Heat Kernels and Spectral Asymptotics for some Random Sierpinski Gaskets

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HPL-BRIMS-1999-01
February, 1999

random recursive fractals, heat kernel, spectral counting function, Dirichlet form, general branching process

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Heat kernels and spectral asymptotics for some random Sierpinski gaskets [1]

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January 29, 1999

Abstract

We discuss two types of randomization for nested fractals based upon the d-dimensional Sierpinski gasket. One type, called homogeneous random fractals, are spatially homogeneous but scale irregular, while the other type, called random recursive fractals are spatially inhomogeneous. We use Dirichlet form techniques to construct Laplace operators on these fractals. The properties of the two types of random fractal differ and we extend and unify previous work to demonstrate that, though the homogeneous random fractals are well behaved in space, the behaviour in time of their on-diagonal heat kernels and their spectral asymptotics is more irregular than that of the random recursive fractals.

1 Introduction

The mathematical work in the area of analysis on fractals has been principally concerned with the study of the Laplace operator on classes of exactly self-similar deterministic fractals. Most emphasis has been on finitely ramified fractals where it is possible to decompose the fractal into pieces only connected by a finite number of points. A large class of such exactly self-similar finitely ramified deterministic fractals can be treated in the framework of p.c.f. fractals, introduced in [15]. There are very natural graph approximations to such fractals and Laplace operators can be constructed either directly, as limits of discrete Laplacians on the approximations, or probabilistically, as the generator of Brownian motion, the limit of a sequence of random walks on the approximations. There are still open questions about existence and uniqueness of the Laplacian for general p.c.f. fractals, but a number of properties are known [18], [13].

The initial physical motivation for this work lies in the use of fractals as models for disordered media. The study of the transport properties of such media leads

[1] AMS subject classification: 60J60, 28A80, 31C25, 35P20, 58G25
Keywords: Random recursive fractals, heat kernel, spectral counting function, Dirichlet form, general branching process

naturally to the mathematical issues of defining and solving partial differential equations on fractals. However the typical 'fractal' model considered by physicists arises when a system is near a phase transition, such as the infinite cluster in the percolation model where it is believed that 'at criticality' the infinite cluster has large scale fractal structure. These 'fractals' arise as random subsets of some lattice and are far from exactly self-similar. This suggests introducing randomness into the mathematical models to determine how this affects some of the analytic properties of the fractal.

In this paper we will work in the class of compact nested fractals F, a subclass of p.c.f. fractals, embedded in $\mathbb{R}^d$, as defined in [21]. The existence and uniqueness of the Laplacian is known for these fractals, [24]. We will begin by defining two natural randomizations for finitely ramified fractals and concentrate on Sierpinski gaskets as the existence problem is easy to solve and these provide a natural test bed for more general results. The two randomizations lead firstly to scale irregular but spatially homogeneous fractals, [10],[2],[14], which we call homogeneous random fractals, and secondly to spatially irregular fractals, first constructed in [8],[5],[22] and discussed in [11],[12], which we call random recursive fractals. In Section 2 we will construct these two types of random fractal and describe them using random trees.

The random fractals are finitely ramified and based on the Sierpinski gasket which enables us to approximate the fractal via a sequence of resistor networks. The Laplacian on the fractal is then constructed via its Dirichlet form, as a suitable limit of forms associated with the discrete Laplacians o n these approximations. In the spatially homogeneous case it is clear what we mean by a Laplacian, however in the random recursive case we need to consider what the 'natural' Laplace operator should be.

The analytic properties of the sets we consider are the spectral properties of the Laplacian and the behaviour of the heat kernel. We recall that there are two fundamental exponents which arise in analysis on fractals; the Hausdorff dimension of the set, denoted d_f, and the spectral dimension, d_s, which describes the asymptotic scaling in the eigenvalues of the 'natural' Laplacian. A third exponent, the walk dimension, d_w, often appears in the literature. This describes the rate at which the 'natural' Brownian motion moves through the fractal and in all the cases so far considered, where these exponents can be defined, they are related by $d_s/2 = d_f/d_w$.

Using the connection between the heat equation and Brownian motion, we know that the heat kernel is the transition density of the Brownian motion on the fractal. There have been a number of results on the transition density for Brownian motion on deterministic finitely ramified fractals, [4, 19, 6, 13] which show that, in the compact case, where the set F is symmetric, there are short time bounds of the form

$$p_t(x,y) \approx t^{-d_s/2} \exp\left(-c\left(\frac{d(x,y)^{d_w}}{t}\right)^{1/(d_w-1)}\right), \quad \forall x,y \in F, 0<t<1,$$

where $d(x,y)$ is a shortest path metric on the fractal and the constants differ in the upper and lower bounds. Note that in Euclidean space $\mathbb{R}^d$ we have $d_f = d, d_s = d$ and $d_w = 2$ and we recover the usual Gaussian heat kernel bounds. If the fractal is not symmetric, then the off diagonal term is not so clean and it is possible for there to be directions in which the decay is different [13]. Here we extend the on-diagonal heat kernel bounds of homogeneous random and random recursive fractals to nested d-dimensional Sierpinski gaskets. We will avoid discussing the off diagonal bounds for random fractals and refer to [2] and [11] for results in this direction. For the homogeneous random fractals the on-diagonal heat kernel has no spatial variation, while in the random recursive case, the best bounds that we have suggest that there is spatial variability and uniform upper and lower bounds are not tight. The temporal oscillations in the homogeneous case are greater than the random recursive case.

The other quantity that we discuss here is the spectral or eigenvalue counting function for the Laplacian. In the Euclidean, if $D \subset \mathbb{R}^d$ is a bounded open subset, then the Laplacian is a compact operator on D and hence has a discrete spectrum consisting of eigenvalues. If we let $N(\lambda)$ denote the eigenvalue counting function, the number of eigenvalues of either the Dirichlet or Neumann Laplacian less than λ, then the classical result of Weyl states that

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{B_d|D|}{(2\pi)^d},$$

where $|D|$ denotes the d-dimensional volume of the set D and B_d the volume of the unit ball in $\mathbb{R}^d$. This has led to extensive investigation of the effect of the boundary of the domain on the second term in the asymptotic expansion of $N(\lambda)$. For a discussion of results in this direction and various conjectures about the behaviour of the spectral counting function for fractals and domains with fractal boundary, see [20].

We will be concerned with the behaviour of the function $N(\lambda)$, for the natural Laplacian on our random fractal subsets of $\mathbb{R}^d$. As a consequence of [7] we have the following result for $N(\lambda)$ on the compact Sierpinski gasket,

$$0 < \liminf_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d_s/2}} < \limsup_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d_s/2}} < \infty, \tag{1.1}$$

where $d_s = 2\log 3/\log 5$. Indeed, this provides us with a justification for the definition of the spectral dimension of the fractal as

$$d_s := 2 \lim_{\lambda\to\infty} \frac{\log N(\lambda)}{\log \lambda}.$$

The fact that the limit in (1.1) does not exist is directly related to localization phenomena for the eigenfunctions of the Laplacian on the Sierpinski gasket, [3]. For the Sierpinski gasket it is the eigenvalues corresponding to localized eigenfunctions which grow at the rate determined by the spectral dimension [17]. For the class

of p.c.f. fractals, it has been shown [18] that the existence of the limit in (1.1) is the generic case. However, whenever there is a lot of symmetry in the fractal this limit will not necessarily exist. Intuitively the reason that the limit does not exist is that, in the symmetric case, there are many localized eigenfunctions with the same eigenvalue, producing large jumps in the spectral counting function.

For the two types of random fractal the spectral counting function exhibits rather different behaviour. For the homogeneous randomization the spatial symmetry leads to large oscillations in $N(\lambda)$, giving even wilder behaviour than shown in (1.1). In the case of random recursive fractals our results will provide natural analogues of those of [18], in showing that the limit in (1.1) will typically exist. It appears to be possible to have non-constant behaviour but as yet there are no known non-trivial examples. The constant which appears when the limit in (1.1) exists, will be a natural extension of that in [18] multiplied by a mean one random variable. This random variable is a function of the limiting random variable for the normalized population size of a general branching process and is a measure of the volume of the fractal.

We conclude this introduction with an example to demonstrate the results. Consider the following two random fractals constructed from the original Sierpinski gasket, SG(2), and the fractal SG(3), as defined initially in [10] and illustrated in Figure 1. As can be seen SG(2) is constructed from a family of 3 similitudes with

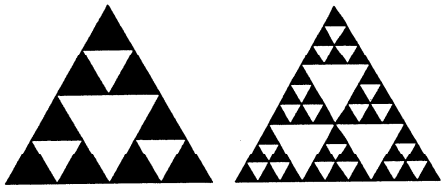

Figure 1: The first two levels in the construction of SG(2) and SG(3)

contraction factor 2, and SG(3) from a family of 6 similitudes with contraction factor 3. The fractals are defined as the fixed points for these families of similitudes acting on the set of compact subsets of $\mathbb{R}^2$. The Hausdorff and spectral dimensions of each fractal can be computed using standard approaches, [10], and are given by

$$d_f(SG(2)) = \frac{\log 3}{\log 2}, \qquad d_s(SG(2)) = 2\frac{\log 3}{\log 5},$$
$$d_f(SG(3)) = \frac{\log 6}{\log 3}, \qquad d_s(SG(3)) = 2\frac{\log 6}{\log (15/7)}.$$

We now construct our two types of random Sierpinski gasket from these two families of similitudes. For both we construct a random tree which describes the fractal. In the homogeneous case we take a sequence $\{a_i, i \in \mathbb{N}\}$, called the environment sequence, taking values in $\{2, 3\}$. We will assume that this sequence is generated by an iid sequence of random variables taking the value 2 with probability p and the value 3 with probability $1-p$ (though it is possible to work with any

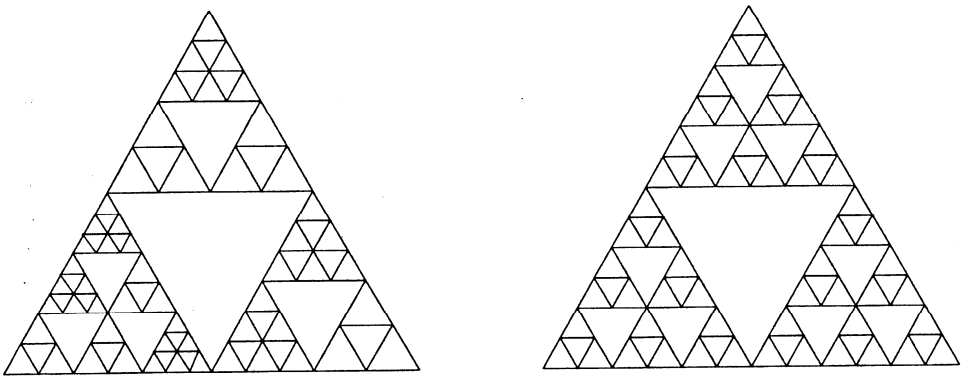

Figure 2: The graph approximation to the random recursive and homogeneous random fractals built from SG(2) and SG(3)

sequence [2]). To construct the fractal we start with an initial equilateral triangle and go from level F^h_{n-1} to F^h_n by looking at a_n and if $a_n = b$ we divide *all the triangles in* F^h_{n-1} into b^2 triangles and remove the downward pointing ones. Thus we generate a random tree by adding 3 branches to every node at generation n, or 6 branches depending on whether $a_n = 2$ or 3. The homogeneous random fractal is then defined as $F^h = \cap_n F^h_n$.

For the random recursive case we start with an equilateral triangle but now to go from F^r_{n-1} to F^r_n we choose independently *for each triangle in* F^r_{n-1} to divide it according to SG(2) with probability p and according to SG(3) with probability $(1-p)$. This generates a random tree in which at each node we either have 3 or 6 branches and such a tree corresponds to the sample path of a Galton-Watson branching process. An extension to a general branching process will be useful in describing the properties of the fractal. Again the fractal is defined to be $F^r = \cap_n F^r_n$. These two random fractals are shown in Figure 2. There is an underlying probability space of possible random fractals and the two randomizations arise from different probability measures on this space. From now on we will refer to quantities associated with either the homogeneous random gasket or the random recursive gasket with a superscript h or r respectively. If a quantity associated with a random fractal is without a superscript h, r, then it applies to either fractal.

In order to state our results we will establish the dimensional exponents for the two sets. Our results will show that for this homogeneous random Sierpinski gasket

$$d^h_f = \frac{p\log 3 + (1-p)\log 6}{p\log 2 + (1-p)\log 3}, \quad d^h_s = \frac{p\log 3 + (1-p)\log 6}{p\log 5 + (1-p)\log (15/7)},$$

while for this random recursive Sierpinski gasket

$$\begin{aligned} d^r_f &= \{s : p3(\frac{1}{2})^s + (1-p)6(\frac{1}{3})^s = 1\}, \\ d^r_s &= 2\frac{\alpha}{\alpha+1}, \text{ where } \alpha = \{s : p3(\frac{3}{5})^s + (1-p)6(\frac{7}{15})^s = 1\}. \end{aligned}$$

In order to state our main results for these two fractals we define, for the homogeneous case, the function $\zeta(s) = \sqrt{s \log\log s}$. Firstly, we consider the on-diagonal heat kernel, $p_t(x,x)$.

Theorem 1.1 *There exist positive constants $c_{1.1}, c_{1.2}, c_{1.3}, c_{1.4}$ such that*

$$c_{1.1}t^{-d_s^h/2}e^{-c_{1.2}\zeta(\log(1/t))} \leq p_t^h(x,x) \leq c_{1.3}t^{-d_s^h/2}e^{c_{1.4}\zeta(\log(1/t))}, \ \forall x \in F^h, \ \mathbb{P}^h - a.s.$$

and there exist positive constants $c_{1.5}, c_{1.6}, \underline{\beta}, \bar{\beta}$, such that

$$c_{1.5}t^{-d_s^r/2}(\log(1/t))^{-\underline{\beta}} \leq p_t^r(x,x) \leq c_{1.6}t^{-d_s^r/2}(\log(1/t))^{\bar{\beta}}, \ \forall x \in F^r, \ \mathbb{P}^r - a.s.$$

Note that the oscillation in the homogeneous random case is wilder than in the random recursive case as $\exp(\sqrt{\log(1/t)}) \geq (\log(1/t))^{\beta}$ as $t \to 0$ for all $\beta > 0$. In Lemmas 4.1 and 4.4 we will describe this oscillation in a form which is best possible up to constants. For the asymptotics of the spectral counting function we also see that there is more regularity in the random recursive case.

Theorem 1.2 *For the homogeneous random fractal, there exist positive constants $c_{1.7}, c_{1.8}$ such that*

$$0 < \liminf_{\lambda\to\infty} \frac{N^h(\lambda)}{\lambda^{d_s^h/2}e^{c_{1.7}\zeta(\log\lambda)}} < \infty, \ \ 0 < \limsup_{\lambda\to\infty} \frac{N^h(\lambda)e^{c_{1.8}\zeta(\log\lambda)}}{\lambda^{d_s^h/2}} < \infty, \ \ \mathbb{P}^h - a.s.$$

For the random recursive Sierpinski gasket there exists a mean 1 random variable $0 < W < \infty$ and a deterministic constant $0 < c_{1.9} < \infty$ such that

$$\lim_{\lambda\to\infty} \frac{N^r(\lambda)}{\lambda^{d_s^r/2}} = c_{1.9}W^{1-d_s^r/2}, \ \ \mathbb{P}^r \ a.s.$$

The constant $c_{1.9}$ is the direct analogue of that arising in the p.c.f. case, [18] and we will give an explicit expression for it in Theorem 5.6. The outline of the paper is that in Section 2 we define the random fractals we will work with. In Section 3 we give a general construction technique for Laplace operators on finitely ramified fractals and use this to construct natural Laplacians for the two cases. In Section 4 we introduce the heat kernels for the Laplacians and study their properties and finally in Section 5 we discuss the spectral asymptotics for the Laplacian. Through out the paper we will use $c_{n.i}$ to denote a fixed constant in section n. Constants written c_i only remain fixed within a proof and have different values between proofs.

2 Random Sierpinski gaskets

As both the homogeneous random and random recursive Sierpinski gaskets will be constructed from nested fractals, we begin by recalling from [21], the definition of a nested fractal.

For $l > 1$, an *l-similitude* is a map $\psi : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\psi(x) = l^{-1}H(x) + x_0, \tag{2.1}$$

where H is a unitary, linear map and $x_0 \in \mathbb{R}^d$. Let $\Psi = \{\psi_1, \ldots, \psi_m\}$ be a finite family of maps where ψ_i is an l-similitude. For $B \subset \mathbb{R}^d$, define

$$\Phi(B) = \cup_{i=1}^m \psi_i(B),$$

and let

$$\Phi_n(B) = \Phi \circ \ldots \circ \Phi(B).$$

The map Φ on the set of compact subsets of $\mathbb{R}^d$ has a unique fixed point F, which is a self-similar set satisfying $F = \Phi(F)$.

As each ψ_i is a contraction, it has a unique fixed point. Let F_0' be the set of fixed points of the mappings ψ_i, $1 \leq i \leq m$. A point $x \in F_0'$ is called an *essential fixed point* if there exist $i, j \in \{1, \ldots, m\}$, $i \neq j$ and $y \in F_0'$ such that $\psi_i(x) = \psi_j(y)$. We write F_0 for the set of essential fixed points. Now define

$$\psi_{i_1,\ldots,i_n}(B) = \psi_{i_1} \circ \ldots \circ \psi_{i_n}(B), \quad B \subset \mathbb{R}^D.$$

We will call the set $F_{i_1,\ldots,i_n} = \psi_{i_1,\ldots,i_n}(F_0)$ an *n-cell* and $E_{i_1,\ldots,i_n} = \psi_{i_1,\ldots,i_n}(F)$ an *n-complex*. The lattice of fixed points F_n is defined by

$$F_n = \Phi_n(F_0), \tag{2.2}$$

and the set F can be recovered from the essential fixed points by setting

$$F = cl(\cup_{n=0}^{\infty} F_n).$$

We can now define a nested fractal as follows.

Definition 2.1 The set F is a nested fractal if $\{\psi_1, \ldots, \psi_m\}$ satisfy:
(A1) *(Connectivity)* For any 1-cells C and C', there is a sequence $\{C_i : i = 0, \ldots, n\}$ of 1-cells such that $C_0 = C, C_n = C'$ and $C_{i-1} \cap C_i \neq \emptyset$, $i = 1, \ldots, n$.
(A2) *(Symmetry)* If $x, y \in F_0$, then reflection in the hyperplane $H_{xy} = \{z : |z-x| = |z-y|\}$ maps F_n to itself.
(A3) *(Nesting)* If $\{i_1, \ldots, i_n\}, \{j_1, \ldots, j_n\}$ are distinct sequences, then

$$\psi_{i_1,\ldots,i_n}(F) \bigcap \psi_{j_1,\ldots,j_n}(F) = \psi_{i_1,\ldots,i_n}(F_0) \bigcap \psi_{j_1,\ldots,j_n}(F_0).$$

(A4) *(Open set condition)* There is a non-empty, bounded, open set V such that the $\psi_i(V)$ are disjoint and $\cup_{i=1}^m \psi_i(V) \subset V$.

Nested fractals were extended to the class of affine nested fractals in [6], by allowing the similitudes to have different scale factors but still preserving the symmetry. A number of the results proved here can be extended to this class but we

concentrate on nested fractals as the results are simpler to state and reveal the key differences in the two randomizations.

Firstly we require a family of families of similitudes to construct our random fractals. Let A be a finite set and we work in $\mathbb{R}^d$ for a fixed $d > 1$. For each $a \in A$, let

$$\psi^a = \{\psi_i^a; i = 1, \ldots, m_a\},$$

denote a set of m_a l_a-similitudes in $\mathbb{R}^d$, with $d+1$ essential fixed points and satisfying the axioms for nested fractals. As above there is a unique compact subset $F^{(a)}$ of $\mathbb{R}^d$ which satisfies

$$F^{(a)} = \bigcup_{i=1}^{m_a} \psi_i^a(F^{(a)}).$$

Under the open set condition (A4), this set has Hausdorff dimension $d_f(F^{(a)}) = \log m_a / \log l_a$.

We are now ready to define the two random fractals we will consider. Both will be described via random trees which are contained in the one-sided shift space on $\bar{M} = \sup_{a \in A} m_a$ symbols (a rooted and labelled $\bar{M}$-ary tree). Let $\bar{I}_n = \cup_{k=0}^n \bar{M}^k$ and $\bar{I} = \cup_n \bar{I}_n$ and write $\mathbf{i} = (i_1, i_2, \ldots)$ for an element of $\bar{I}$. We will write $\mathbf{i}, \mathbf{j}$ for concatenation of sequences. For a point $\mathbf{i} \in \bar{I} \backslash \bar{I}_n$ denote by $[\mathbf{i}]_n = (i_1, \ldots, i_n)$, the sequence of length n such that $\mathbf{i} = [\mathbf{i}]_n, \mathbf{k}$ for a sequence $\mathbf{k}$. We write $\mathbf{j} \leq \mathbf{i}$, if $\mathbf{i} = \mathbf{j}, \mathbf{k}$ for some $\mathbf{k}$, which provides a natural ordering on sequences. Also denote by $|\mathbf{i}|$ the length of the sequence $\mathbf{i}$.

We will define the space of all possible trees constructible from our families of similitudes, which is equivalent to the sample space for the random recursive fractal. We define an infinite random tree, I^r as follows. Let the root be $I_0^r = \bar{I}_0$, the empty sequence. Let $U_{\mathbf{i}}, \mathbf{i} \in I^r$ be A-valued random variables, indicating the family ψ^a of m_a-similitudes to be used. Then $\mathbf{i} \in I^r$ if $[\mathbf{i}]_n \in I_n^r \subset \bar{I}_n$ for each $n \leq |\mathbf{i}|$, where $[\mathbf{i}]_n \in I_n^r$ if

1. $[\mathbf{i}]_{n-1} \in I_{n-1}^r$,
2. there is a $j : 1 \leq j \leq m(U_{[\mathbf{i}]_{n-1}})$ such that $[\mathbf{i}]_{n-1}, j = [\mathbf{i}]_n$.

Let Ω^r denote the set of all random trees I^r. Now define σ-algebras

$$\mathcal{B}_n^r = \sigma(U_{\mathbf{i}}; \mathbf{i} \in I_{n-1}^r), \quad \mathcal{B}^r = \bigcup_{n=1}^{\infty} \mathcal{B}_n^r,$$

and a probability measure, $\mathbb{P}^r$, by choosing the random variables U independently according to $P(U_{\mathbf{i}} = a) = p_a > 0$, $\forall a \in A$. This makes $(\Omega^r, \mathcal{B}^r, \mathbb{P}^r)$ into a probability space for the Galton-Watson process in which an individual has m_a offspring with probability p_a for each $a \in A$. For these random recursive fractals the branching process is clearly supercritical with no possibility of extinction. In the case of the random recursive example discussed in the introduction and shown in Figure 2, we have generating function for the offspring distribution $f(u) = pu^3 + (1-p)u^6$.

We now define a sequence of sets with addresses given by the branches of the tree and we drop the reference to the underlying probability space. Let $E = E_0$ be the unit equilateral tetrahedron, and let G_0 denote the complete graph on F_0, the vertices of E_0. Then set $E_{\mathbf{i}}$, $\mathbf{i} \in I_n^r$, geometrically similar to E, to be

$$E_{\mathbf{i}} = \psi_{\mathbf{i}}(E) = \psi_{[\mathbf{i}]_1}^{U_{[\mathbf{i}]_1}}(\cdots(\psi_{[\mathbf{i}]_n}^{U_{[\mathbf{i}]_n}}(E))).$$

Thus for all possible trees I^r we can define a random recursive Sierpinski gasket, written $F^r(I^r)$ or F^{I^r}, as

$$F^r(I^r) = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i} \in I_n^r} E_{\mathbf{i}}.$$

Under the probability measure $\mathbb{P}^r$ we can determine the almost sure properties of random recursive fractals. The Hausdorff dimension of the set F can be found by applying the results of [5], [22], [8] and is given by,

$$d_f^r(F^r) = \inf\{\alpha : \mathbb{E}\left(m(U_0)l(U_0)^{-\alpha}\right) = 1\}, \quad \mathbb{P}^r - a.s. \tag{2.3}$$

For the homogeneous random fractal we have a random tree with the property that at each generation every node has the same number of branches. Thus Ω^h is a subspace of Ω^r and the elements of the space are sequences. We fix one, an environment sequence $\{a\}$, and will drop a from our notation. The homogeneous random shift space I^h can be associated with the particular fractal F by

$$I^h = \bigotimes_{i=1}^{\infty} \{1, \ldots, m_{a_i}\} = \{(i_1, i_2, \ldots) : 1 \le i_j \le m_{a_j}\}. \tag{2.4}$$

We write $I_n^h = \{(i_1, \ldots, i_n) : 1 \le i_j \le m_{a_j}, 1 \le j \le n\}$ for the set of sequences of length n and write

$$\psi_{[\mathbf{i}]_n} = \psi_{i_1}^{a_1} \circ \ldots \circ \psi_{i_n}^{a_n}. \tag{2.5}$$

As above we define $F_n^h = \cup_{\mathbf{i} \in I_n^h} \psi_{\mathbf{i}}(F_0)$, and for any $a \in \Omega^h$, we define the homogeneous random fractal, written $F^h(I^h)$ or $F^h(a)$, as

$$F^h(a) = \bigcap_{n=0}^{\infty} F_n^h$$

defines the homogeneous random fractal. This fractal is defined for any environment sequence $\{a\}$. If we restrict to the case where the sequence is generated by independent and identically distributed random variables we can think of the fractal as an element in a probability space $(\Omega^h, \mathcal{B}^h, \mathbb{P}^h)$ of homogeneous random fractals. Define the mass scale $M_n = \prod_{i=1}^n m_{a_i}$ and the length scale $L_n = \prod_{i=1}^n l_{a_j}$. The Hausdorff dimension of the fractal is easily seen to be given by $\lim_{n\to\infty} \log M_n / \log L_n$. Using the convergence of the proportions of each type we have

$$d_f(F^h) = \frac{\sum_{a \in A} p_a \log m_a}{\sum_{a \in A} p_a \log l_a}, \quad \mathbb{P}^h - a.s.$$

2.1 General branching processes

A useful tool for proving results about the random recursive Sierpinski gasket is to describe the fractal with a general branching process. The Galton-Watson process introduced in the construction of the random fractal contains information about the number of sets in the fractal. However there is no information about the size of the sets after a certain number of generations. We enlarge the probability space to include more information about the set by using a general or C-M-J branching processes.

In the general branching process a typical individual in the population has a reproduction point process, $\xi(t)$ which describes the birth events, as well as a life-length L, and a function ϕ, on $[0, \infty)$, called a random characteristic of the process. We make no assumptions about the joint distributions of these quantities. The basic probability space for these processes will be the one underlying our random fractals and is given by

$$(\Omega^r, \mathcal{B}^r, \mathbb{P}^r) = \prod_{\mathbf{i} \in \bar{I}} (\Omega_\mathbf{i}, \mathcal{B}_\mathbf{i}, \mathbb{P}_\mathbf{i}),$$

where the spaces $(\Omega_\mathbf{i}, \mathcal{B}_\mathbf{i}, \mathbb{P}_\mathbf{i})$ are identical and contain independent copies of (ξ, L, ϕ). We now denote a random tree by $I \in \Omega$ and we will write $\theta_\mathbf{i}(I)$ for the subtree of I rooted at individual $\mathbf{i}$.

The individuals in the population are ordered according to their birth times σ_n. As we can have multiple births this will not be a strictly increasing sequence. We denote the attributes of the n-th individual by (ξ_n, L_n, ϕ_n). At time 0 we have an initial ancestor so that $\sigma_1 = 0$. We will also denote the attributes of the individual with ancestry $\mathbf{i}$ by $(\xi_\mathbf{i}, L_\mathbf{i}, \phi_\mathbf{i})$ and birth time $\sigma_\mathbf{i}$. The general branching process with random characteristic is then written as

$$Z^\phi(t) = \sum_{n:\sigma_n \leq t} \phi_n(t - \sigma_n).$$

That is the individuals in the population are counted according to the random characteristic ϕ. We easily see that Z^ϕ must satisfy

$$Z^\phi(t) = \phi(t) + \sum_{i=1}^{\xi_1(t)} Z_i^\phi(t - \sigma_i), \tag{2.6}$$

where Z_i^ϕ are iid copies of Z^ϕ. We denote by z_t the branching process with random characteristic given by

$$\phi(t) = I_{\{L > t\}},$$

so that z_t is the total number of individuals alive at time t. The process $Z^\varphi(t)$, with characteristic $\varphi(t) = 1$ for all t, counts the total number of individuals born up to time t. Later we will allow characteristics to be defined for negative time in order to count eigenvalues.

Let $\xi_\alpha(t) = \int_0^t e^{-\alpha s}\xi(ds)$, and define the mean reproduction measure $\nu(t) = E\xi(t)$. We will assume that $\nu(0) = 0$ and that the process is strictly supercritical with Malthusian parameter $\alpha > 0$ such that

$$E(\xi_\alpha(\infty)) = \int_0^\infty e^{-\alpha t}\nu(dt) = 1.$$

We will write

$$v_\alpha^\phi(t) = E(e^{-\alpha t}Z^\phi(t)),$$

for the discounted mean of the process with random characteristic ϕ. We now introduce a martingale, analogous to the standard branching process martingale, which will enable us to discuss the asymptotic growth of this process. This plays an important role in the discussion of random recursive fractals.

Let

$$\mathcal{A}_n = \sigma((\xi_k, \tau_k, \phi_k) : 1 \le k \le n).$$

As the birth time of an individual is determined by their parent's reproduction process, the birth times σ_k are $\mathcal{A}_{k-1}$ measurable. Now define

$$R_n = \sum_{l=n+1}^{\infty} e^{-\alpha\sigma_l} I_{\{l \text{ is a child of } 1 \ldots n\}}.$$

Then we have the following theorem, an amalgamation and simplification of the results of [23], which is all that we require here. We assume that the random characteristic can depend upon the whole line of descent, as discussed in [23] Section 7.

Theorem 2.2 *The quantity $\{R_n\}_{n=1}^\infty$ is a non-negative martingale with respect to $\mathcal{A}_n$ and*

$$W = \lim_{n\to\infty} R_n \quad \textit{exists.}$$

Also $W > 0$ almost surely if and only if

$$E\left(\xi_\alpha(\infty)\log^+ \xi_\alpha(\infty)\right) < \infty,$$

otherwise $W = 0$, a.s. If the characteristic ϕ is bounded, and ν is non-lattice, then

$$\lim_{t\to\infty} e^{-\alpha t}Z_t^\phi = v_\alpha^\phi(\infty)W, \quad a.s.$$

There is also a lattice case for this Theorem which we will state later. We note that in the general branching processes considered here $\xi(\infty)$ is bounded and hence $W > 0$ almost surely.

The random recursive fractals can be described in more detail by these branching processes. Let the general branching process have reproduction and lifelength given by

$$(\xi, L) = (m_a\delta_{\log l_a}, \log l_a) \text{ with probability } p_a,$$

where δ denotes the Dirac delta function, then the process z_n counts the number of sets in the fractal of radius roughly e^{-n}. To be precise, if we let ϕ denote the characteristic

$$\phi_i(t) = \xi_i(\infty), \tag{2.7}$$

which counts the offspring of individuals alive at time t, this will encode a e^{-n}-cover for the fractal. With a little effort the Hausdorff dimension of the fractal given in (2.3) can be recovered as the Malthusian parameter in the general branching process.

3 Laplacians on random Sierpinski gaskets

We now define a Laplace operator on the two types of random fractal. As the space of homogeneous random fractals is a subset of the space of random recursive fractals we consider $\Omega = \Omega^r$ and define a Laplacian for each $I \in \Omega$. Our construction works for all sequences and all possible random trees and enables us to construct a Laplacian with respect to any Borel measure with full support. The usual Laplace operator is characterised as being invariant under the natural symmetries in the space, however in the random recursive case there are no symmetries. We therefore need to choose a natural measure for our Laplacian and this is done using the idea that the movement of Brownian motion through a medium is determined by the resistance of the medium. It is also the case that, for a p.c.f. fractal, the measure determined by the resistance is the one which maximises the spectral exponent, [18].

We will begin by outlining the general scheme of [16] for constructing Laplace operators on fractals via Dirichlet forms, for the case of nested fractals based on the Sierpinski gasket. We consider the graph G_0 as a resistor network with unit resistance on each edge. Define a Dirichlet form on G_0 by

$$\mathcal{E}_0^{(a)}(f,g) = \frac{1}{2}\sum_{x,y\in F_0}(f(x)-f(y))(g(x)-g(y)),$$

for $f, g \in C(F_0)$. If we let

$$\tilde{\mathcal{E}}_1^{(a)}(f,f) = \sum_{i=1}^{m_a}\mathcal{E}_0^{(a)}(f\circ\psi_i, f\circ\psi_i),$$

for $f \in C(F_1^a)$, then, as there is a unique solution [24] to the fixed point problem of [21], with each edge in G_0 of equal resistance, there is a constant ρ_a such that

$$\mathcal{E}_0^{(a)}(f,f) = \rho_a \inf\{\tilde{\mathcal{E}}_1^{(a)}(g,g) : g = f|_{F_0}\}.$$

We will call the constant ρ_a the resistance scale factor; it is the effective conductance between two vertices of F_0 in the resistor network G_1, when each edge in the network has conductance 1. This allows us to define the Dirichlet form for each fractal in

our family A, for details see [1], [16]. We note here that in [2] it was shown that $\rho_a \geq \frac{3}{2}$.

Our aim is to construct a Dirichlet form $\mathcal{E}$ on an appropriate $L^2(F,\mu)$ for each of the random fractals $I \in \Omega$. As in [16] we build this up from a sequence of approximating forms on the graph approximations to the fractal. Let

$$\mathcal{E}_0(f,g) = \frac{1}{2}\sum_{x,y\in F_0}(f(x)-f(y))(g(x)-g(y)),$$

for $f,g \in C(F_0)$. Let $\rho(U_{\mathbf{i}})$ denote the resistance scale factor for the set of type $U_{\mathbf{i}}$ at address $\mathbf{i}$. Now define the resistance of the triangle with address $\mathbf{i}$ to be

$$R(\mathbf{i}) = \prod_{j=1}^{|\mathbf{i}|}\rho(U_{[\mathbf{i}]_j})^{-1}.$$

We let $G_n = \cup_{\mathbf{i}\in I_n^r}\psi_{\mathbf{i}}(G_0)$ denote the n-th graph approximation to the fractal. For $f,g \in C(G_n)$ we can write

$$\mathcal{E}_n^I(f,g) = \sum_{\mathbf{i}\in I_n^r} R(\mathbf{i})^{-1}\mathcal{E}_0(f\circ\psi_{\mathbf{i}}, g\circ\psi_{\mathbf{i}}).$$

By the definition of the resistance scaling we see that the sequence of Dirichlet forms is monotone increasing as for $f : F \to \mathbb{R}$ as it has the property that

$$\mathcal{E}_n^I(f|_{G_n}, f|_{G_n}) = \inf\{\mathcal{E}_{n+1}^I(g,g) : g \in C(G_{n+1}), g = f|_{G_n}\}.$$

Once we have such a sequence we can clearly define the limiting object as the limit of the sequence. However, in order to define the associated Laplace operator, we need to show this is a Dirichlet form and set it on an appropriate L^2 space and this requires us to define a measure on the random fractal F. Any Borel measure with full support can be used for the L^2 space but we will be concerned with choosing a natural measure so that the operator is the 'natural' Laplacian on the fractal F.

The measure μ that we choose is determined by the resistance as it is the limit of the sequence of invariant measures for the Markov chains on the graph approximations. For an m-cell $E_{\mathbf{i}} \subset F^I$, define

$$\mu_n^I(E_{\mathbf{i}}) = \frac{\sum_{\mathbf{j}\in I_{n-m}^r} R(\mathbf{i},\mathbf{j})^{-1}}{\sum_{\mathbf{i}\in I_n^r} R(\mathbf{j})^{-1}}. \tag{3.1}$$

As the fractal F^I is compact, the sequence of measures μ_n^I is tight and there is a subsequence which converges weakly to a limit measure μ^I on the fractal F^I. We can then define the Dirichlet form $(\mathcal{E}^I, \mathcal{F}^I)$ on $L^2(F^I,\mu^I)$ for each $I \in \Omega$.

3.1 The Dirichlet form and its properties

For the rest of the section we omit reference to the sample point $I \in \Omega$ when it is not required. For all the random fractals we can define a symmetric bilinear form $(\mathcal{E}, \mathcal{F})$ on the space $L^2(F, \mu)$ as

$$\mathcal{F} = \{f : \sup_n \mathcal{E}_n(f, f) < \infty\},$$

and

$$\mathcal{E}(f, f) = \lim_{n \to \infty} \mathcal{E}_n(f, f), \quad \forall f \in \mathcal{F}.$$

In order to show that this is a Dirichlet form, we define the effective resistance between two points in the random fractal F, by

$$r(x, y) = (\inf\{\mathcal{E}(f, f) : f(x) = 0, f(y) = 1\})^{-1}.$$

As in [11] we have the following estimate on the effective resistance

Lemma 3.1 *There exist constants $c_{3.1}, c_{3.2}$ such that for each edge $(x, y) \in \psi_{\mathbf{i}}(G_0)$,*

$$c_{3.1} R(\mathbf{i}) \leq r(x, y) \leq c_{3.2} R(\mathbf{i}).$$

Using the definition of effective resistance and Lemma 3.1, we can prove the following estimate on the continuity of functions in the domain $\mathcal{F}$.

Lemma 3.2 *There exists a constant $c_{3.3}$ such that*

$$\sup_{x,y \in E_{\mathbf{i}}} |f(x) - f(y)|^2 \leq c_{3.3} R(\mathbf{i}) \mathcal{E}(f, f), \quad \forall f \in \mathcal{F}, \quad \forall \mathbf{i} \in I_m.$$

By construction we have $R(\mathbf{i}) \leq (3/2)^{-m}$ for $\mathbf{i} \in I_m$ and this shows that the domain $\mathcal{F} \subset C(F)$. The following theorem can be proved in our setting, as in [11].

Theorem 3.3 *The bilinear form $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on $L^2(F, \mu)$ and has the property that there exists a constant $c_{3.4}$ such that*

$$\sup_{x,y \in F} |f(x) - f(y)|^2 \leq c_{3.4} \mathcal{E}(f, f), \quad \text{for all } f \in \mathcal{F} \tag{3.2}$$

We can also observe a scaling property of this Dirichlet form and a Poincaré inequality.

Lemma 3.4 *We can write for all $f, g \in \mathcal{F}^I$,*

$$\mathcal{E}^I(f, g) = \sum_{i=1}^{m(U_0)} \rho(U_0) \mathcal{E}^{\theta_i(I)}(f \circ \psi_i, g \circ \psi_i). \tag{3.3}$$

There exists a constant $c_{3.5}$ such that

$$\|f - \int_F f d\mu\|_2^2 \leq c_{3.5} \mathcal{E}(f, f), \quad \forall f \in \mathcal{F}. \tag{3.4}$$

Proof: For (3.3) we use the definition of the approximating form $\mathcal{E}_n^I$ to write

$$\mathcal{E}_n^I(f,g) = \sum_{i=1}^{m(U_0)} \rho(U_0)\mathcal{E}_{n-1}^{\theta_i(I)}(f\circ\psi_i, g\circ\psi_i).$$

Now let $n \to \infty$. The Poincaré inequality (3.4) follows from (3.2), as in [2]. □

Note that we can define the Laplacian Δ with respect to the measure μ, for the fractal F, by setting

$$\mathcal{E}(f,g) = -(\Delta f, g), \quad \forall f,g \in \mathcal{F},$$

where we have taken the inner product on $L^2(F,\mu)$. As we are dealing with a compact fractal we will also need to consider the boundary conditions. To do this we need to define a normal derivative at the boundary for our fractal. We follow [15] and set

$$(du)_x = -\lim_{m\to\infty} \Delta_m u(x),$$

where Δ_m is the discrete Laplacian associated with the Dirichlet form $\mathcal{E}_m$. The existence of this limit follows as in [15]

For the homogeneous case we have a description of the fractal via the environment sequence $\{a\}$. The scale factors are also generated by $\{a\}$. Firstly, recall the mass scale factor $M_n = \prod_{i=1}^n m_{a_i}$ is the number of maps after n generations. The resistance scale factor defined above becomes $R_n = \prod_{i=1}^n \rho_{a_i}$ and we can also define a time scale as $T_n = R_n M_n$. As in [2] we can also construct a shortest path scale factor B_n, however we will not refer to this quantity here.

3.2 Natural measures on random fractals

The previous section dealt with the general case and defined the natural measure for each possible fractal $F^I, I \in \Omega$. In the case of the homogeneous random fractal we can again use the measure defined for any sequence $\{a\}$ as the weak limit of the sequence of measures with the property that $\mu_n(E_{\mathbf{i}}) = M_n^{-1}$ for $\mathbf{i} \in I_n^h$. For the random recursive case we introduce the natural measure on our random fractals specified by the probability measure $\mathbb{P}$.

As in [11], [12] we consider the graph G_n as a resistor network approximating the fractal and find the invariant measure for the Markov chain on G_n and take a limit. To do this we define a new set of approximations to the fractal where each is determined by keeping the resistance of each edge in the graph of approximately the same resistance. The resulting measure is equivalent to the Hausdorff measure in the effective resistance metric.

We can modify the general branching process description of the fractal, introduced at the end of Section 3, to describe this new approximation to the fractal. It is now the resistance of a set rather than its length that is crucial. Let

$$(\xi(ds), L) = (m_a\delta_{\log\rho_a}, \log\rho_a) \text{ with probability } p_a,$$

so that individuals now live for a time given by $\log \rho_a$. As in (2.7) we let ϕ denote the number of offspring of the individuals in the population at time t and consider the process z_n^ϕ.

Let

$$\Lambda_n = \{\mathbf{i} \in z_n^\phi\},$$

where we label each individual in the population z_n^ϕ by their branch in the random tree, and then define

$$\tilde{G}_n = \bigcup_{\mathbf{i}\in\Lambda_n} \psi_{\mathbf{i}}(G_0).$$

The graph $\tilde{G}_n$ has the resistance of each edge satisfying $c_1 e^{-n} \le R(\mathbf{i}) \le e^{-n}$. We refer to the sets $E_{\mathbf{i}}$ for $\mathbf{i} \in \Lambda_n$ as n-cells.

We will now work with a subset $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ where the normalized limit of the general branching process exists. On this set we can describe the limit measure for our fractal using the general branching process. By Theorem 2.2 we have that, as $t \to \infty$,

$$e^{-\alpha t} z_t^\phi \to \nu_\alpha^\phi(\infty) W,$$

where α satisfies the equation

$$\sum_{a\in A} m_a \rho_a^{-\alpha} p_a = 1. \tag{3.5}$$

Thus, if we multiply (2.6) by $e^{-\alpha t}$ to get

$$e^{-\alpha t} z_t^\phi = e^{-\alpha t}\phi_{\mathbf{i}}(t) + \sum_{\mathbf{i}:\sigma_{\mathbf{i}}\le t} e^{\alpha\sigma_{\mathbf{i}}} \left(e^{-\alpha(t-\sigma_{\mathbf{i}})} z_{t-\sigma_{\mathbf{i}}}^\phi\right),$$

and use the boundedness of ϕ and the definition of Λ_m we see that

$$W = \sum_{\mathbf{i}\in\Lambda_m} R(\mathbf{i})^\alpha W_{\mathbf{i}}.$$

Hence, for an m-cell $E_{\mathbf{i}}$, the measure μ defined in (3.1) can be written

$$\mu(E_{\mathbf{i}}) = \frac{R(\mathbf{i})^\alpha W(\theta_{\mathbf{i}}(I))}{W(I)}. \tag{3.6}$$

The random variable W has been studied closely for the Galton-Watson process. For the general branching process which appears here the tails of W can be estimated as in Section 3 of [11] where the proof of the following can be found.

Lemma 3.5 *There exist constants $c_{3.6}, c_{3.7}, c_{3.8}, c_{3.9}, \beta_1, \beta_2$ such that*

$$P(W \le \delta) \le c_{3.6} \exp(-c_{3.7}\delta^{-\beta_1}),$$

and

$$P(W \ge \delta) \le c_{3.8} \exp(-c_{3.9}\delta^{\beta_2}).$$

From the calculations it is clear that $0 < \beta_1 < \infty$ and $1 < \beta_2 < \infty$.

Using these tail estimates we can obtain a result on the oscillation in the measure akin to [11] Theorem 5.5.

Lemma 3.6 *There exist random constants $c_{3.10}, c_{3.11}$ such that*

$$c_{3.10} n^{-1/\beta_1} \leq \inf_{\mathbf{i} \in \Lambda_n} \frac{W_{\mathbf{i}}}{W} \leq \sup_{\mathbf{i} \in \Lambda_n} \frac{W_{\mathbf{i}}}{W} \leq c_{3.11} n^{1/\beta_2}, \ \ \mathbb{P}^r - a.s.$$

and hence there exist constants $c_{3.12}, c_{3.13}$ such that

$$c_{3.12} n^{-1/\beta_1} e^{-\alpha n} \leq \inf_{\mathbf{i} \in \Lambda_n} \mu(E_{\mathbf{i}}) \leq \sup_{\mathbf{i} \in \Lambda_n} \mu(E_{\mathbf{i}}) \leq c_{3.13} n^{1/\beta_2} e^{-\alpha n}, \ \ \mathbb{P}^r - a.s. \tag{3.7}$$

This shows that α is the Hausdorff dimension of the set in the effective resistance metric.

4 Estimates for the heat kernel

Let P_t denote the semigroup of positive operators associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(F, \mu)$. The form constructed in Theorem 3.3 is local and regular and hence there exists a Feller diffusion $\{X_t; t \geq 0\}$ with semigroup P_t on F. By (3.2) we see that the resolvent $G_\lambda = \int \exp(-\lambda t) P_t dt$ will have a bounded symmetric density. As this density will be continuous as in [1] we find that P_t will have a bounded symmetric density $p_t(x, y)$ with respect to μ and that $p_t(x, y)$ will satisfy the Chapman-Kolmogorov equations.

We obtain an upper bound on the transition density using ideas in [2].

Lemma 4.1 *(1) There exist constants $c_{4.1}$ such that if $T_n^{-1} \leq t \leq T_{n-1}^{-1}$, then for all $I^h \in \Omega^h$,*

$$\|P_t^h\|_{1 \to \infty} \leq c_{4.1} M_n.$$

(2) There exist constants $c_{4.2}, \beta_3$ such that for almost every $I \in \Omega^r$,

$$\|P_t^r\|_{1 \to \infty} \leq c_{4.2} t^{-\alpha/(\alpha+1)} |\log t|^{\beta_3}.$$

Proof: We show a general technique for heat kernel upper bounds. For $\mathbf{i} \in \Lambda_n$ write $f_{\mathbf{i}} = f \circ \psi_{\mathbf{i}}$ and

$$\bar{f}_{\mathbf{i}} = \int_{F^{\theta_{\mathbf{i}}(I)}} f_{\mathbf{i}}(x) \mu^{\theta_{\mathbf{i}}(I)}(dx).$$

Note that for $v \in C(F_n)$, $\bar{v} = \int v d\mu = \sum_{\mathbf{i} \in \Lambda_n} \mu(E_{\mathbf{i}}) \bar{v}_{\mathbf{i}}$.

Let $u_0 \in \mathcal{D}(\Delta)$ with $u_0 \geq 0$ and $\|u_0\|_1 = 1$. Set $u_t(x) = (P_t u_0)(x)$ and $g(t) = \|u_t\|_2^2$, so that $\sup_{u_0} g(t) = \|P_t\|_{1 \to \infty}$. We remark that g is continuous and decreasing

and, as the semigroup is Markov, $\|u_t\|_1 = 1$. Iterating the decomposition (3.3) and applying the Poincaré inequality (3.4), we have

$$\begin{aligned}
\frac{d}{dt}g(t) &= -2\mathcal{E}(u_t, u_t) \\
&= -2\sum_{\mathbf{i}\in\Lambda_n} R(\mathbf{i})^{-1}\mathcal{E}^{\theta_{\mathbf{i}}(I)}(u_t\circ\psi_{\mathbf{i}}, u_t\circ\psi_{\mathbf{i}}) \\
&\leq -2c_{3.5}\sum_{\mathbf{i}} R(\mathbf{i})^{-1}\int (u_{t,\mathbf{i}} - \bar{u}_{t,\mathbf{i}})^2 d\mu^{\theta_{\mathbf{i}}(I)} \\
&= -c_1 \min_{\mathbf{i}\in\Lambda_n} R(\mathbf{i})^{-1}\mu(E_{\mathbf{i}})^{-1}\int u_t^2 d\mu + \\
&\qquad c_1 \max_{\mathbf{i}\in\Lambda_n} R(\mathbf{i})^{-1}\mu(E_{\mathbf{i}})^{-2}\left(\sum_{\mathbf{i}} \bar{u}_{t,\mathbf{i}}\mu(E_{\mathbf{i}})\right)^2. \quad (4.1)
\end{aligned}$$

For the homogeneous case we have $R(\mathbf{i}) = R_n$, which we can replace in (4.1) to get

$$\frac{d}{dt}g(t) \leq -c_1 T_n(g(t) - c_2 M_n).$$

integrating this and following [2] we obtain the first result.

For the random recursive case we substitute the bounds on the measure from (3.7) into (4.1) to get

$$\frac{d}{dt}g(t) \leq -c_3 n^{-1/\beta_2} e^{(\alpha+1)n}\|u_t\|_2^2 + c_4 n^{2/\beta_1} e^{(2\alpha+1)n}.$$

Thus we have $g'(t) \leq -c_3 n^{-1/\beta_2} e^{(\alpha+1)n}(g(t) - c_5 n^{2/\beta_1} e^{\alpha n})$, for all $n \geq 0$. Then we have

$$-\frac{d}{dt}\log(g(t) - c_5 n^{2/\beta_1} e^{\alpha n}) \geq c_3 n^{-1/\beta_2} e^{(\alpha+1)n},$$

integrating this and following the ideas of [2] gives

$$g(c_6 n^{1/\beta_2} e^{(\alpha+1)n}) \leq c_5 n^{2/\beta_1} e^{\alpha n}, \quad \forall n \in \mathbb{N}$$

and hence by setting $c_6 n^{1/\beta_2} e^{(\alpha+1)n} \approx t$ we have the result. □

In order to obtain lower bounds on the heat kernel we need a probabilistic result. This is an estimate on the crossing and exit times for the associated diffusion process. Firstly we define the crossing times

$$\begin{aligned}
S_0^n(X) &= \inf\{t \geq 0 : X_t \in \tilde{G}_n\}, \\
S_{i+1}^n(X) &= \inf\{t > S_i^n : X_t \in \tilde{G}_n \backslash \{X_{S_i^n}\}\}, \\
\tilde{S}_i^n(X) &= S_i^n(X) - S_{i-1}^n(X).
\end{aligned}$$

and also the exit time of a set A as

$$S_A(X) = \inf\{t : X_t \in A^c\}.$$

Lemma 4.2 *In the homogeneous case there exist positive constants* $c_{4.3}, c_{4.4}, c_{4.5}$ *such that for all* $I^h \in \Omega^h$,

$$P^x(\tilde{S}_i^n(X^h) < \delta T_n^{-1}) \leq c_{4.3} \exp(-c_{4.4}\delta^{-c_{4.5}}), \quad \delta > 0. \tag{4.2}$$

In the random recursive case, there exist positive constants $c_{4.6}, c_{4.7}, c_{4.8}, c_{4.9}, \beta_4$ *such that*

$$P^x(\tilde{S}_i^n(X^r) < \delta \theta_n^{-1}) \leq c_{4.6} \exp(-c_{4.7}\delta^{-c_{4.8}} |\log \delta|^{-c_{4.9}}), \quad \delta > 0, \ \mathbb{P}^r - a.s. \tag{4.3}$$

where

$$\theta_n = n^{\beta_4} e^{(\alpha+1)n}.$$

Proof: The bound in (4.2) follows from [2] Lemma 4.3 and (4.3) is proved in [11] Lemma 7.7. □

However once we have such results we can obtain lower bounds on the heat kernel via the following lemma. We let $D_{[\mathrm{i}]_n}(x)$ denote the union of the set $E_{[\mathrm{i}]_n}$, containing the point $x \in F$, and its neighbouring n-cells.

Lemma 4.3 *If there is an increasing continuous function* h *on* $[0,1]$, *such that* $h(t) \to 0$ *as* $t \to 0$ *and there is a sequence* $\{b_n\}$ *such that*

$$P^x(S_{D_{[\mathrm{i}]_n}(x)} < b_n^{-1} t) \leq h(t), \quad 0 < t \leq 1, \ \textit{for all } n \geq 0,$$

then there exists a constant $c_{4.10}$ *such that for* $b_{n+1}^{-1} < t < b_n^{-1}$

$$p_t(x,x) \geq c_{16} \mu(D_{[\mathrm{i}]_n}(x))^{-1} \quad 0 < t \leq 1.$$

Proof: We follow [1]. Let $0 < a < t_0$ be such that $h(a) \leq 1/2$. Then, if $n = \sup\{k : b_k t < a\}$, then

$$P^x(X_t \in D_{[\mathrm{i}]_n}(x)) \geq P^x(S_{D_{[\mathrm{i}]_n}(x)} > t) \geq 1/2.$$

For this value of n we have $b_{n+1} \geq a/t$. By Cauchy-Schwartz

$$\begin{aligned} 1/4 &\leq P^x(X_t \in D_{[\mathrm{i}]_n}(x))^2, \\ &= \left(\int_{D_{[\mathrm{i}]_n}(x)} p_t(x,y) \mu(dy) \right)^2, \\ &\leq \mu(D_{[\mathrm{i}]_n}(x)) p_{2t}(x,x). \end{aligned} \tag{4.4}$$

Thus, with a suitable adjustment of constants, we have the result. □

Comparing the hitting and exit times we can use the hitting time estimates (4.2), (4.3), with a suitable choice of b_n in Lemma 4.3 and the estimate on the measure from (3.7), to obtain the on-diagonal heat kernel lower bounds.

Lemma 4.4 *There exist positive constants* $c_{4.11}, c_{4.12}, \beta_5$ *such that*

$$\inf_{x\in F^r} p_t^r(x,x) \geq c_{4.11} t^{-\alpha/(\alpha+1)} |\log t|^{-\beta_5}, \quad 0 < t < 1,$$

and for $T_n^{-1} \leq t < T_{n-1}^{-1}$,

$$p_t^h(x,x) \geq c_{4.12} M_n, \quad \forall x \in F^h, \quad 0 < t < 1.$$

The spectral dimension for the homogeneous case can be determined explicitly when the sequence $\{a\}$ is iid, as $d_s^h = 2\lim_{n\to\infty} \log M_n / \log T_n$, giving

$$d_s^h = 2\frac{\sum_{a\in A} p_a \log m_a}{\sum_{a\in A} p_a \log m_a \rho_a}, \quad \mathbb{P}^h - a.s.$$

To find the exact oscillation in the heat kernel for this case we need the following law of the iterated logarithm, arising from our iid sequence $\{a_n\}$. If we let $\mathbb{I}_A$ denote the indicator of the set A, then there exist positive constants $c_{4.13}, c_{4.14}$ such that $\mathbb{P}^h$-almost surely,

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{I}_{\{a_i=a\}} - np_a}{\sqrt{n\log\log n}} = -c_{4.13}, \quad \limsup_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{I}_{\{a_i=a\}} - np_a}{\sqrt{n\log\log n}} = c_{4.14}. \tag{4.5}$$

By using this we can estimate the oscillation in the sequences T_n and M_n and using, Lemmas 4.1 and 4.3, derive the first part of Theorem 1.1. The second part of Theorem 1.1 also follows from Lemmas 4.1 and 4.3.

An application of (3.2) and the upper bound of Lemma 4.1 allow us to conclude that, as in [11] Lemma 8.4, there exists a constant $c_{4.15}$ such that

$$\sup_{x,y\in E_{\mathbf{i}}} |p_t^r(x,y) - p_t^r(x',y)| \leq c_{4.15} e^{-m} \sqrt{t^{-1} t^{-\alpha/(\alpha+1)} |\log t|^{\beta}},$$

for $\mathbf{i} \in \Lambda_m$. A similar estimate in the homogeneous case shows that for both random fractals the transition density is uniformly continuous and hence that the semigroup is compact.

5 The spectral counting function

As the semigroup is compact, the Laplacian will also be compact and hence it will have a discrete spectrum consisting of eigenvalues. We begin by defining the Dirichlet and Neumann eigenvalue problems for our random fractals. Recall that for each $I \in \Omega'$ there is a random fractal F^I and we have a measure μ^I satisfying (3.6). We will prove results about the counting function for all $I \in \Omega'$, giving almost sure statements on Ω.

Firstly the Dirichlet eigenvalues are defined to be the numbers λ, each with eigenfunction u, such that

$$\begin{aligned} \Delta^I u &= -\lambda u, \\ u(x) &= 0, \quad x \in \partial F. \end{aligned} \tag{5.1}$$

This eigenvalue problem can be reformulated for the Dirichlet form. Firstly we change the domain and define $\mathcal{F}_0^I = \{f \in \mathcal{F}^I : f(x) = 0,\ x \in \partial F\}$, setting $\mathcal{E}_0^I(f,f) = \mathcal{E}^I(f,f)$ for $f \in \mathcal{F}_0^I$. Then λ is a Dirichlet eigenvalue with eigenfunction u if

$$\mathcal{E}_0^I(u,v) = \lambda(u,v),$$

for all $v \in \mathcal{F}_0^I$.

As the operator is compact we can write the spectrum as an increasing sequence of eigenvalues given by $0 < \lambda_0 < \lambda_1 \leq \ldots$. We define the associated eigenvalue counting function to be

$$N_0^I(x) = \max\{i : \lambda_i \leq x, \lambda_i \text{ solves } (5.1)\}.$$

Analogously we can define the Neumann eigenvalues to be the numbers λ, each associated with an eigenfunction u, such that

$$\begin{aligned} \Delta^I u &= -\lambda u \\ (du)_x &= 0,\ x \in \partial F \end{aligned} \tag{5.2}$$

This eigenvalue problem can also be reformulated for the Dirichlet form. We say λ is a Neumann eigenvalue with eigenfunction u if

$$\mathcal{E}^I(u,v) = \lambda(u,v),$$

for all $v \in \mathcal{F}^I$.

Again, we write the spectrum as an increasing sequence of eigenvalues with $0 = \lambda_0 < \lambda_1 \leq \ldots$, and define the associated eigenvalue counting function to be

$$N^I(x) = \max\{i : \lambda_i \leq x, \lambda_i \text{ solves } (5.2)\}.$$

We can use the heat kernel estimates to get a preliminary result on the asymptotics of the spectral counting function. We can use the Mercer expansion theorem to write

$$p_t(x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} u_i(x) u_i(y),$$

where we assume the eigenfunctions u_i are orthonormal in $L^2(F,\mu)$. Thus we can observe that

$$\int_F p_t(x,x)\mu(dx) = \sum_{i=0}^{\infty} e^{-\lambda_i t} = \int_0^{\infty} e^{-\lambda t} N(d\lambda).$$

By using the heat kernel estimates to bound the term on the left we have bounds on the Laplace-Stieltjes transform of $N(\lambda)$ and then bounds on $N(\lambda)$ follow from Tauberian theorems.

For the homogeneous random fractal case the heat kernel technique gives the following result which is sharp result up to constants,

$$c_1 T_n \leq N^h(M_n) \leq c_2 T_n. \tag{5.3}$$

In order to obtain the result given in Theorem 1.2, we use the law of the iterated logarithm (4.5) again to estimate the oscillations in T_n and M_n.

For the random recursive case we have uniform bounds on the heat kernel which lead to

$$t^{-d_s/2}|\log t|^{\beta_1} \leq \int_F p_t^r(x,x)\mu(dx) \leq t^{-d_s/2}|\log t|^{\beta_2}.$$

Using the Tauberian theorem gives

$$\lambda^{d_s/2}(\log\lambda)^{\beta_1} \leq N^r(\lambda) \leq \lambda^{d_s/2}(\log\lambda)^{\beta_2}. \tag{5.4}$$

We will follow [12] to show that, by using a detailed description of the eigenvalues, we can improve upon (5.4) to show that there are typically no oscillations for the random recursive case. The techniques are an extension of the Dirichlet-Neumann bracketing idea developed by [18] for p.c.f. self-similar sets. We will establish a decimation property of the eigenfunctions which differs from the usual decimation property for fractals which expresses the eigenfunctions for the Laplacian in terms of other eigenfunctions for the Laplacian. Instead we can build an eigenfunction for a particular random Laplacian in terms of eigenfunctions for other random Laplacians. The key relationship is provided by the following Lemma.

Lemma 5.1 *For all $x > 0$ and each $I \in \Omega'$, we have*

$$\sum_{i=1}^{m(U_0)} N_0^{\theta_i(I)}(x\rho(U_0)^{-1}\mu(E_i)) \leq N_0^I(x) \leq N^I(x) \leq \sum_{i=1}^{m(U_0)} N^{\theta_i(I)}(x\rho(U_0)^{-1}\mu(E_i)) \tag{5.5}$$

and

$$N_0^I(x) \leq N^I(x) \leq N_0^I(x) + M. \tag{5.6}$$

In order to establish this key result we begin by defining some closely related Dirichlet forms. Let $(\tilde{\mathcal{E}}^I, \tilde{\mathcal{F}}^I)$ be defined by setting

$$\tilde{\mathcal{F}}^I = \{f : F\backslash F_1 \to \mathbb{R} | f \circ \psi_i = f_i \text{ on } F\backslash F_0, \text{ for some } f_i \in \mathcal{F}^{\theta_i(I)}\},$$

and

$$\tilde{\mathcal{E}}^I(f,g) = \sum_{i=1}^{m(U_0)} \rho(U_0)\mathcal{E}^{\theta_i(I)}(f\circ\psi_i, g\circ\psi_i).$$

As in [18] we can prove that

Proposition 5.2 *(1) $\mathcal{F}^I \subset \tilde{\mathcal{F}}^I$ and $\mathcal{E}^I = \tilde{\mathcal{E}}^I|_{\mathcal{F}\times\mathcal{F}}$.*
(2) $(\tilde{\mathcal{E}}^I, \tilde{\mathcal{F}}^I)$ is a local regular Dirichlet form on $L^2(F^I, \mu^I)$.
(3) $\tilde{\mathcal{F}}^I \hookrightarrow L^2(F^I, \mu^I)$ is a compact operator.
(4) If $\tilde{N}^I(x)$ denotes the eigenvalue counting function for the eigenvalues of $\tilde{\mathcal{E}}^I$, then

$$\tilde{N}^I(x) = \sum_{i=1}^{m(U_0)} N^{\theta_i(I)}(x\rho(U_0)^{-1}\mu(E_i)).$$

Proof: (1),(2),(3) follow in the same way as [18] Proposition 6.2. We demonstrate (4). Assume that we have an eigenfunction f of $\mathcal{E}^I$ with eigenvalue λ. By using the decomposition of the Dirichlet form we have

$$\sum_{i=1}^{m(U_0)} \rho(U_0)\mathcal{E}^{\theta_i(I)}(f\circ\psi_i, g\circ\psi_i) = \mathcal{E}^I(f,g) = \lambda(f,g) = \lambda\sum_{i=1}^{m(U_0)}(f\circ\psi_i, g\circ\psi_i)\mu(E_i).$$

Thus for all $h \in \mathcal{F}^{\theta_i(I)}$ we have

$$\mathcal{E}^{\theta_i(I)}(f\circ\psi_i, h) = \lambda\rho(U_0)^{-1}\mu(E_i)(f\circ\psi_i, h),$$

and we have that $\lambda\rho_1^{-1}\mu(E_i)$ is an eigenvalue of $\Delta^{\theta_i(I)}$ with eigenfunction $f_i = f\circ\psi_i$. Now setting

$$\tilde{f}(x) = \begin{cases} f_i(x), & x \in \text{int}(E_i), \\ 0, & x \notin \text{int}(E_i), \end{cases}$$

we have an eigenfunction with eigenvalue λ of $(\tilde{\mathcal{E}}^I, \tilde{\mathcal{F}}^I)$. Hence we have that

$$\tilde{N}^I(x) = \sum_{i=1}^{m(U_0)} N_0^{\theta_i(I)}(x\rho(U_0)^{-1}\mu(E_i)),$$

as required. □

There is a similar proof to the following proposition. Let $(\tilde{\mathcal{E}}_0^I, \tilde{\mathcal{F}}_0^I)$ be defined by setting

$$\tilde{\mathcal{F}}_0^I = \{f : f \in \mathcal{F}_0^I, f|_{F_1} = 0\},$$

and

$$\tilde{\mathcal{E}}_0^I(f,g) = \mathcal{E}^I|_{\mathcal{F}_0^I\times\mathcal{F}_0^I}.$$

Proposition 5.3 *(1)* $\tilde{\mathcal{F}}_0^I \subset \mathcal{F}_0^I$.
(2) $(\tilde{\mathcal{E}}_0^I, \tilde{\mathcal{F}}_0^I)$ *is a local regular Dirichlet form on* $L^2(F^I, \mu^I)$.
(3) If $\tilde{N}_0^I(x)$ *denotes the eigenvalue counting function for the eigenvalues of* $\tilde{\mathcal{E}}_0^I$, *then*

$$\tilde{N}_0^I(x) = \sum_{i=1}^{m(U_0)} N_0^{\theta_i(I)}(x\rho(U_0)^{-1}\mu(E_i)).$$

To conclude the proof of the key inequalities we require the Dirichlet-Neumann bracketing results given in [18]. We give here a version of [18] Corollary 4.7.

Lemma 5.4 *If* (E,F) *and* (E',F') *are two Dirichlet forms on* $L^2(F,\mu)$ *and* F' *is a closed subspace of* F *and* $E' = E|_{F'\times F'}$, *then*

$$N_{E'}(x) \le N_E(x) \le N_{E'}(x) + Dim(F/F').$$

Proof of Lemma 5.1: Using the left inequality of Lemma 5.4 twice with the two propositions gives (5.5). As the space of harmonic functions for finitely ramified fractals is finite dimensional Lemma 5.4 gives $\mathrm{Dim}(\mathcal{F}/\tilde{\mathcal{F}}) = |F_0| = d+1$ and hence we have (5.6). □

In order to analyse the eigenvalue counting function we extend the general branching process, by using characteristics defined for all $t \in \mathbb{R}$. We begin by writing the left inequality in (5.5) in the same way as the equation for a general branching process. By definition $\mu(E_i) = \rho(U_0)^{-\alpha} W_i/W$ and we can write (5.5) as

$$\sum_{i=1}^{m(U_0)} N_0^{\theta_i(I)}(x\rho(U_0)^{-1-\alpha} W_i/W) \leq N_0^I(x).$$

We will make the substitution $X_0^{I'}(t) = N_0^{I'}(e^t W(I'))$ for all $I' \in \Omega$, and consider

$$\sum_{i=1}^{m(U_0)} X_0(t - \log \tau_1) \leq X_0(t),$$

where we write $\tau_1 = \rho(U_0)^{1+\alpha}$ and suppress the I dependence.

This suggests that we extend the class of general branching processes to $\{X^\phi(t) : -\infty < t < \infty\}$, defined by

$$X^\phi(t) = \sum_{\mathrm{i} \in I^r} \phi(t - \sigma_\mathrm{i}),$$

where the class of characteristics $\{\phi^I(t) : -\infty < t < \infty\}$, is defined for all time. For our purposes it is enough that the $\phi(t)$, which can be random, are bounded and $\phi^I(t) = 0$ for $t < t_0(I)$, some $t_0(I) \in \mathbb{R}$. As the process is defined by summing over the entire tree I^r we see that the existence of the process requires that the sum is finite for all $t \in \mathbb{R}$. This is clear for the case of X_0 by its construction. It is also easy to see that the process satisfies the usual evolution equation for a general branching process

$$X^\phi(t) = \phi(t) + \sum_{i=1}^{\xi_1} X_i^\phi(t - \sigma_i), \quad \forall t \in \mathbb{R},$$

where the X_i^ϕ are iid copies of X^ϕ. The process $\{X^\phi(t) : t \in \mathbb{R}\}$ considered here is obtained by adding together a number of time shifted copies of itself. The time shifts are the birth times of individuals in the general branching process z_t which starts from a single individual at time 0 and has a lifelength and reproduction point process given by

$$(\xi_x(ds), L_x) = (m_a \delta_{(1+\alpha)\log \rho_a}, (1+\alpha)\log \rho_a) \text{ with probability } p_a.$$

Now define the function η by

$$\eta(t) = X_0(t) - \sum_{i=1}^{\xi_1} X_0(t - \log \tau_1),$$

Clearly we have for all $t \in \mathbb{R}$,

$$X_0(t) = \eta(t) + \sum_{i=1}^{\xi_1} X_0(t - \log \tau_1), \tag{5.7}$$

As for the general branching process we will write $v_\gamma^\phi(t) = \mathbb{E}e^{-\gamma t}X^\phi(t)$. The growth of the process X_0 is described by the following.

Lemma 5.5 *The process $\{X_0(t) : t \in \mathbb{R}\}$ has Malthusian parameter $\gamma = \alpha/(\alpha+1)$ where α satisfies the equation*

$$\mathbb{E}(m(U_0)\rho(U_0)^{-\alpha}) = 1.$$

If the mean lifelength distribution ν is non-lattice, then

$$\lim_{t\to\infty} X_0(t)e^{-\gamma t} = v_\gamma^\eta(\infty)W, \quad a.s.,$$

where

$$v_\gamma^\eta(\infty) = \frac{\int_{-\infty}^{\infty} e^{-\gamma t}\mathbb{E}\eta(t)dt}{\int_0^\infty te^{-\gamma t}\nu(dt)}.$$

If the support of the measure ν lies in a discrete subgroup of $\mathbb{R}$, then, if T is the generator of the support, then

$$X_0(t) = (G(t) + o(1))\, e^{\gamma t}W,$$

where G is a positive periodic function with period T given by

$$G(t) = \frac{\sum_{j=-\infty}^{\infty} e^{-\gamma(t+jT)}\mathbb{E}\eta(t+jT)}{\int_0^\infty te^{-\gamma t}\nu(dt)}.$$

Proof: We sketch the main ideas and refer to [12] for more details. The Malthusian parameter is γ the rate of growth for this process. We derive a renewal equation for the mean by multiplying (5.7) by $e^{-\gamma t}$, taking expectations and letting $v(t) = \mathbb{E}(e^{-\gamma t}X_0(t))$. Also let $w(t) = \mathbb{E}(e^{-\gamma t}\eta(t))$, and note that $w(t) \to 0$ as $t \to \pm\infty$, then

$$v(t) = w(t) + \int_0^\infty e^{-\gamma s}v(t-s)\nu(ds) = w(t) + \int_0^\infty v(t-s)\nu_\gamma(ds). \tag{5.8}$$

In order for this to be a renewal equation the Malthusian parameter γ must be the solution to the equation

$$\mathbb{E}\int_0^\infty e^{-\gamma t}\xi(dt) = 1.$$

Thus

$$1 = \sum_{a \in A} m_a \rho_a^{-\gamma(1+\alpha)} p_a.$$

By the definition of α in (3.5) we see that $\alpha = \gamma(1+\alpha)$, giving $\gamma = \alpha/(\alpha+1)$.

Equation (5.8) is the renewal equation of [18] and hence we can apply the renewal theorem given in [18]. If ν is not lattice, then

$$v(\infty) = \frac{\int_{-\infty}^{\infty} w(x)dx}{\int_0^{\infty} x\nu(dx)}.$$

Otherwise, if the support of ν lies in some discrete subgroup of $\mathbb{R}$, then if T is the greatest common divisor of the support of ν, then $G(t) = \lim_{n\to\infty} m(t+nT)$ exists for every t and

$$G(t) = \frac{\sum_{j=-\infty}^{\infty} w(t+jT)}{\int_0^{\infty} x\nu(dx)}.$$

To complete the proof we will indicate how to establish the almost sure convergence in the non-lattice case. By Corollary 5.8 of [12], we just need convergence down a particular subsequence of times $t_k = t_0 + ck$, where $k \in \mathbb{Z}, c \in \mathbb{R}_+, t_0 \in [0,c]$. It is also enough to consider the truncated characteristic η^c defined by

$$\eta^c(t) = \begin{cases} \eta(t), & t < n_0 c, \\ 0, & t \geq n_0 c. \end{cases}$$

We will write $\mathcal{I}_t = \{\mathbf{i} = (\mathbf{j}, i) : \sigma_\mathbf{j} < t, \sigma_\mathbf{i} > t\}$ and $\mathcal{I}_{t,c} = \{\mathbf{i} = (\mathbf{j}, i) : \sigma_\mathbf{j} < t, \sigma_\mathbf{i} > t + c\}$. Now for $n \geq n_0$, we have, writing X^c for X^{η^c} and $a_\mathbf{i}(t) = e^{-\gamma(t-\sigma_\mathbf{i})} X_0^c(t - \sigma_\mathbf{i}) - v_\gamma^c(t - \sigma_\mathbf{i})$, that

$$\begin{aligned} |e^{-\gamma t_{k+n}} X_0^c(t_{k+n}) - v_\gamma^c(\infty) W| &\leq \left| \sum_{\mathbf{i} \in \mathcal{I}_{t_k} \setminus \mathcal{I}_{t_k,nc}} e^{-\gamma\sigma_\mathbf{i}} a_\mathbf{i}(t_{k+n}) \right| \\ &\quad + \left| \left(\sum_{\mathbf{i} \in \mathcal{I}_{t_k} \setminus \mathcal{I}_{t_k,nc}} e^{-\gamma\sigma_\mathbf{i}} v_\gamma^c(t_{k+n} - \sigma_\mathbf{i}) \right) - v_\gamma^c(\infty) W \right| \\ &= S_1(t_k) + S_2(t_k). \end{aligned}$$

For the first term we note that $a_\mathbf{i}$ are mean 0 random variables and we can apply the version of the strong law of large numbers proved as Lemma 4.1 in [23]. For this we use boundedness of η, finiteness of the total population at fixed times and exponential growth of $|\mathcal{I}_{t_k} \setminus \mathcal{I}_{t_k,nc}|$. Using [23] Proposition 4.3 we have $S_1(t_k) \to 0$, $a.s.$ as $k \to \infty$.

The second term $S_2(t_k)$ depends purely on the general branching process and by [23] (5.53) we can prove that there is an $n \geq n_0$ such that $S_2(t_k) \to 0$ almost surely as $k \to \infty$. Removing the truncation using a dominated convergence argument gives the required result. □

We can now state and prove our main theorem. Let

$$\eta_0^I(t) = N_0^I(e^t) - \sum_{i=1}^{m(U_0)} N^{\theta_i(I)}(e^t \rho(U_0)^{-1}\mu(E_i)).$$

By replacing X by N in the convergence result of Lemma 5.5 and rearranging, we obtain our final Theorem.

Theorem 5.6 *For the random recursive Sierpinski gasket if the mean lifelength distribution ν is non-lattice, then*

$$\lim_{x\to\infty} N_0^I(x)x^{-d_s^r/2} = v^{\eta_0}_{d_s^r/2}(\infty)W^{1/(\alpha+1)}(I), \quad a.e.\ I \in \Omega,$$

where $d_s^r = 2\alpha/(\alpha+1)$ and

$$v^{\eta_0}_{d_s^r/2}(\infty) = \frac{\int_{-\infty}^{\infty} e^{-td_s^r/2} E\eta_0(t)dt}{\int_0^{\infty} te^{-td_s^r/2}\nu(dt)}.$$

If the support of the measure ν lies in a discrete subgroup of $\mathbb{R}$, then if T is the generator of the support, then for a.e. $I \in \Omega$,

$$N_0^I(x) = (G(\log(x/W)) + o(1))\, x^{d_s^r/2} W^{1/(\alpha+1)},$$

where G is a positive periodic function with period T given by

$$G(t) = \frac{\sum_{j=-\infty}^{\infty} e^{-d_s^r(t+jT)} E\eta_0(t+jT)}{\int_0^{\infty} te^{-td_s^r/2}\nu(dt)}.$$

By (5.6) we know that the spectral asymptotics for both the Dirichlet and Neumann Laplacians will be the same and hence we have

Corollary 5.7 *For the random recursive Sierpinski gasket, if the mean lifelength distribution is non-lattice, then*

$$\lim_{x\to\infty} N^I(x)x^{-d_s^r/2} = v^{\eta_0}_{d_s^r/2}(\infty)W^{1-d_s^r/2}(I), \quad a.e.\ I \in \Omega.$$

As $\rho = 5/3$ for SG(2) and $\rho = 15/7$ for SG(3) the limit of the spectral counting function for the random recusive fractal of the introduction exists and we have the second part of Theorem 1.2.

Corollary 5.8 *For the random recursive Sierpinski gasket constructed from SG(2) and SG(3) we have for almost every $I \in \Omega^r$, that*

$$\lim_{x\to\infty} N^I(x)x^{-d_s^r/2} = \frac{\int_{-\infty}^{\infty} e^{-td_s^r/2} E\eta_0(t)dt}{p3\log(5/3)(5/3)^{-d_s^r/2} + (1-p)6\log(15/7)(15/7)^{-d_s^r/2}} W^{1-d_s^r/2}(I).$$

Remark 5.9 (1) We remark that as yet there are no non-trivial examples in which there is a periodic function G.
(2) Using the relationship between the partition function $\int_F p_t(x,x)\mu(dx)$ and the spectral counting function we have the existence of a constant C_1 such that

$$\lim_{t\to 0}\int_F p_t(x,x)t^{d_s^r/2}\mu(dx) = C_1 W^{1-d_s^r/2}, \ \ \mathbb{P}-a.s.$$

For the homogeneous case a similar analysis to the above leads to the fundamental inequality

$$M_n N_0^{\theta_n(I^h)}(xT_n^{-1}) \le N_0^{I^h}(x) \le N^{I^h}(x) \le M_n N^{\theta_n(I^h)}(xT_n^{-1}).$$

This is the same as that derived from the heat kernel in (5.3), and gives the first result in Theorem 1.2.

References

[1] M.T. Barlow. Diffusions on fractals *St Flour Lecture Notes 1995*, 1998.

[2] M.T. Barlow and B.M. Hambly. Transition density estimates for Brownian motion on scale irregular Sierpinski gaskets. *Ann. Inst. H. Poincar/'e*, 33, 531-556, 1997.

[3] M.T. Barlow and J. Kigami. Localized eigenfunctions of the Laplacian on p.c.f. self similar sets *J. London Math. Soc.*, 56, 320–332, 1997.

[4] M.T. Barlow and E.A. Perkins. Brownian motion on the Sierpinski gasket. *Probab. Theory Related Fields*, 79, 543–624, 1988.

[5] K.J. Falconer, Random graphs. *Math. Proc. Cambridge Philos. Soc.*, 100, 559–582, 1986.

[6] P.J. Fitzsimmons, B.M. Hambly and T. Kumagai. Transition density estimates for diffusion on affine nested fractals. *Comm. Math. Phys.*, 165, 595–620, 1994.

[7] M. Fukushima and T. Shima. On a spectral analysis for the Sierpinski gasket. *Potential Analysis*, 1, 1–35, 1992.

[8] S. Graf. Statistically self-similar fractals. *Probab. Theory Related Fields*, 74, 357–392, 1987.

[9] S. Graf, R.D. Mauldin, and S.C. Williams. The exact Hausdorff dimension in random recursive constructions. *Memoirs Am. Maths. Soc.*, 381, 1988.

[10] B.M. Hambly. Brownian motion on a homogeneous random fractal. *Probab. Theory Related Fields*, 94, 1–38, 1992.

[11] B.M. Hambly. Brownian motion on a random recursive Sierpinski gasket *Ann. Probab.*, 25, 1059-1102, 1997.

[12] B.M. Hambly. On the asymptotics of the eigenvalue counting function for random recursive Sierpinski gaskets *Preprint*, 1998.

[13] B.M. Hambly and T. Kumagai. Transition density estimates for diffusion processes on post critically finite self-similar fractals. *Proc. London Math. Soc.*, 78, 431–458, 1999.

[14] B.M. Hambly, T. Kumagai, S. Kusuoka and X.Y. Zhou. Transition density estimates for diffusion processes on homogeneous random Sierpinski carpets. *Preprint*, 1998.

[15] J. Kigami. Harmonic calculus on p.c.f. self-similar sets. *Trans. Amer. Math. Soc.*, 335, 721–755, 1993.

[16] J. Kigami. Harmonic calculus on limits of networks and its application to dendrites. *J. Funct. Anal.*, 128, 48–86, 1995.

[17] J. Kigami. Distributions of localized eigenvalues of Laplacians on post critically finite self-similar sets *J. Funct. Anal.*, 156, 170–198, 1998.

[18] J. Kigami and M. Lapidus. Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals. *Comm. Math. Phys.*, 158, 92-125, 1993.

[19] T. Kumagai. Estimates of transition densities for Brownian motion on nested fractals *Prob. Theory Related Fields*, 96, 205–224, 1993.

[20] M. Lapidus. Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media and the Weyl-Berry conjecture. *In: Ordinary and partial differential equations, vol IV, proc. 12th Int. Conf. on theory of ordinary and partial differential equations, Dundee, 1992, Research Notes in Maths: Longman, London*, 126–209, 1993.

[21] T. Lindstrøm. Brownian motion on nested fractals. *Mem. Amer. Math. Soc.*, 420, 1990.

[22] R.D. Mauldin and S.C. Williams. Random recursive constructions: asymptotic geometric and topological properties. *Trans. Am. Maths. Soc.*, 295, 325–346, 1986.

[23] O. Nerman. On the convergence of supercritical general (C-M-J) branching processes *Zeit. Wahr.*, 57, 365-395, 1981.

[24] C. Sabot. Existence and uniqueness of diffusions on finitely ramified self-similar fractals. *Ann. Scient. Ecole Norm. Sup*, 30, 605–673, 1997.

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