\omega}$,*

$$\mathcal{E}^{\omega}(f,g) = \sum_{i=1}^{\xi_1(\infty)} \rho_1(s(i)) \mathcal{E}^{\theta_i(\omega)}(f \circ \psi_i, g \circ \psi_i).$$

Proof: We write the version of this result for the approximating form $\mathcal{E}_n^\omega$ as

$$\mathcal{E}_n^\omega(f,g) = \sum_{i=1}^{\xi_1(\infty)} \rho_1(s(i))\mathcal{E}_{n-1}^{\theta_i(\omega)}(f \circ \psi_i, g \circ \psi_i).$$

Now let $n \to \infty$. □

Let P_t denote the semigroup of positive operators associated with the Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(F,\mu)$. The form is local and regular and hence there exists a Feller diffusion $\{X_t; t \geq 0\}$ with semigroup P_t on F. By (4.4) we see that the resolvent $G_\lambda = \int \exp(-\lambda t)P_t dt$ will have a bounded symmetric density. As this density will be continuous as in [2] we find that P_t will have a bounded symmetric density $p_t(x,y)$ with respect to μ and that $p_t(x,y)$ will satisfy the Chapman-Kolmogorov equations. Some estimates for the transition density were obtained in [11].

Note that we can define the Laplacian Δ with respect to the measure μ, for the fractal F, by setting

$$\mathcal{E}(f,g) = -(\Delta f, g), \quad \forall f,g \in \mathcal{F},$$

where we have taken the inner product on $L^2(F,\mu)$. As we are dealing with a compact fractal we will also need to consider the boundary conditions. For Neumann boundary conditions we need to define a normal derivative at the boundary for our fractal. We follow [13] and set

$$(du)_x = -\lim_{m\to\infty} \Delta_m u(x),$$

where Δ_m is the discrete Laplacian associated with the Dirichlet form $\mathcal{E}_m$. The existence of this limit follows as in [13]

In order to prove compactness of the Laplacian it is enough to show that the natural inclusion map from $\mathcal{F}$ into $L^2(F,\mu)$ is compact. We follow [15] in proving the following.

Lemma 4.6 *The natural inclusion map from $(\mathcal{F}, \mathcal{E} + ||.||_2)$ to $L^2(F,\mu)$ is a compact operator.*

Proof: Let U be a bounded set in $(\mathcal{F}, \mathcal{E} + ||.||_2)$. By (4.4) we have the equicontinuity of U.

We can also use this to show that U is uniformly bounded. Let $h_p(x), x \in F, p \in \partial F$ denote the harmonic function with boundary values 1 at p and 0 for all other points of ∂F. It is easy to see by (4.4) that, if $\bar{f} = \sum_{p\in\partial F} f(p)h_p(x)$, the harmonic function with the same boundary values as f, then

$$|f(x) - \bar{f}(x)| \leq \sum_{p\in\partial F} h_p(x)|f(x) - f(p)| \leq c_4^{1/2}\mathcal{E}(f,f)^{1/2}.$$

As the space of harmonic functions is finite dimensional, the L^2 and L^∞ norms are equivalent and thus

$$\begin{aligned} ||f||_\infty &\leq ||f-\bar{f}||_\infty + ||\bar{f}||_\infty \\ &\leq ||f-\bar{f}||_\infty + ||\bar{f}||_2 \\ &\leq 2||f-\bar{f}||_\infty + ||f||_2 \\ &\leq 2c_4^{1/2}\mathcal{E}(f,f)^{1/2} + ||f||_2. \end{aligned}$$

Thus there exists a constant c_5 such that, for $f \in U$, we have $||f||_\infty \leq c_5$ and hence U is uniformly bounded.

We then apply the Arzela-Ascoli Theorem to see that U is relatively compact in $C(F)$ and hence in $L^2(F,\mu)$. □

By this result the Laplacian will also be a compact operator and it will have a discrete spectrum consisting of eigenvalues. Our aim is to discuss the behaviour of the eigenvalue counting function for this operator.

5 The spectral counting function

We begin by defining the Dirichlet and Neumann eigenvalue problems for our random fractals. Recall that for each $\omega \in \Omega'$ there is a random fractal F^ω and we have a measure μ^ω satisfying (4.3). We will prove results about the counting function for all $\omega \in \Omega'$, giving almost sure statements on Ω. The techniques are based upon the Dirichlet-Neumann bracketing idea developed by [15] for p.c.f. self-similar sets.

Firstly the Dirichlet eigenvalues are defined to be the numbers λ, each with eigenfunction u, such that

$$\begin{aligned} \Delta^\omega u &= -\lambda u \\ u(x) &= 0, \quad x \in \partial F \end{aligned} \tag{5.1}$$

Reformulating this eigenvalue problem for the Dirichlet form, we firstly change the domain and define $\mathcal{F}_0^\omega = \{f \in \mathcal{F}^\omega : f(x) = 0, \ x \in \partial F\}$, setting $\mathcal{E}_0^\omega(f,f) = \mathcal{E}^\omega(f,f)$ for $f \in \mathcal{F}_0^\omega$. Then λ is a Dirichlet eigenvalue with eigenfunction u if

$$\mathcal{E}_0^\omega(u,v) = \lambda(u,v),$$

for all $v \in \mathcal{F}_0^\omega$.

As the operator is compact we can write the spectrum as an increasing sequence of eigenvalues given by $0 < \lambda_0 < \lambda_1 \leq \dots$. We define the associated eigenvalue counting function to be

$$N_0^\omega(x) = \max\{i : \lambda_i \leq x, \lambda_i \text{ solves } (5.1)\}.$$

Analogously we can define the Neumann eigenvalues to be the numbers λ, each associated with an eigenfunction u, such that

$$\begin{aligned} \Delta^\omega u &= -\lambda u \\ (du)_x &= 0, \quad x \in \partial F \end{aligned} \tag{5.2}$$

This eigenvalue problem can be reformulated for the Dirichlet form as λ is a Neumann eigenvalue with eigenfunction u if

$$\mathcal{E}^\omega(u,v) = \lambda(u,v),$$

for all $v \in \mathcal{F}^\omega$.

Again, we write the spectrum as an increasing sequence of eigenvalues with $0 = \lambda_0 < \lambda_1 \leq \dots$, and define the associated eigenvalue counting function to be

$$N^\omega(x) = \max\{i : \lambda_i \leq x, \lambda_i \text{ solves } (5.2)\}.$$

The technique that we will use is a decimation property of the eigenfunctions. This is not the usual decimation property for exactly self-similar fractals, which expresses the eigenfunctions for the Laplacian in terms of other eigenfunctions for the Laplacian. Instead we can build an eigenfunction for a particular random Laplacian in terms of eigenfunctions for other random Laplacians. The key relationship is provided by the following Lemma.

Lemma 5.1 *For all $x > 0$ and each $\omega \in \Omega'$, we have*

$$\sum_{i=1}^{\xi_1(\infty)} N_0^{\theta_i(\omega)}(x\rho_1^{-1}(s(i))\mu(E_i)) \leq N_0^\omega(x) \leq N^\omega(x) \leq \sum_{i=1}^{\xi_1(\infty)} N^{\theta_i(\omega)}(x\rho_1^{-1}(s(i))\mu(E_i)) \tag{5.3}$$

and there exists a constant $M < \infty$ such that for all $\omega \in \Omega$,

$$N_0^\omega(x) \leq N^\omega(x) \leq N_0^\omega(x) + M. \tag{5.4}$$

In order to establish this key result we begin by defining some closely related Dirichlet forms. Let $(\tilde{\mathcal{E}}^\omega, \tilde{\mathcal{F}}^\omega)$ be defined by setting

$$\tilde{\mathcal{F}}^\omega = \{f : F\backslash F_1 \to \mathbb{R} | f \circ \psi_i = f_i \ \text{ on } \ F\backslash F_0, \ \text{ for some } f_i \in \mathcal{F}^{\theta_i(\omega)}\},$$

and

$$\tilde{\mathcal{E}}^{\omega}(f,g) = \sum_{i=1}^{\xi_1(\infty)} \rho_1(s(i))\mathcal{E}^{\theta_i(\omega)}(f\circ\psi_i, g\circ\psi_i).$$

As in [15] we can prove that

Proposition 5.2 *(1)* $\mathcal{F}^{\omega} \subset \tilde{\mathcal{F}}^{\omega}$ *and* $\mathcal{E}^{\omega} = \tilde{\mathcal{E}}^{\omega}|_{\mathcal{F}\times\mathcal{F}}$.
(2) $(\tilde{\mathcal{E}}^{\omega}, \tilde{\mathcal{F}}^{\omega})$ *is a local regular Dirichlet form on* $L^2(F^{\omega}, \mu^{\omega})$.
(3) $\tilde{\mathcal{F}}^{\omega} \hookrightarrow L^2(F^{\omega}, \mu^{\omega})$ *is a compact operator.*
(4) If $\tilde{N}^{\omega}(x)$ *denotes the eigenvalue counting function for the eigenvalues of* $\tilde{\mathcal{E}}^{\omega}$*, then*

$$\tilde{N}^{\omega}(x) = \sum_{i=1}^{\xi_1(\infty)} N^{\theta_i(\omega)}(x\rho_1(s(i))^{-1}\mu(E_i)).$$

Proof: (1),(2), follow easily from the definitions. The proof of (3) will follow in the same way as [15] Proposition 6.2. The one part that we need to prove is (4). Assume that we have a Neumann eigenfunction f of $\mathcal{E}^{\omega}$ with eigenvalue λ. By using the decomposition of the Dirichlet form we have

$$\begin{aligned}\sum_{i=1}^{\xi_1(\infty)} \rho_1(s(i))\mathcal{E}^{\theta_i(\omega)}(f\circ\psi_i, g\circ\psi_i) &= \mathcal{E}^{\omega}(f,g)\\ &= \lambda(f,g)\\ &= \lambda\sum_{i=1}^{\xi_1(\infty)}(f\circ\psi_i, g\circ\psi_i)\mu(E_i).\end{aligned}$$

Thus for all $h \in \mathcal{F}^{\theta_i(\omega)}$ we have

$$\mathcal{E}^{\theta_i(\omega)}(f\circ\psi_i, h) = \lambda\rho_1^{-1}(s(i))\mu(E_i)(f\circ\psi_i, h),$$

and we have that $\lambda\rho_1^{-1}(i)\mu(E_i)$ is an eigenvalue of $\Delta^{\theta_i(\omega)}$ with eigenfunction $f_i = f\circ\psi_i$. Now we can construct from this an eigenfunction with eigenvalue λ of $(\tilde{\mathcal{E}}^{\omega}, \tilde{\mathcal{F}}^{\omega})$. This is just done by setting

$$\tilde{f}(x) = \begin{cases} f_i(x), & x \in \mathrm{int}(E_i),\\ 0, & x \in \mathrm{int}(E_j)\ j \neq i.\end{cases}$$

It is easy to check that each of these functions is an eigenfunction of $(\tilde{\mathcal{E}}^{\omega}, \tilde{\mathcal{F}}^{\omega})$ with eigenvalue λ. Hence it is obvious that

$$\tilde{N}^{\omega}(x) = \sum_{i=1}^{\xi_1(\infty)} N_0^{\theta_i(\omega)}(x\rho_1^{-1}(s(i))\mu(E_i)),$$

as required. □

There is a similar proof to the following proposition. Let $(\tilde{\mathcal{E}}_0^{\omega}, \tilde{\mathcal{F}}_0^{\omega})$ be defined by setting

$$\tilde{\mathcal{F}}_0^{\omega} = \{f : f \in \mathcal{F}_0^{\omega}, f|_{F_1} = 0\},$$

and

$$\tilde{\mathcal{E}}_0^{\omega}(f,g) = \mathcal{E}^{\omega}|_{\mathcal{F}_0^{\omega}\times\mathcal{F}_0^{\omega}}.$$

Proposition 5.3 *(1)* $\tilde{\mathcal{F}}_0^{\omega} \subset \mathcal{F}_0^{\omega}$.
(2) $(\tilde{\mathcal{E}}_0^{\omega}, \tilde{\mathcal{F}}_0^{\omega})$ *is a local regular Dirichlet form on* $L^2(F^{\omega}, \mu^{\omega})$.
(3) $\tilde{\mathcal{F}}_0^{\omega} \hookrightarrow L^2(F^{\omega}, \mu^{\omega})$ *is a compact operator.*
(4) If $\tilde{N}_0^{\omega}(x)$ *denotes the eigenvalue counting function for the eigenvalues of* $\tilde{\mathcal{E}}_0^{\omega}$*, then*

$$\tilde{N}_0^{\omega}(x) = \sum_{i=1}^{\xi_1(\infty)} N_0^{\theta_i(\omega)}(x\rho_1^{-1}(s(i))\mu(E_i)).$$

To conclude the proof of the key inequalities we require the Dirichlet-Neumann bracketing results given in [15]. We give here a version of [15] Corollary 4.7.

Lemma 5.4 *If (E,F) and (E',F') are two Dirichlet forms on $L^2(F,\mu)$ and F' is a closed subspace of F and $E' = E|_{F'\times F'}$, then*

$$N_{E'}(x) \leq N_E(x) \leq N_{E'}(x) + Dim(F/F').$$

Proof of Lemma 5.1: Using the left inequality of Lemma 5.4 twice with the two propositions gives (5.3).

As the space of harmonic functions for finitely ramified fractals is finite dimensional Lemma 5.4 gives $\text{Dim}(\mathcal{F}/\bar{\mathcal{F}}) = |F_0| = d+1$ and hence we have (5.4) for all $\omega \in \Omega$. □

We can now state and prove our main theorem. In order to do this we define the following function,

$$\eta_0(t) = N_0^\omega(e^t) - \sum_{i=1}^{\xi_1(\infty)} N^{\theta_i(\omega)}(e^t \rho_1^{-1}(s(i))\mu(E_i)),$$

which will act as a characteristic for a process closely related to the general branching process.

Theorem 5.5 *For the random recursive Sierpinski gasket the spectral dimension d_s is given by*

$$d_s = 2 \lim_{x\to\infty} \frac{\log N(x)}{\log x} = \frac{2\alpha}{\alpha+1},$$

where α satisfies the equation

$$E\Big(\sum_{i=1}^{\xi_1(\infty)} \rho_1(s(i))^{-\alpha}\Big) = 1.$$

If the mean lifelength distribution ν is non-lattice, then

$$\lim_{x\to\infty} N_0^\omega(x) x^{-d_s/2} = m(\infty) W^{1/(1+\alpha)}(\omega), \quad a.e.\ \omega \in \Omega,$$

where

$$m(\infty) = \frac{\int_{-\infty}^{\infty} e^{-td_s/2} E\eta_0(t)dt}{\int_0^\infty te^{-td_s/2}\nu(dt)}.$$

If the support of the measure ν lies in a discrete subgroup of $\mathbb{R}$, then, if T is the generator of the support, then for a.e. $\omega \in \Omega$, for large x

$$N_0^\omega(x) = (G(\log(x/W(\omega))) + o(1))\, x^{d_s/2} W^{1/(1+\alpha)}(\omega),$$

where G is a positive periodic function with period T given by

$$G(t) = \frac{\sum_{j=-\infty}^{\infty} e^{-d_s(t+jT)/2} E\eta_0(t+jT)}{\int_0^\infty te^{-td_s/2}\nu(dt)}.$$

The technique used to prove this result is to express the problem of finding the spectral dimension and the limit of the eigenvalue counting function as determining a characteristic of an extended general branching process. The spectral dimension will be the Malthusian parameter for the process and the limit result will be an extension of Theorem 3.2.

We begin by writing the left inequality in (5.3) in the same way as the equation for a general branching process. By definition $\mu(E_i) = \rho_1^{-\alpha}(i) W_i/W$ and we can write (5.3) as

$$\sum_{i=1}^{\xi_1(\infty)} N_0^{\theta_i(\omega)}(x\rho^{-1-\alpha}(s(i)) W_i/W) \leq N_0^\omega(x).$$

We will make the substitution $X_0^{\omega'}(t) = N_0^{\omega'}(e^t W(\omega'))$ for all $\omega' \in \Omega$, and consider

$$\sum_{i=1}^{\xi_1(\infty)} X_0(t - \log \tau_1(s(i))) \leq X_0(t),$$

where we write $\tau_1(j) = \rho_1^{1+\alpha}(j)$ and suppress the ω dependence. The counting process $\{X_0(t) : -\infty < t < \infty\}$ differs from the general branching process as it is defined for negative values of time and evolves as a counting process but time changed by the random constant W, so that it is started with an individual arising at some random time point.

Define the function η by

$$\eta(t) = X_0(t) - \sum_{i=1}^{\xi_1(\infty)} X_0(t - \log \tau_1(s(i))),$$

and note that $\eta_0(t) = \eta(t - \log W)$. Clearly we have for all $t \in \mathbb{R}$,

$$X_0(t) = \eta(t) + \sum_{i=1}^{\xi_1(\infty)} X_0(t - \log \tau_1(s(i))), \tag{5.5}$$

which is almost the equation for the evolution of a general branching process with characteristic η as in (3.1). The difference lies in the time scales. The process $\{X_0(t) : t \in \mathbb{R}\}$ considered here is obtained by adding together a number of time shifted copies of itself. The time shifts are the birth times of individuals in the general branching process z_t which starts from a single individual at time 0 and has a lifelength and reproduction point process given by

$$(\xi_x(ds), L_x) = \Big(\sum_{j=1}^{k_a} m_a(j) I_{\{(1+\alpha)\log x_j \in ds\}}, \max_j (1+\alpha)\log x_j\Big)$$
$$\text{with probability } p_a d\Phi_a(x_1, \ldots, x_{k_a}).$$

Note that the first Dirichlet eigenvalue is some $\lambda_\omega^D > 0$, and hence we see that almost surely $t_0 := \inf\{t : X_0(t) = 1\} > -\infty$.

We will follow the argument of [19] for the non-lattice case; the extension to the lattice case will be clear. We begin by examining the mean behaviour. Multiplying (5.5) by $e^{-\gamma t}$, taking expectations and letting $m(t) = E(e^{-\gamma t} X_0(t)), u(t) = E(e^{-\gamma t}\eta(t))$, we have a renewal equation

$$m(t) = u(t) + \int_0^\infty e^{-\gamma s} m(t-s)\nu(ds) = u(t) + \int_0^\infty m(t-s)\nu_\gamma(ds), \tag{5.6}$$

provided the Malthusian parameter γ is a solution to the equation

$$E\int_0^\infty e^{-\gamma t}\xi(dt) = 1.$$

Thus, with this choice, we have

$$1 = \sum_{a\in A} \int_{S_a} \sum_{j=1}^{k_a} m_a(j) x_j^{-\gamma(1+\alpha)} d\Phi(x_1, \ldots, x_{k_a}) p_a.$$

By the definition of α in (4.2) we see that $\alpha = \gamma(1+\alpha)$, giving $\gamma = \alpha/(\alpha+1)$.

Equation (5.6) is the renewal equation of [15] and hence we can conclude from the version of the renewal theorem given in [15], that

Lemma 5.6 *If ν is not lattice, then*

$$m(\infty) = \frac{\int_{-\infty}^{\infty} u(x)dx}{\int_0^{\infty} x\nu(dx)}.$$

Otherwise, if the support of ν lies in some discrete subgroup of $\mathbb{R}$, then if T is the greatest common divisor of the support of ν, then $G(t) = \lim_{n\to\infty} m(t+nT)$ exists for every t and

$$G(t) = \frac{\sum_{j=-\infty}^{\infty} u(t+jT)}{\int_0^{\infty} x\nu(dx)}.$$

This determines the mean behaviour of the limits in Theorem 5.5. In order to prove the existence of the almost sure limit we will try to establish a similar result to the general branching process result from Theorem 3.2. For this we set up a little more notation.

Recall that $Y_t = R_{Z^{\varphi}(t)}$ denotes the continuous time martingale which converges to W almost surely as $t \to \infty$. Also write

$$\begin{aligned}\mathcal{I}_t &= \{\mathbf{i} = (\mathbf{j}, i) : \sigma_{\mathbf{j}} < t, \sigma_{\mathbf{i}} > t\},\\ \mathcal{I}_{t,c} &= \{\mathbf{i} = (\mathbf{j}, i) : \sigma_{\mathbf{j}} < t, \sigma_{\mathbf{i}} > t + c\}.\end{aligned}$$

We now define a class of processes $\{X^{\phi}(t) : -\infty < t < \infty\}$ constructed from a class of characteristics $\{\phi_{\omega}(t) : -\infty < t < \infty\}$, which can be random but are independent for offspring of the same parent and where $\phi_{\omega}(t) = 0$ for $t < t_0(\omega)$. We define

$$X^{\phi}(t) = \sum_{\mathbf{i}\in T(\omega)} \phi_{\mathbf{i}}(t - \sigma_{\mathbf{i}}),$$

where we sum over the entire tree T. Note that the existence of the process requires that the sum is finite for all $t \in \mathbb{R}$. This is clear for the case of X_0 by its construction. It is also easy to see that the process satisfies the evolution equation

$$X^{\phi}(t) = \phi(t) + \sum_{i=1}^{\xi_1(\infty)} X_i^{\phi}(t - \sigma_i),$$

where the $X_i^{\phi}(t - \sigma_i)$ are iid copies of X^{ϕ}. We will write $m^{\phi}(t) = Ee^{-\gamma t}X^{\phi}(t)$.

The proof of Theorem 5.5 will establish the almost sure convergence on certain lattices which we define as follows. Let $c > 0$, take $t_0 \in [0, c]$ and set $t_k = t_0 + kc$ for $k \in \mathbb{Z}$. Also we write $t_{k,n} = kc/n$ for $k \in \mathbb{Z}$ and $n = 1, 2, \ldots$.

Lemma 5.7 *For each $c > 0, t_0 \in [0, c]$ we have*

$$e^{-\gamma t_k} X_0(t_k) \to m(\infty)W, \quad a.s.$$

as $k \to \infty$.

Proof: We follow the proof of [19] Lemma 5.10. Firstly truncate η to η^c where

$$\eta^c(t) = \begin{cases} \eta(t), & t < n_0 c,\\ 0, & t \geq n_0 c.\end{cases}$$

Then, for $n \geq n_0$, we have, writing X^c for X^{η^c} and $a_{\mathbf{i}}(t) = e^{-\gamma(t-\sigma_{\mathbf{i}})}X_0^c(t - \sigma_{\mathbf{i}}) - m^c(t - \sigma_{\mathbf{i}})$, that

$$\begin{aligned}|e^{-\gamma t_{k+n}} X_0^c(t_{k+n}) - m^c(\infty)W| &\leq \left|\sum_{\mathbf{i}\in\mathcal{I}_{t_k}\setminus\mathcal{I}_{t_k,nc}} e^{-\gamma\sigma_{\mathbf{i}}} a_{\mathbf{i}}(t_{k+n})\right|\\ &\quad + \left|\left(\sum_{\mathbf{i}\in\mathcal{I}_{t_k}\setminus\mathcal{I}_{t_k,nc}} e^{-\gamma\sigma_{\mathbf{i}}} m^c(t_{k+n} - \sigma_{\mathbf{i}})\right) - m^c(\infty)W\right|\\ &= S_1(t_k) + S_2(t_k).\end{aligned}$$

The second term $S_2(t_k)$ depends purely on the general branching process and by [19] (5.53) we can prove that for any $\epsilon > 0$, there is an $n \geq n_0$ such that

$$\limsup_{k\to\infty} S_2(t_k) \leq W\epsilon.$$

The second term can be written $S_1(t_k) = S_{11}(t_k)S_{12}(t_k)$ where

$$\begin{aligned} S_{11}(t_k) &= e^{-\gamma t_k}|\mathcal{I}_{t_k}\backslash\mathcal{I}_{t_k,nc}|, \\ S_{12}(t_k) &= \frac{1}{|\mathcal{I}_{t_k}\backslash\mathcal{I}_{t_k,nc}|}\left|\sum_{i\in\mathcal{I}_{t_k}\backslash\mathcal{I}_{t_k,nc}} e^{-\gamma(\sigma_i-t_k)}a_i(t_{k+n})\right|. \end{aligned}$$

Then $S_{11}(t_k) \leq e^{-\gamma t_k}Z^{\varphi(t_k)}$ and hence is almost surely bounded by a constant. For the final term $S_{12}(t_k)$ we note that a_i are mean 0 random variables and we can apply the version of the strong law of large numbers proved as Lemma 4.1 in [19]. For this we use boundedness of η, finiteness of the total population at fixed times and exponential growth of $|\mathcal{I}_{t_k}\backslash\mathcal{I}_{t_k,nc}|$. Using [19] Proposition 4.3 we have

$$S_1(t_k) \to 0, \quad a.s. \text{ as } k \to \infty.$$

Both parts obtain results which are independent of c. We then use the fact that $X_0 = X_0^c + X_0'$, where X_0' satisfies

$$X'(t) = \eta(t)I_{\{t>n_0c\}} + \sum_{i=1}^{\xi(\infty)} X_0'(t-\sigma_i).$$

Now from this equation, there exists a constant C such that

$$\begin{aligned} \limsup_{t\to\infty} e^{-\gamma(t+c)}X'(t+c) &= \limsup_{t\to\infty} e^{-\gamma(t+c)}\sum_{i=1}^{Z^\varphi(t)} \eta(t+c-\sigma_i) \\ &\leq \limsup_{t\to\infty} e^{-\gamma(t+c)}Z^\varphi(t)M \\ &\leq e^{-\gamma c}CW, \quad a.s. \end{aligned}$$

From this we use dominated convergence to show that we can take $c \to \infty$ and remove the truncation to get the result for η. □

Corollary 5.8 *For each fixed n*

$$e^{-\gamma t_{k,n}}X_0(t_{k,n}) \to m(\infty)W, \quad a.s.$$

Proof: This follows from the previous Lemma as in [19] Corollary 5.11. □

Lemma 5.9 *The process $\{X_0(t) : t \in \mathbb{R}\}$ has Malthusian parameter $\gamma = \alpha/(\alpha+1)$ where α satisfies the equation*

$$E\left(\sum_{i=1}^{\xi_1(\infty)} \rho_1(s(i))^{-\alpha}\right) = 1.$$

If the mean lifelength distribution ν is non-lattice, then

$$\lim_{t\to\infty} X_0(t)e^{-\gamma t} = m(\infty)W, \quad a.s.,$$

where

$$m(\infty) = \frac{\int_{-\infty}^{\infty} e^{-\gamma t}E\eta(t)dt}{\int_0^\infty te^{-\gamma t}\nu(dt)}.$$

If the support of the measure ν lies in a discrete subgroup of $\mathbb{R}$, then, if T is the generator of the support, then for large t,

$$X_0(t) = (G(t) + o(1))\, e^{\gamma t} W, \quad a.s.$$

where G is a positive periodic function with period T given by

$$G(t) = \frac{\sum_{j=-\infty}^{\infty} e^{-\gamma(t+jT)} E\eta(t+jT)}{\int_0^\infty t e^{-\gamma t} \nu(dt)}.$$

Proof: The discussion prior to Lemma 5.6 shows that the Malthusian parameter is γ and the expression for $m(\infty)$ comes from Lemma 5.6. Thus we just need to demonstrate convergence as in Corollary 5.8. We begin by defining

$$\begin{aligned} \eta^\epsilon(t) &= \sup_{|s-t|<\epsilon} \eta(s) \\ \eta_\epsilon(t) &= \inf_{|s-t|<\epsilon} \eta(s). \end{aligned}$$

As the paths of $\eta(t)$ are bounded and cadlag we see that $E\eta(t)$ is continuous and, as $\epsilon \to 0$, we have

$$E\eta^\epsilon(t) \downarrow E\eta(t), \quad E\eta_\epsilon(t) \uparrow E\eta(t),$$

for almost every t. Thus $E\eta^{c/n}(t), E\eta_{c/n}(t)$ are continuous for almost every t. It is clear that the processes $X^{\eta_{c/n}}, X^{\eta^{c/n}}$ will exist and by definition

$$X_0^{\eta_{c/n}}(t) \leq X_0(t) \leq X_0^{\eta^{c/n}}(t).$$

Again using the boundedness of the function η we have

$$\begin{aligned} e^{-\gamma c/n} m^{\eta_{c/n}}(\infty) W &\leq \liminf_{t\to\infty} e^{-\gamma t} X_0(t) \\ &\leq \limsup_{t\to\infty} e^{-\gamma t} X_0(t) \leq e^{\gamma c/n} m^{\eta^{c/n}}(\infty) W. \end{aligned}$$

Using dominated convergence and the renewal equation we can deduce that $m^{\eta^{c/n}}, m^{\eta_{c/n}} \to m(\infty)$ and hence we have the result on letting $n \to \infty$. □

Proof of Theorem 5.5: We can now complete the proof of the theorem by replacing X_0 in the almost sure convergence result given in Lemma 5.9, by the counting function $N_0(x)$,

$$\lim_{t\to\infty} e^{-\gamma t} N_0(e^t W) = m(\infty) W.$$

Finally substituting $t = \log(x/W)$ we see that $\gamma = d_s/2$ and the results of Lemma 5.9 complete the proof. □

By (5.4) we know that the spectral asymptotics for both the Dirichlet and Neumann Laplacians will be the same.

Corollary 5.10 *For the random recursive Sierpinski gasket we have*

$$\lim_{x\to\infty} N^\omega(x) x^{-d_s/2} = m(\infty) W^{1/(1+\alpha)}(\omega), \quad a.e.\ \omega \in \Omega.$$

Remark 5.11 (1). In order to obtain the lattice case we need to find Sierpinski gasket based fractals with resistances which are related via their logarithms, in that $\log \rho_1 = n \log \rho_2$. We could take SG(2) and the second level version of SG(2), in which case $\rho_1 = 5/3, \rho_2 = 25/9$, and we are in the lattice case as we are just providing a random method of generating the Sierpinski gasket. It would be interesting to find a non-trivial example.

(2). It is clear that the only way it is possible to get the lattice case is if the family of fractals is finite. As soon as we use a distribution which is not purely atomic, it is not possible for the mean offspring distribution to be lattice.
(3). The deterministic case can be recovered if we take our probability distribution to be a point mass on a particular fractal in the family. As the limiting distribution will become degenerate we have $W = 1$ and then the value of $m(\infty)$ will be the same as that for the p.c.f. case discussed in [15].
(4). Using the fact that the trace of the heat kernel is the Laplace transform of the eigenvalue counting function, as in [3] Section 7, we can apply a Tauberian theorem to obtain a constant limit result for the quantity $\int_F p_t(x,x)\mu(dx)$ as

$$\lim_{t\to 0}\int_F t^{d_s/2} p_t(x,x)\mu(dx) = m(\infty)W^{1-d_s/2}\Gamma(1+d_s/2) \ \ \mathbb{P}-a.s.$$

From the results in [11] for the first random recursive fractal mentioned in the introduction, there are pointwise bounds on the on-diagonal heat kernel. There exist constants c_6, c_7, β, β' such that

$$c_6 t^{-d_s/2}|\log t|^{-\beta'} \leq p_t(x,x) \leq c_7 t^{-d_s/2}|\log t|^{\beta}, \ \ 0 < t < 1, \ \forall x \in G, \ \mathbb{P}-a.s.$$

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