Traditional Signal Processing Seen From the Morphological Poset-Theoretical Point of View

Renato Keshet (Kresch)
HP Laboratories Israel*
HPL-1999-39(R.1)
March, 2000

E-mail: renato@hpli.hpl.hp.com

Mathematical Morphology, partially ordered sets, complete semilattices, signal processing, image processing

In this paper, we prove that several key signal processing tasks (linear filtering, quantization, decimation, JPEG coding, and others) are particular cases of morphological operators. This is obtained as a consequence of the recent extension of MM's framework, from complete lattices to complete inf-semilattices, and a further extension to partially ordered sets (posets) presented in this paper.

The above results motivated us to pose the question: "Could MM serve as a general framework for signal processing?" Seeking an answer to the above question, we revisited signal processing foundations, looking at them from the point of view of MM fundaments. In this paper, we also present this brief study, which points towards a positive answer to the question. Further work is still required, however, in order to answer a second question: "Can a MM framework for signal processing provide new algorithms and applications, or improve existing ones?"

* HP Labs-Israel, Technion City, Haifa 32000, Israel

1 Introduction

A comparison between linear processing (LP; see for e.g., [1, 2]) and mathematical morphology (MM) [3] shows two theories that lie on different mathematical fundaments. LP is based on vector spaces, and uses addition as fundamental operation, whereas MM is based on complete lattices and uses infimum and supremum as basic operations. LP is closely related to Fourier analysis, whereas MM has a strong geometric nature. Finally, they differ also on their processing results; LP deals mainly with smooth spatial variations, whereas MM usually deals with abrupt edges.

On the other hand, algebraic similarities between the two theories have been pointed out in the past years. Marago's work [4] finds analogies to many linear processing concepts in MM, including translation invariant systems, impulse response, and transforms based on eigen-functions (Fourier transform, in the linear theory, and slope transform in the morphological one). In [5, chapter 9], structural similarities between the (linear) Laplacian pyramid and the morphological skeleton were presented. Also, correspondence between the linear and morphological approaches to inverse problems was drawn. Finally, Heijmans and Goutsias [6] have recently extended the concept of pyramids and Wavelets, and proposed new morphological versions of these structures.

Despite all the structural and algebraic similarities between LP and MM, the two theories are widely considered unconnected and independent. I.e., none "contains" the other.

Correlations between MM and other signal processing theories and techniques have also been reported. Dilations and erosions are particular cases of order-statistics filters [8], a category to which the well-known median filter also belongs. In [9, 10], binary dilations and erosions are shown to be particular cases of curve evolution. In [5, chapter 5], quadtree and bit-plane decomposions are shown to be particular cases of morphological skeletons on complete semilattices. However, similarly to LP, no widely accepted unification of all the above concepts currently exists.

In summary, MM is usually considered as yet another signal processing theory, besides LP, order-statistics filtering, curve evolution, etc.

The results presented in this work, however, encourages us to contradict the above perception. Following the extension of MM's theory to complete inf-semilattices (in [7]), and further extension to *partial ordered sets* (posets) presented in this paper, we demonstrate that many key tasks in traditional signal processing are morphological operations on specific posets. For example, linear translation-invariant filtering is shown here to be an erosion on a complete inf-semilattice. Ideal (idempotent) linear filters are morphological openings on the same inf-semilattice. Quantization, decimation, and coding are erosions on specific posets, whereas dequantization, interpolation, and decoding are their adjoint dilations, respectively. Also, the structural similarity between Laplacian pyramids and morphological skeletons mentioned above is shown to be of a deeper nature – the Laplacian pyramid is indeed a skeleton on the appropriate poset. Other examples are considered throughout the paper.

In our opinion, the above observations are more than just curious remarks. We believe that the very nature of MM constitutes a suitable framework for signal processing in general, as explained through the following arguments (and detailed in Section 4). Typically, the aim of a signal processing task is to address (e.g., to produce, reconstruct from, etc.) "simpler versions" of a given signal. The concept of simplicity varies with the application and approach, and is associated to a *partial ordering* on the set of signals. This partial ordering is usually closely related to some signal measure, like energy, entropy, noise, error, variance, information, smoothness, compressibility, etc. In the resulting signal poset, ideal filters become algebraic openings, and operator/pseudo-inverse pairs become morphological adjunctions, i.e. erosion/dilation pairs.

In several situations, one can find a signal that is the "simplest" one, according to some given measure. E.g., the signal with least energy is the null function, which is also the signal with the greatest compressibility. However, there does not seem to exist one single signal that corresponds to the greatest energy or the lowest compressibility. In this case, the corresponding partial ordering usually induces a complete semilattice structure to the signal poset (where either a least or greatest element exists, but not both).

In this report, we address the nature of common signal processing tasks, seen from the theoretical point of view of MM on posets, and on complete semilattices, in particular.

We first extend the basic concepts of MM to posets (Section 2), and show some examples of traditional LP tasks as morphological operations (Section 3). Then, we address signal processing in general (Section 4), and some of its basic traditional tasks (Section 5), from the point of view of MM.

2 Morphology on Posets

In this section, we review basic notions of mathematical morphology, with two main extensions: *i)* The fundamental framework is now posets, instead of complete lattices, and *ii)* the domain and range of the operators may now be distinct. The proofs to some of the propositions in this section are straightforward extensions of their well-known lattice counterparts (see e.g., [3, 11]), and are given in this article only for the completeness of the theory. The proofs for all the propositions of these section are given in the appendix.

2.1 Posets, Semilattices, and Lattices

A *partially ordered set* (poset) $\mathcal{A}$ is a set associated to a binary operator $\leq$, satisfying the following properties for any $X, Y, Z \in \mathcal{A}$: reflexivity ($X \leq X$), anti-symmetry ($X \leq Y, Y \leq$

$X \Rightarrow X = Y$), and transitivity ($X \leq Y, Y \leq Z \Rightarrow X \leq Z$).

In a poset $\mathcal{A}$, the greatest minorant $\wedge\mathcal{X}$ (also called *infimum*) of a subset $\mathcal{X} \subseteq \mathcal{A}$ is defined as an element $X_0 \in \mathcal{A}$, such that: *i)* $X \geq X_0, \forall X \in \mathcal{X}$, and *ii)* if $Y \in \mathcal{A}$ satisfies $X \geq Y$ for all $X \in \mathcal{X}$, then $Y \leq X_0$. One defines the least majorant $\vee\mathcal{X}$ (also called *supremum*) of $\mathcal{X}$, dually.

A poset $\mathcal{S}$ is an *inf semilattice* (resp. *sup semilattice*) if every two-element subset $\{X_1, X_2\}$ in $\mathcal{S}$ has an infimum $X_1 \wedge X_2$ (resp., a supremum $X_1 \vee X_2$) in $\mathcal{S}$. An inf semilattice (resp., sup semilattice) is *complete* when *every* non-empty subset $\mathcal{B} \subset \mathcal{S}$ has an infimum $\wedge\mathcal{B}$ (resp. supremum $\vee\mathcal{B}$). In this case, there exists in the semilattice an unique element 0, called zero element, (resp., U, called universe), such that, for any $X \in \mathcal{S}$, $0 \wedge X = 0$ (resp. $U \vee X = U$).

A complete lattice is a lattice which is both a complete inf and a complete sup semilattice.

2.1.1 Increasing Operators

Increasing operators preserve the partial ordering, by definition.

Definition 1 *An operator $\mathcal{T} : \mathcal{A} \mapsto \mathcal{B}$ is increasing iff:*

$$Y_1 \overset{\mathcal{A}}{\leq} Y_2 \Rightarrow \mathcal{T}(Y_1) \overset{\mathcal{B}}{\leq} \mathcal{T}(Y_2), \tag{1}$$

for all $Y_1, Y_2 \in \mathcal{A}$.

2.2 Adjunctions

The concept of *adjunction* is defined for complete lattices by Heijmans and Ronse in [11, page 264] (see, alternatively, the concept of *morphological duality* in [12, Chapter 1]). We consider here the following extended version of that definition.

Definition 2 *Consider two posets $\mathcal{A}$ and $\mathcal{B}$, with corresponding partial orders $\overset{\mathcal{A}}{\leq}$ and $\overset{\mathcal{B}}{\leq}$, and two operators $\rho : \mathcal{A} \mapsto \mathcal{B}$ and $\sigma : \mathcal{B} \mapsto \mathcal{A}$. The pair (ρ, σ) is an adjunction between $\left(\mathcal{A}, \overset{\mathcal{A}}{\leq}\right)$ and $\left(\mathcal{B}, \overset{\mathcal{B}}{\leq}\right)$ iff, for all $X \in \mathcal{B}, Y \in \mathcal{A}$:*

$$X \overset{\mathcal{B}}{\leq} \rho(Y) \Leftrightarrow \sigma(X) \overset{\mathcal{A}}{\leq} Y. \tag{2}$$

The operator ρ is then called the upper adjoint of σ, and σ is the lower adjoint of ρ. If $\mathcal{A} = \mathcal{B}$ and $\overset{\mathcal{A}}{\leq} \equiv \overset{\mathcal{B}}{\leq}$, then we say that (ρ, σ) is an adjunction in $\left(\mathcal{A}, \overset{\mathcal{A}}{\leq}\right)$.

The concept of *adjoint* can be viewed as an extension of that of *inverse*. This is because of the similarity between equation (2) and the equation $X = \mathcal{T}(Y) \Leftrightarrow \mathcal{T}^{-1}(X) = Y$, which characterizes the inverse $\mathcal{T}^{-1}$ of a given operator $\mathcal{T}$. Notice, however, a pair of operator/inverse is an adjunction iff they are increasing.

Proposition 1 *If (ρ, σ) is an adjunction, then:*

1. $\sigma\rho(Y) \overset{\mathcal{A}}{\leq} Y$, $\forall Y \in \mathcal{A}$, *and* $X \overset{\mathcal{B}}{\leq} \rho\sigma(X)$, $\forall X \in \mathcal{B}$.
2. $\rho\sigma\rho = \rho$ *and* $\sigma\rho\sigma = \sigma$.
3. σ *and* ρ *are increasing.*
4. $\sigma(X) = \overset{\mathcal{A}}{\wedge} \{Y \in \mathcal{A} \mid\ X \overset{\mathcal{B}}{\leq} \rho(Y)\}$, $\forall X \in \mathcal{B}$.
5. $\rho(Y) = \overset{\mathcal{B}}{\vee} \{X \in \mathcal{B} \mid\ Y \overset{\mathcal{A}}{\geq} \sigma(X)\}$, $\forall Y \in \mathcal{A}$.

2.2.1 Erosions and Dilations

In traditional morphology on complete lattices, erosion is defined as any operator that distributes over the infimum, and preserves the universe. Similarly, a dilation is any operator that distributes over the supremum, and preserves the zero element. We extend its definition to posets in a similar manner, except that no element preservation is required. The details of the extension are presented in the Appendix A, section 8.1.

As on posets, there is a one-to-one relationship between adjunctions and pairs of dilations/erosions. I.e., if (ρ, σ) is an adjunction, then ρ is an erosion, and σ is a dilation. Conversely, for every erosion ρ (resp. dilation σ), there is a dilation σ (resp. erosion ρ), for which (ρ, σ) is an adjunction. Pairs of erosion/dilation forming an adjunction are called adjoint.

2.3 Openings

Definition 3 (Algebraic Opening): *An operator γ in a poset $(\mathcal{A}, \leq)$ is an* algebraic opening *iff it is idempotent ($\gamma\gamma = \gamma$), increasing ($X \leq Y \Rightarrow \gamma(X) \leq \gamma(Y)$), and anti-extensive ($\gamma(X) \leq X$).*

The above definition is a direct extension of the concept of algebraic openings on lattices. As discussed in the sequel (Section 4.1.2), openings are considered ideal filters.

The extension of morphological openings is as follows:

Definition 4 (Morphological Opening): *The morphological opening γ_ε associated to an erosion (upper adjoint) between posets $\varepsilon : \mathcal{A} \mapsto \mathcal{B}$ is defined, for any $X \in \mathcal{A}$, by:*

$$\gamma_\varepsilon(X) \triangleq \bigwedge^{\mathcal{A}} \left\{ Y \in \mathcal{A} \mid \; \varepsilon(X) \overset{\mathcal{B}}{\leq} \varepsilon(Y) \right\}. \tag{3}$$

Remark: Equation (3) is a well-known iff-property of morphological openings on complete lattices. The lattice property is therefore used here as a definition for the poset counterpart.

Proposition 2 *Given any erosion between posets $\left(\mathcal{A}, \overset{\mathcal{A}}{\leq}\right)$ and $\left(\mathcal{B}, \overset{\mathcal{B}}{\leq}\right)$, the associated morphological opening $\gamma_\varepsilon(X)$ of any element $X \in \mathcal{A}$ exists in $\mathcal{A}$ and is unique. Moreover, γ_ε is an algebraic opening, and satisfies:*

1. $\varepsilon \gamma_\varepsilon = \varepsilon$,
2. $\gamma_\varepsilon = \delta_\varepsilon \varepsilon$.

3 Some Traditional Approaches Viewed as Morphological Operations

In this section, we demonstrate that some key traditional approaches for image processing can be seen as particular cases of morphological operations. The methodology is as follows: To each example of traditional image processing operation, we associate a partial ordering that provides the set of signals with a complete inf-semilattice structure. Then we show that the operation in consideration is an erosion in that complete inf-semilattice. We also consider the corresponding adjoint dilation, and sometimes the associated morphological opening. In the case of down-scaling, we show that the corresponding skeleton decomposition is the well-known Laplacian pyramid.

3.1 Linear Filtering

Let us define the partial ordering $\dot{\leq}$ on the set $\mathcal{P}_f^D$ of continuous D-dimensional signals as follows. For all $f, g \in \mathcal{P}^D$:

$$\begin{gathered} f \;\dot{\leq}\; g \Leftrightarrow \forall \Omega \in I\!R^D, \\ \begin{cases} |F(\Omega)| \leq |G(\Omega)|, \text{ and} \\ \text{phase}\left\{F(\Omega)\right\} = \text{phase}\left\{G(\Omega)\right\}, \end{cases} \end{gathered} \tag{4}$$

where $F(\Omega)$ and $G(\Omega)$ are the D-dimensional Fourier transforms of $f(x)$ and $g(x)$, respectively.

The poset $(\mathcal{P}^D, \dot{\leq})$ is actually a complete inf semilattice, where the least element is the null function. The infimum operation is given by the following relation:

$$(f \mathbin{\dot{\wedge}} g)(x) = \mathcal{F}^{-1}\{W\}(x), \tag{5}$$

where $\mathcal{F}^{-1}$ denotes inverse D-dimensional Fourier transform, and $W(\Omega)$ is defined by:

$$W(\Omega) \triangleq \begin{cases} \min\{|F(\Omega)|, |G(\Omega)|\} \exp\left(j \cdot \text{phase}\{F(\Omega)\}\right), \\ \quad \text{if phase}\{F(\Omega)\} = \text{phase}\{G(\Omega)\}, \\ 0, \quad \text{otherwise.} \end{cases} \tag{6}$$

Proposition 3 *Linear filtering (convolution by a given D-dimensional kernel) is an erosion in $(\mathcal{P}^D, \dot{\leq})$.*

Proof *Consider the linear filtering $\mathcal{L}_h(f)$ of a signal $f(x)$ by a given kernel $h(x)$, given by the convolution:*

$$\mathcal{L}_h(f) \triangleq f * h. \tag{7}$$

We have to prove that $\mathcal{L}_h\left(f \mathbin{\dot{\wedge}} g\right) = \mathcal{L}_h(f) \mathbin{\dot{\wedge}} \mathcal{L}_h(g)$, for any pair of signals f and g in $\mathcal{P}_f^D$.

It is well known that linear filtering can be performed in the Fourier domain as follows:

$$\mathcal{L}_h(f) = \mathcal{F}^{-1}\left\{F(\Omega)H(\Omega)\right\}. \tag{8}$$

Therefore:

$$\begin{aligned} \mathcal{F}\left\{\mathcal{L}_h\left(f \mathbin{\dot{\wedge}} g\right)\right\} &= W(\Omega)H(\Omega) \\ &= \begin{cases} \min\{|F(\Omega)|, |G(\Omega)|\} \exp\left(j \cdot \textit{phase}\{F(\Omega)\}\right) H(\Omega), \\ \quad \textit{if phase}\{F(\Omega)\} = \textit{phase}\{G(\Omega)\}, \\ 0, \quad \textit{otherwise.} \end{cases} \\ &= \begin{cases} \min\{|F(\Omega)H(\Omega)|, |G(\Omega)H(\Omega)|\} \exp\left(j \cdot \textit{phase}\{F(\Omega)H(\Omega)\}\right), \\ \quad \textit{if phase}\{F(\Omega)H(\Omega)\} = \textit{phase}\{G(\Omega)H(\Omega)\}, \\ 0, \quad \textit{otherwise.} \end{cases} \\ &= \mathcal{F}\left\{\mathcal{L}_h(f) \mathbin{\dot{\wedge}} \mathcal{L}_h(g)\right\}, \end{aligned} \tag{9}$$

which proves the proposition. □

Proposition 4 *The adjoint dilation of a linear filtering operation is its pseudo-inverse (also called generalized-inverse) filtering operation.*

Proof *The pseudo-inverse* $\mathcal{L}_h^*$ *of a linear filtering operator* $\mathcal{L}_h$ *is given by:* $\mathcal{L}_h^*(f) \triangleq \mathcal{F}^{-1}\{F(\Omega)H^*(\Omega)\}$*, where*

$$H^*(\Omega) \triangleq \begin{cases} 1/H(\Omega), & |H(\Omega)| > 0, \\ 0, & \textit{otherwise.} \end{cases} \tag{10}$$

We have to show that $\mathcal{L}_h^*$ *is the adjoint dilation of* $\mathcal{L}_h$*. One can verify that* $\mathcal{L}_h^*$ *is increasing in* $\left(\mathcal{P}^D, \dot{\leq}\right)$*.*

According to Proposition 19, the adjoint dilation δ *of* $\mathcal{L}_h$ *satisfies:* $\delta(f) = \dot{\bigwedge} \mathcal{A}_f^{\mathcal{L}_h}$, $\forall f \in \mathcal{P}^D$*, where*

$$\mathcal{A}_f^{\mathcal{L}_h} \triangleq \left\{ g \in \mathcal{P}^D \mid \ f \dot{\leq} \mathcal{L}_h(g) \right\}. \tag{11}$$

Notice that $\mathcal{L}_h^*(f)$ *belongs to* $\mathcal{A}_f^{\mathcal{L}_h}$*, since* $\mathcal{L}_h \mathcal{L}_h^*(f) = f$*.*

And $\mathcal{L}_h^*(f)$ *is the infimum of* $\mathcal{A}_f^{\mathcal{L}_h}$*, because for every* $g \in \mathcal{A}_f^{\mathcal{L}_h}$*:* $f \dot{\leq} \mathcal{L}_h(g) \Rightarrow \mathcal{L}_h^*(f) \dot{\leq} \mathcal{L}_h^* \mathcal{L}_h(g) \dot{\leq} g$*.* □

Corollary 1 *The morphological opening associated to a given linear filtering operation in* $(\mathcal{P}^D, \dot{\leq})$ *is an ideal (idempotent) linear filtering operation.*

3.2 Quantization and JPEG Coding

Consider the complete inf-semilattice $\left(\mathbb{R}, \overset{\mathrm{T}}{\leq}\right)$ of real numbers with the partial ordering:

$$x \overset{\mathrm{T}}{\leq} y \Leftrightarrow 0 \leq x \leq y \text{ or } y \leq x \leq 0, \tag{12}$$

where $\leq$ in the right-hand side denotes the standard scalar ordering. The infimum operation $\overset{\mathrm{T}}{\wedge}$ associated to $\overset{\mathrm{T}}{\leq}$ is given by: $x \overset{\mathrm{T}}{\wedge} y = \text{median}\{x, y, 0\}$.

Now, consider the quantization by truncation with step $q > 0$, $Q_q^{(T)}$, defined by:

$$Q_q^{(T)}(x) = \begin{cases} \lfloor x/q \rfloor, & \text{if } x \geq 0, \\ \lceil x/q \rceil, & \text{if } x < 0. \end{cases} \tag{13}$$

Proposition 5 *For any* $q > 0$, $Q_q^{(T)}$ *is an erosion in* $\left(\mathbb{R}, \overset{\mathrm{T}}{\leq}\right)$*. The adjoint dilation of* $Q_q^{(T)}$ *is the dequantization* $D_q^{(T)}$ *given by:*

$$D_q^{(T)}(x) \triangleq \begin{cases} q\lceil x \rceil, & \textit{if } x \geq 0, \\ q\lfloor x \rfloor, & \textit{if } x < 0. \end{cases} \tag{14}$$

The proof is given in the appendix.

Consider now the quantization by rounding, $Q_q^{(R)}$, defined by:

$$Q_q^{(R)}(x) = \begin{cases} \lfloor x/q + 0.5 \rfloor, & \text{if } x \geq 0, \\ \lceil x/q + 0.5 \rceil, & \text{if } x < 0. \end{cases} \tag{15}$$

And assume a dequantization operation, $D_q^{(R)}$, identical to the dequantization for truncation, i.e., $D_q^{(R)} \equiv D_q^{(T)}$.

Proposition 6 *$Q_q^{(R)}$ is an erosion in the complete inf-semilattice $\left(\mathbb{R}, \overset{R,q}{\leq}\right)$, where the partial ordering $\overset{R,q}{\leq}$ is given by:*

$$x \overset{R,q}{\leq} y \Leftrightarrow \begin{cases} Q_q^{(R)}(x) \overset{\mathrm{T}}{\leq} Q_q^{(R)}(y), & \text{if } Q_q^{(R)}(x) \neq Q_q^{(R)}(y), \\ x - q \cdot Q_q^{(R)}(x) \overset{\mathrm{T}}{\leq} y - q \cdot Q_q^{(R)}(y), & \text{if } Q_q^{(R)}(x) = Q_q^{(R)}(y). \end{cases} \tag{16}$$

$D_q^{(R)}$ is the adjoint dilation of $Q_q^{(R)}$ in $\left(\mathbb{R}, \overset{R,q}{\leq}\right)$.

Although more labourous, the proof of Proposition 6 is similar to that of Proposition 5, and therefore is omitted.

Let us consider now the Joint Photography Experts Group (JPEG) baseline image coding standard [18]. We argue next that it can be considered as an erosion operation between complete inf-semilattices. Moreover, its adjoint dilation is the JPEG decoding procedure. Therefore, the decoding of a coded JPEG file returns the corresponding morphological opening of the original image.

The JPEG coding consists of three main steps: *i)* Discrete cosine transform (DCT) of 8×8-pixel disjoint blocks of the input image, *ii)* quantization (by rounding) in the DCT domain using a quantization matrix Q, and *iii)* a Huffman coding of the zig-zag stream of the resulting DCT coefficients.

Let us denote the 8×8 DCT blocks of an image f by F_{mn}, $m, n \in \mathbb{Z}$, and define the following partial ordering between images:

$$f \overset{\mathrm{JPEG}}{\leq} g \Leftrightarrow \text{for all } m, n,$$

$$F_{mn}(i,j) \overset{R,q_{ij}}{\leq} G_{mn}(i,j), \qquad \forall i,j = 0, \ldots, 7, \tag{17}$$

where the quantization steps $\{q_{ij}\}$ consist of the elements of Q. In addition, denote by $\mathcal{B}$ the set of all possible binary sequences.

Now, consider the complete inf-semilattices $\left(\mathcal{P}_f^2, \overset{\text{JPEG}}{\leq}\right)$ of 2D images, and $\left(\mathcal{B}, \overset{\text{JPEG}}{\leq}\right)$ of binary sequences, where the JPEG ordering on binary sequences corresponds to the ordering of the images that they represent after JPEG decoding.

Proposition 7 *The JPEG coding procedure is an erosion between* $\left(\mathcal{P}_f^2, \overset{\text{JPEG}}{\leq}\right)$ *and* $\left(\mathcal{B}, \overset{\text{JPEG}}{\leq}\right)$. *The JPEG decoding procedure is the adjoint dilation of the JPEG coding procedure.*

The proof is given in the appendix.

3.3 Scaling and Pyramids

Consider the sets $\mathcal{P}_{RM \times RN}$ of all $R \cdot M \times R \cdot N$ discrete images and $\mathcal{P}_{M \times N}$ of all $M \times N$ discrete images. The $R:1$ decimation operation $f_d = \mathcal{D}_R(f)$, mapping $\mathcal{P}_{RM \times RN}$ into $\mathcal{P}_{M \times N}$, is characterized by:

$$f_d(m,n) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} h_{i,j} \cdot f(R \cdot m - i, R \cdot n - j), \tag{18}$$

where $\{h_{i,j}\}$ is the set of coefficients of the ideal π/R-cutoff low-pass filter.

Consider also a discrete version of $\dot{\leq}$ (defined in (4)):

$$\begin{gathered} f \mathrel{\dot{\leq}} g \Leftrightarrow \forall k, \ell, \\ \begin{cases} |F(k,\ell)| \leq |G(k,\ell)|, \text{ and} \\ \text{phase}\{F(k,\ell)\} = \text{phase}\{G(k,\ell)\}, \end{cases} \end{gathered} \tag{19}$$

where F and G are the discrete Fourier transforms (DFT) of f and g, respectively.

Proposition 8 $\mathcal{D}_R$ *is an erosion between the complete inf-semilattices* $\left(\mathcal{P}_{RM \times RN}, \dot{\leq}\right)$ *and* $\left(\mathcal{P}_{M \times N}, \dot{\leq}\right)$. *The adjoint dilation of* $\mathcal{D}_R$ *is the the interpolation operation* $\mathcal{I}_R : \mathcal{P}_{M \times N} \mapsto \mathcal{P}_{RM \times RN}$, *given by the up-sampling of* f_d, *followed by filtering with the same ideal* π/R*-cutoff low-pass filter* $\{h_{i,j}\}$ *as above.*

The proof is similar to that of Propositions 3 and 4, and is omitted.

Corollary 2 *The corresponding morphological opening is the filtering by* $\{h_{i,j}\}$.

Proposition 9 *The laplacian pyramid, calculated with the decimation and interpolation operations $\mathcal{D}_2$ and $\mathcal{I}_2$, is a morphological skeleton in the complete inf-semilattices $\left(\mathcal{P}_{2^i \times 2^i}, \dot{\leq}\right)$, $i = 0, \ldots, i_{max}$.*

Proof *Consider the image $f \in \mathcal{P}_{2^{i_{max}} \times 2^{i_{max}}}$. The laplacian pyramid $\{\ell_i\}$ associated to f is given by:*

$$\begin{aligned} \ell_i &= f_i - \mathcal{I}_2\left(f_{i+1}\right) && (20) \\ &= f_i - \mathcal{I}_2 \mathcal{D}_2\left(f_i\right), && (21) \end{aligned}$$

where $\{f_i\}$ characterizes the gaussian pyramid:

$$f_i = \begin{cases} f, & i = 0, \\ \mathcal{D}_2(f_{i-1}), & i = 1, 2, \ldots \end{cases} \qquad (22)$$

By comparing (57) with (21) and (58) with (22), one concludes that these pairs of equations are identical, once one replaces $S_i(X)$ by f_i, ε_i by $\mathcal{D}_2$, and δ_i by $\mathcal{I}_2$ for all i (thus γ_i becomes $\mathcal{I}_2\mathcal{D}_2$).

It should also be noted that the corresponding supremum operation $\overset{i}{\vee}$ for all i becomes a standard summation when the signals involved in the supremum operation have disjoint Fourier spectra. For this reason, $[f_i - \mathcal{I}_2\mathcal{D}_2(f_i)] \overset{i}{\vee} \mathcal{I}_2\mathcal{D}_2(f_i) = [f_i - \mathcal{I}_2\mathcal{D}_2(f_i)] + \mathcal{I}_2\mathcal{D}_2(f_i) = f_i$, which means that standard subtraction operation $-$ assumes the rôle of $\overset{i}{-}$ (as characterized in Section 8.2) in this case. □

4 Signal Processing on Posets

The various examples in Section 3 motivates us to look at MM theoretical fundaments as a possible, appropriate framework for signal processing in general. In this section, we investigate signal processing from the point of view of MM on posets.

4.1 Hierarchy and Morphological Operations

4.1.1 Partial Orders

Typically, the aim of a signal processing operation is to address a *simplified version* of a given signal. By "address a simplified version" we mean either to *produce* a simplified version of a given signal, to *recover* a signal from a simplified version of it, or to *transform* the signal in order to ultimately assess simplified versions of it.

The term "simplified version" implies the existence of a partial ordering $\preceq$ on the set of signals $\mathcal{S}$, where a signal $f \in \mathcal{S}$ is a simplified version of another $g \in \mathcal{S}$ iff $f \preceq g$. This partial order constitutes a hierarchy in $\mathcal{S}$.

Often (but not necessarily), one can also associate to this hierarchy a simplicity *measure* μ, which is a non-negative real function that quantifies the simplicity of each signal. Examples of useful simplicity measures of signals include energy, entropy, the inverse of smoothness (roughness), variance, error, area, length, etc. The measure and the partial ordering should be related as follows: $f \prec g \Rightarrow \mu(f) \leq \mu(g)$[1]. Notice that $\mu(f) < \mu(g) \not\Rightarrow f \prec g$, which means that being simpler than another signal does not necessarily means being a simpler *version* of it.

4.1.2 Ideal Filters and Openings

One conceptually important way to produce a simplified version of a given signal is by using a *filter*. J. Serra compares the functionality of a filter with hand washing [13]: *i)* It does not add dirt to the hands, *ii)* once the hands are clean, further cleaning produces nothing, and *iii)* washing only one hand cleans less than washing both. The above features are mathematically described as *anti-extensivity*, *idempotence*, and *increasingness*, respectively. Serra also points out that these correspond exactly to the definition of algebraic opening. The conclusion is that filtering is performing an algebraic opening, and vice-versa.

A filter is conceptually important because it defines *components* of a signal: A component is the output of a given filter. If "subtraction" is also defined on the poset, then the difference between the original signal and a component can also be considered a component. The concept of component is the basis of most signal processing tasks. For instance, filtering, quantization, dimensionality/resolution reduction, and lossy compression correspond to *component removal*; detection is *component extraction*; classification and recognition are *component analysis*; enhancement is *different processing of each component followed by signal reconstruction*; synthesis is *component fusion*, etc.

4.1.3 Generalized Inverse and Adjunctions

Definition 5 (Generalized Inverse): *Let T be an operator defined in $(\mathcal{S}, \preceq)$. We say that T has a* generalized-inverse *(also called* pseudo-inverse*) operator, denoted $\tilde{T}^{-1}$, if:*

1. $T\tilde{T}^{-1}(f) = f$, $\forall f \in T(\mathcal{S})$, *where $T(\mathcal{S})$ is the image of $\mathcal{S}$ under T.*
2. $\tilde{T}^{-1}T(f) \preceq f$, $\forall f \in \mathcal{S}$.

Proposition 10 *If T has a generalized-inverse $\tilde{T}^{-1}$, then:*

$$\tilde{T}^{-1}(f) = \bigwedge \{g \in \mathcal{S} \mid \; f = T(g)\}, \qquad \forall f \in T(\mathcal{S}), \tag{23}$$

where $\wedge$ denotes the infimum operation associated to the given partial ordering $\preceq$.

[1] If $\mathcal{S}$ is discrete, or if the signals f and g are "regular," then $f \prec g \Rightarrow \mu(f) < \mu(g)$

In other words, the generalized-inverse of a signal f is the "simplest" signal g among all those having f as their image under T.

Proof (Proposition 10) *Consider the set* $\mathcal{T}_f \triangleq \{g \in \mathcal{S} \mid f = T(g)\}$. *One can see that* $\tilde{T}^{-1}(f) \in \mathcal{T}_f$, *because* $T\tilde{T}^{-1}(f) = f$ *(item 1 in Definition 5). Moreover, because of item 2 in Definition 5,* $\forall g \in \mathcal{T}_f$: $f = T(g) \Rightarrow \tilde{T}^{-1}(f) = \tilde{T}^{-1}T(g) \preceq g$, *which proves that* $\tilde{T}^{-1}(f)$ *is the infimum of* $\mathcal{T}_f$. □

It is our opinion that sound and meaningful signal processing schemes are related to operators T that:

1. preserve the structure of $\mathcal{S}$, i.e., are increasing,
2. have an increasing generalized-inverse $\tilde{T}^{-1}$.

The reason is: To each such operator T corresponds a filter, obtained by the composition of the operator with its generalized inverse: $\tilde{T}^{-1}T$ (see Corollary 3 below). I.e., each such operator T is associated to signal simplification by removal of a well defined component.

For instance, pairs of simplification and generalized-inverse operators are given by down-scaling/up-scaling, quantization/dequantization, coding/decoding, filtering/pseudo-filtering, decomposition/representation, etc, where each task might take place in a different signal hierarchy.

Proposition 11 *If the generalized-inverse* $\tilde{T}^{-1}$ *of a given increasing operator* T *exists and is increasing, then the pair* $(T, \tilde{T}^{-1})$ *is an adjunction between* $(\mathcal{S}, \preceq)$ *and* $(T(\mathcal{S}), \preceq)$.

Proof *On one hand,* $f \preceq T(g) \Rightarrow \tilde{T}^{-1}(f) \preceq \tilde{T}^{-1}T(g) \preceq g$. *On the other hand,* $\tilde{T}^{-1}(f) \preceq g \Rightarrow T\tilde{T}^{-1}(f) \preceq T(g) \Rightarrow f \preceq T(g)$. □

Corollary 3 *If the generalized-inverse* $\tilde{T}^{-1}$ *of a given increasing operator* T *exists and is increasing, then* T *is an erosion between* $(\mathcal{S}, \preceq)$ *and* $(T(\mathcal{S}), \preceq)$, $\tilde{T}^{-1}$ *is its adjoint dilation, and* $\tilde{T}^{-1}T$ *is the associated morphological opening.*

In the above framework, the closing operator (dilation followed by erosion) has no intrinsic meaning or function. The composition $T\tilde{T}^{-1}$ coincides by definition with the identity operator, when applied to eroded elements, and is sometimes not well defined otherwise. In this context, the classical closing on a complete lattice, which usually does have a meaning and function, is seen as an opening in the dual complete lattice. This is because such closing is used for filtering, and therefore functions as an opening.

4.1.4 Erosions as Simplifying Operators

There is a one-to-one correspondence between the subsets $T(\mathcal{S})$ and the filter $\tilde{T}^{-1}T(\mathcal{S})$, because the function $\tilde{T}^{-1} : T(\mathcal{S}) \mapsto \tilde{T}^{-1}T(\mathcal{S})$ is invertible, the inverse being $T : \tilde{T}^{-1}T(\mathcal{S}) \mapsto T(\mathcal{S})$.

Conceptually, the above bijective correspondenc means that the information removed by an increasing operator T having an increasing pseudo-inverse is *identical* to the information removed by a filter $(\tilde{T}^{-1}T)$. The difference lies only on the "format" assumed by the information that remains. It is often correct to say that the eroded signal is a *compact, lossless representation* of the opened one.

Notice also that every opening has a generalized inverse, which is given by the identity operator $\mathcal{I}$. Therefore, the openings themselves are particular cases of erosions. Those readers that are familiar with the traditional theory of MM may find the above sentence incongruent, since it seems to contradict common knowledge. Nevertheless, there is no contradiction. Indeed, an opening γ in $(\mathcal{S}, \preceq)$ is usually not an erosion in $(\mathcal{S}, \preceq)$, however it is an erosion between $(\mathcal{S}, \preceq)$ and $(\gamma(\mathcal{S}), \preceq)$. As a corollary, we get that the adjoint dilation of an opening is the identity operation.

Finally, notice that the generalized inverse of an invertible operator is obviously the corresponding inverse operator. In this case, the associated morphological opening is the identity, which means that invertible operators do not remove information from a signal. This includes transforms, translations, rotations, and group operations in general, etc.

4.1.5 Adjoint Dilations and its Rôles

One motivation for using the generalized inverse $\tilde{T}^{-1}$ might be to reconstruct, approximate or predict a signal from its eroded version (which, as we stressed, does not add new information to the signal). We also mentioned that it may serve as a representation scheme of the filtered signal (more about that in Section 5). In all of the above cases, the corresponding dilation $\tilde{T}^{-1}$ is defined only on $T(\mathcal{S})$.

In other cases, one might be interested in extending the dilation domain to signals outside the set $T(\mathcal{S})$. For example, for enhancement, interpolation, super-resolution, etc. Part[2] of the extrapolation can be done by using the definition of adjoint dilation:

$$\tilde{T}^{-1}(f) \stackrel{\triangle}{=} \bigwedge \{g \in \mathcal{S} \mid \; f \preceq T(g)\}, \tag{24}$$

instead of the more restricted formula in (23).

[2] The signals that still have no well-defined image under $\tilde{T}^{-1}$ after the modification should be heuristically assigned.

4.1.6 Signal Processing Between Different Sets

For simplicity, the above discussion assumed that the signal set $\mathcal{S}$ and its image $T(\mathcal{S})$ by an erosion T share the same partial ordering $\preceq$. In several cases, however, the image domain $T(\mathcal{S})$ may be of a different nature, and $\preceq$ is not defined in it. In this case, we consider a second partial ordering $\sqsubseteq$, defined in $T(\mathcal{S})$, and require that T and its pseudo-inverse form an adjunction between $(\mathcal{S}, \preceq)$ and $(T(\mathcal{S}), \sqsubseteq)$. All concepts and propositions presented above are then extended to the pair of posets $(\mathcal{S}, \preceq)$ and $(T(\mathcal{S}), \sqsubseteq)$.

4.1.7 Summary

In this subsection, we pondered that signal processing consists of accessing simple versions of signals, or equivalently, manipulating signal components. The concept of simple version is equivalent to the definition of a partial ordering between signals, and the concept of signal components assumes the existence of ideal filters (openings) on the corresponding poset. In general, signal processing operators that deal with signal components are those having an increasing generalized inverse on the poset.

In the remaining of this section, we consider the construction of partial orderings.

4.2 Partial Ordering Construction

In [17, Section 2.8], Cover and Thomas demonstrate that, through data processing, information can only decrease or be preserved; never increase. The following proposition could be seen as an MM counterpart of the above.

Proposition 12 *Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary sets. For every operator $\varphi : \mathcal{A} \overset{\text{onto}}{\mapsto} \mathcal{B}$, there exists a pair of partial orderings $\overset{\mathcal{A}}{\preceq}$ and $\overset{\mathcal{B}}{\preceq}$, for which φ is an erosion between $\left(\mathcal{A}, \overset{\mathcal{A}}{\preceq}\right)$ and $\left(\mathcal{B}, \overset{\mathcal{B}}{\preceq}\right)$.*

In other words, every data processing is an erosion, which is bijectively associated to an opening, i.e., to information loss, or preservation at most. The proof of the proposition is given in the appendix.

Another interesting aspect of information decrease is given by the following proposition.

Proposition 13 *Let $\mathcal{A}$ be an arbitrary set. For every idempotent operator $\varphi : \mathcal{A} \mapsto \mathcal{A}$, there exists a partial ordering $\preceq$, such that φ is an opening in $(\mathcal{A}, \preceq)$.*

I.e., any idempotent operator can be seen as a filter. Proposition 13 is also proven in the appendix.

Despite the above propositions, it is obvious that not every existing operator is appropriate for signal processing. Actually, most possible operators are not. An appropriate signal processing approach requires that the underlying partial ordering matches one's intuition of what "simpler version" means. For instance, one could say that one reason for linear filtering (convolution) being usually regarded as an appropriate signal processing operation is because its underlying partial ordering (4) is associated to *signal energy*[3], which is often recognized a good measure of signal complexity. On the other hand, in many applications (e.g., segmentation), signal energy is not an appropriate measure, and other (non-linear) methods are usually invoked.

In this subsection we address a few ways to produce meaningful partial orders, or to derive such orders from given operators, measures, or adjunctions.

4.2.1 Partial Ordering on Transformed Spaces

In the case of traditional MM, the partial ordering is defined on the canonical representation of the signal, in a point-wise fashion. Note that the quantization orderings $\overset{\mathrm{T}}{\leq}$ [defined in (12)] and $\overset{R,q}{\leq}$ [defined in (16)] are also defined on the canonical representation of the signal. However, this is not true for other signal processing schemes. On the other hand, some signal processing schemes have their underlining partial ordering defined also in a point-wise basis, but on a representation other than the canonical one. I.e., the signal is first transformed into a different space, where point-to-point comparisons are then performed. This is the case of $\dot{\leq}$ [defined by eq. (4), in the Fourier domain] and $\overset{\mathrm{JPEG}}{\leq}$ [defined in (17), in the DCT domain)]. Other several linear processing schemes, including Wavelets, can be considered to belong to this category. But this is not restricted to linear transformation.

A particularly important partial ordering, which one can recognize as a basic component of most of the partial orderings presented in Section 3, is defined as follows:

$$x \preceq y \Leftrightarrow |x| \leq |y| \text{ and } \mathrm{phase}\{x\} = \mathrm{phase}\{y\}, \tag{25}$$

where x and y are complex numbers. Notice that this ordering is present in the definition of $\dot{\leq}$. Moreover, if x and y are real, then $\preceq$ becomes identical to $\overset{\mathrm{T}}{\leq}$, which is a basic component of $\overset{R,q}{\leq}$ and $\overset{\mathrm{JPEG}}{\leq}$.

Composite partial orderings can be constructed by using $\preceq$, as defined in (25), in a point-wise manner.

In general, if an operator A is diagonalized by a certain linear transform T, then A is an erosion in the complete inf-semilattice defined by the point-wise application of $\preceq$ in the

[3]The partial order definition in terms of absolute value of the frequency components in (4) is equivalent to a definition in terms of the energy (square of absolute value) of the frequency components.

transform domain of T. For instance, let A and T be $N \times N$ matrices, T unitary, satisfying:

$$T \cdot A \cdot T^t = \Lambda, \tag{26}$$

where Λ is a diagonal matrix. Consider the operator $\varepsilon(x) = A \cdot x$, where x is an N-dimensional vector. Then, ε is an erosion is the complete inf semilattice characterized by the following partial ordering $\tilde{\leq}$:

$$x \tilde{\leq} y \Leftrightarrow X_i \preceq Y_i, \quad \forall i, \tag{27}$$

where $X = (X_0, X_1, \cdots, X_{N-1}) = T \cdot x$, $Y = (Y_0, Y_1, \cdots, Y_{N-1}) = T \cdot y$, and $\preceq$ is as defined in (25).

4.2.2 Measures

In some cases, one can identify an appropriate *measure* μ of the signal to work with. On the other hand, as stressed before, a measure by itself is not enough to characterize "simple versions" of signals; additional criterions/constraints have to be added, in order to turn the measure into a partial ordering.

For instance, consider the basic partial ordering $\preceq$, defined in (25). One can say that it consists of a complementation of the intensity (absolute value) measure. That is, a signal a is smaller than b if the measure of intensity of a is smaller than that of b ($|a| \leq |b|$), and, in addition, they have the same phase. In this case, the phase criterion is a complementation of the intensity measure.

Measure Complementation with an Operator

Let $\mu(\cdot)$ be a measure on the set of signals $\mathcal{S}$, and φ be an arbitrary operator, with domain $\mathcal{S}$ and range $\mathcal{R} \triangleq \varphi(\mathcal{S})$. Let S_r^φ be the set of all signals in $\mathcal{S}$ that have $r \in \mathcal{R}$ as their image by φ, i.e., $S_r^\varphi \triangleq \{f \in \mathcal{S} \mid \varphi(f) = r\}$. Finally, let us define m_r^φ as the smallest measurement in S_r^φ, i.e., $m_r^\varphi \triangleq \inf_{f \in S_r^\varphi} \mu(f)$.

Proposition 14 *If, for all $r \in \mathcal{R}$, the set S_r^φ has exactly one element with measure m_r^ε, then there exists a pair of partial orderings $\left(\overset{\mathcal{S}}{\preceq}, \overset{\mathcal{R}}{\preceq}\right)$, for which:*

1. *φ is an erosion between $\left(\mathcal{S}, \overset{\mathcal{S}}{\preceq}\right)$ and $\left(\mathcal{R}, \overset{\mathcal{R}}{\preceq}\right)$,*

2. *$f \overset{\mathcal{S}}{\preceq} g \Rightarrow \mu(f) \leq \mu(g)$.*

Several combinations measure/operator in signal processing satisfy the hypothesis of Proposition 14. To state a few examples[4]: Energy with linear filtering, area with binary Minskowski erosion, and quantization error with quantization.

The proof of the proposition, presented next, provides a constructive way of generating the partial orderings $\overset{\mathcal{S}}{\preceq}$ and $\overset{\mathcal{R}}{\preceq}$.

Proof *Assuming that the hypothesis of Proposition 14 holds, let us define the operator φ^* as follows:*

$$\varphi^*(r) = \arg \inf_{f \in \mathcal{S}_r^{\varphi}} \mu(f). \tag{28}$$

Define then the partial ordering $\preceq$:

$$f \preceq g \Leftrightarrow \exists n \geq 0 \mid \ f = (\varphi^*)^n \, \varphi^n(g), \tag{29}$$

where φ^n denotes the n-fold recursive operation by φ, and similarly to $(\varphi^*)^n$. The symbols φ^0 and $(\varphi^*)^0$ refer both to the identity operator.

We set $\overset{\mathcal{S}}{\preceq} = \overset{\mathcal{R}}{\preceq} = \preceq$. Notice that, if $\mathcal{R} \not\subseteq \mathcal{S}$ (i.e., if φ is not defined in $\mathcal{R}$), then the greatest possible value for n in $\left(\mathcal{S}, \overset{\mathcal{S}}{\preceq}\right)$ is one, while the possible greatest value for n in $\left(\mathcal{R}, \overset{\mathcal{R}}{\preceq}\right)$ is zero. This means that, if $\mathcal{R} \not\subseteq \mathcal{S}$, then:

$$f \overset{\mathcal{S}}{\preceq} g \quad \Leftrightarrow \quad f = g \text{ or } f = \varphi^* \varphi(g), \tag{30}$$

$$r_1 \overset{\mathcal{R}}{\preceq} r_2 \quad \Leftrightarrow \quad r_1 = r_2. \tag{31}$$

According to the proof of Proposition 12 (shown in the appendix), (φ, φ^*) is an adjunction. Therefore, φ is an erosion. Moreover, by definition of φ^* and $\overset{\mathcal{S}}{\preceq}$, one obtains $f \overset{\mathcal{S}}{\preceq} g \Rightarrow \mu(f) \leq \mu(g)$. □

4.2.3 Adjunctions

Another approach for inducing partial orderings in signal processing is to start with two functions δ and ε, satisfying $\varepsilon\delta\varepsilon = \varepsilon$.

Proposition 15 *For every pair (ε, δ), defined between two sets $\mathcal{S}$ and $\mathcal{R}$, for which $\varepsilon\delta\varepsilon = \varepsilon$, there is a pair of partial orderings $\left(\overset{\mathcal{S}}{\preceq}, \overset{\mathcal{R}}{\preceq}\right)$, for which (ε, δ) is an adjunction between $\left(\mathcal{S}, \overset{\mathcal{S}}{\preceq}\right)$ and $\left(\mathcal{R}, \overset{\mathcal{R}}{\preceq}\right)$.*

[4] We assume either regular (e.g., continuous functions, topologically open sets, etc.) or discrete signals.

Pairs of adjunctions include down-scaling/up-scaling, quantization/dequantization, coding/decoding, filtering/inverse-filtering, and more.

Proof *The same partial ordering defined in (29) serves as well here for* $\overset{\mathcal{S}}{\preceq}$ *and* $\overset{\mathcal{R}}{\preceq}$*, with* φ *replaced by* ε*, and* φ^* *replaced by* δ*.*

Now, $\varepsilon\delta$ *is the identity in* $\varepsilon(\mathcal{S})$*, because* $\varepsilon\delta\varepsilon = \varepsilon$*, and* $\delta\varepsilon$ *is anti-extensive, by definition of* $\preceq$*. Therefore,* δ *is the generalized-inverse of* ε*. We obtain that* ε *and* δ *are increasing, in the exactly same way we obtained that* φ *and* φ^* *are increasing in the proof of Proposition 12 (shown in the appendix). Therefore, according to Proposition 11,* (ε, δ) *is an adjunction.* □

5 Signal Processing Tasks

In the previous section, we pondered about the morphological aspects of signal processing in general. In this section, we proceed the analysis by considering several specific signal processing tasks.

The analysis in this section is neither deeply detailed, nor does it intend to cover all the signal processing field. It serves only as a first look at signal processing tasks from the MM point of view.

5.1 Filtering

The importance of filtering to signal processing, and its connection to MM, have already been discussed in Section 4.1.2. In this section, we discuss an additional practical aspect.

It is widely common in signal processing to use linear filters, which at their vast majority are *not idempotent*, and therefore do not match the morphological description of a filter (filter = opening). On the other hand, J. Serra writes in [3, page 102]: "(...) it is common practice to consider these filters as band-pass devices, even if this is not exactly true". In other words, linear filters were historically intended to be idempotent, but practical reasons lead us to use *approximations* of ideal filters instead. Semantically, the filter approximations became known simply as filters, and this denomination was extended also to any linear translation-invariant operator, even if its rôle does not exactly consist of filtering. According to the theory presented in Section 4, linear "filters" (i.e., linear translation-invariant operators) are *erosions*, and not *openings* (except for ideal linear filters), which means that they are not exactly filters, but they are related to a filter by means of their generalized inverse.

5.2 Projections

Linear idempotent operators are usually called *projection* operators. They map their domain into a linear *sub-space*, and the projection of an element x is the closest element to x in that sub-space.

We extend the above concept to posets. Any opening in a poset can be referred to as a projection operator, and we call its range as a "sub-space", in analogy to the linear case. More specifically, if $\mathcal{S}$ is a poset, then any set of the form $\{\gamma(s) \mid s \in \mathcal{S}\}$ is a "sub-space" of $\mathcal{S}$, when γ is an opening. A sub-space S is partially closed with respect to the supremum, in the sense that, if x and y belong to S, and $x \vee y$ exists in the poset, then $x \vee y$ belongs to S.

Proposition 16 *For all $f \in \mathcal{S}$, $\gamma(f) = \bigvee \{g \in \gamma(\mathcal{S}) \mid g \preceq f\}$.*

The proof is given in the appendix.

According to Proposition 16, the projection $\gamma(x)$ of an element x is the largest element in the sub-space defined by γ that is smaller than x. If there is a metric $d(\cdot, \cdot)$ in $\mathcal{S}$ satisfying $x \leq y \leq z \Rightarrow d(x, y) \leq d(x, z)$ and $d(y, z) \leq d(x, z)$, then the projection γ maps x to the closest element in the sub-space that is smaller than x.

If the projection γ is a *morphological* opening associated to an erosion $\varepsilon : \mathcal{S} \mapsto \mathcal{D}$, then every element of the sub-space is of the form $\delta(q)$ for some $q \in \mathcal{D}$, where δ is the adjoint dilation of ε, and δ is called the *generator* of the sub-space.

A Linear Example

Let u be an $M \times N$ matrix, $M < N$, satisfying $uu^T = \mathcal{I}_M$, where $\mathcal{I}_M$ denotes the identity square matrix of size $M \times M$. Moreover, the matrix $U \triangleq u^T u$ is an $N \times N$ diagonal matrix, where the first M diagonal elements are equal to one, and the rest are null.

The operator $U(x) \triangleq U \cdot x$, where x is an N-dimensional vector, which can be recognized as a linear projection, is an opening in the complete inf-semilattice $(\mathbb{R}^N, \dot{\leq})$, where $\dot{\leq}$ is defined as follows: $x \dot{\leq} y \Leftrightarrow \hat{x}_i \preceq \hat{y}_i, \forall i = 1, \ldots, M$, where $\hat{x}_i$ is the i-th component of the vector $u \cdot x$ (and similarly for y), and $\preceq$ is defined in (25).

In this context, the operator $u(y) \triangleq u \cdot y$, where y is an N-dimensional vector, is an erosion between $(\mathbb{R}^N, \dot{\leq})$ and $(\mathbb{R}^M, \dot{\leq})$, and $\tilde{u}(y) \triangleq u^T \cdot y$ is the adjoint dilation. The projection $U(\cdot)$ is the associated morphological opening, and the dilation $u(\cdot)$ is the generator of the subspace.

5.3 Representation

Let $\mathcal{S}$ and $\mathcal{D}$ be posets, and $\delta : \mathcal{D} \mapsto \mathcal{S}$ be the adjoint dilation of some erosion $\varepsilon : \mathcal{S} \mapsto \mathcal{D}$.

Now consider the sub-space generated by δ: $S \stackrel{\triangle}{=} \{x \in \mathcal{S} \mid \; x = \delta(d), d \in \mathcal{D}\}$. Then every element x in S is of the form:

$$x = \gamma(x) = \delta(d), \tag{32}$$

where γ is the associated opening, and $d \in \mathcal{D}$. We call $\mathcal{D}$ the parameter poset, and we say that d is a representation of x. The dilation δ is called *representation operator*.

Given an element $x \in S$, there might be more than one representation d in $\mathcal{D}$. The erosion ε provides the largest among them. I.e., if $x \in S$, then $\varepsilon(x) = \bigvee\{d \in \mathcal{D} \mid \; \delta(d) = x\}$.

For a given element $x \in \mathcal{S}$, the operation:

$$d = \varepsilon(x), \tag{33}$$

is called *parameter extraction*.

5.4 Quantization & Dequantization

Quantization/dequantization is a particular case of extraction/representation.

Let A and B be any two sets, and define the function $\varepsilon : A \mapsto B$. We call B the set of region labels, and ε the *neighboring rule*. The function ε groups the elements of A in regions that are labeled by some element in B.

Now define the function $\delta : B \mapsto A$, which we call *representation rule*, such that $\varepsilon\delta\varepsilon = \varepsilon$. The function δ assigns for each region label in B a representation point in A.

As seen in Section 4.2.3, (ε, δ) is an adjunction, for some pair of partial orderings. We call the partial ordering of A the *quantization-error criterion*. Now, each representation point given by the function (dilation) δ is the smallest element within the region to which it belongs, i.e., the one that is associated to the smallest quantization error.

Alternatively, the error criterion or the partial orderings can be defined first, and then an erosion chosen as neighboring rule. In this case, the representation rule is the corresponding adjoint dilation.

5.5 Parameter Estimation and Reconstruction

In parameter estimation and reconstruction problems, one wishes to solve the equation:

$$x = A(d) + n, \tag{34}$$

where A is some given representation operator, d is an unknown parameter element ("vector") in a parameter set $\mathcal{D}$, x is the "measured" signal in $\mathcal{S}$, and n is noise "added," by some "adding" operation $+$, to the signal $A(d)$ represented by d.

A classical way to solve (34) is to perform the minimization:

$$\hat{d} = \arg\min_{d} \text{dist}[A(d), x], \tag{35}$$

where dist$(\cdot,\cdot)$ is some distance measure. When A is a linear operator and the distance is Euclidean, the solution is called *least squared error*, or simply *least squares*.

If the operator A is not invertible, then the set $S(x)$ of all solutions of (35) has more than one element. In this case, it is common to choose the *minimum-energy solution*:

$$\check{d} \triangleq \tilde{A}(x) = \begin{array}{l} \arg\min_{\hat{d}} \|\hat{d}\|, \\ \text{s.t. } \hat{d} \in S(x), \end{array} \tag{36}$$

where $\|\cdot\|$ is some "energy" measure. We assume that $\check{d}$ is unique.

We wish to regard the above solution from the morphological point of view. For this purpose, define the following partial ordering in $\mathcal{S}$:

$$x \overset{\mathcal{S}}{\preceq} y \Leftrightarrow \text{dist}[x, A(\mathcal{D})] = \text{dist}[y, A(\mathcal{D})] + \text{dist}(x, y), \tag{37}$$

where dist$[x, A(\mathcal{D})]$ denotes the distance of x to the range of A, i.e.:

$$\text{dist}[x, A(\mathcal{D})] = \min_{y \in A(\mathcal{D})} \text{dist}(x, y). \tag{38}$$

In $\mathcal{D}$, we define the following partial ordering:

$$d_1 \overset{\mathcal{D}}{\succeq} d_2 \Leftrightarrow d_1 = d_2 \text{ or } \left\{ \begin{array}{l} A(d_1) = A(d_2), \text{ and} \\ d_1 = \arg\min_{d \in S(x)} \|d\|. \end{array} \right. \tag{39}$$

Now, notice that (35) and (36) can be summarized into the following equation[5]:

$$\tilde{A}(x) = \overset{\mathcal{D}}{\bigvee} \left\{ d \in \mathcal{D} \mid \ x \overset{\mathcal{S}}{\succeq} A(d) \right\}, \tag{40}$$

which means that, if A is a dilation, then $\tilde{A}$ is its adjoint erosion.

Let us show now that A is a dilation between $\left(\mathcal{D}, \overset{\mathcal{D}}{\preceq}\right)$ and $\left(\mathcal{S}, \overset{\mathcal{S}}{\preceq}\right)$. Suppose that the supremum $d_1 \overset{\mathcal{D}}{\vee} d_2$ exists in $\mathcal{D}$; therefore, $A(d_1) = A(d_2)$. It follows:

$$A\left(d_1 \overset{\mathcal{D}}{\vee} d_2\right) = A(d_1) = A(d_2) = A(d_1) \overset{\mathcal{S}}{\vee} A(d_2). \tag{41}$$

The associated morphological opening $A\tilde{A}$ acts as a projection. It projects a "noisy" signal x onto the range of "valid signals" $A(\mathcal{D})$.

[5] The symbol $\overset{\mathcal{D}}{\bigvee}$ denotes the supremum associated to $\overset{\mathcal{D}}{\preceq}$.

5.6 Feature Detection and Extraction by Template Matching

A solid theoretical basis for template matching in binary images has already been developed based on MM [14, 15]. In our opinion, general template matching could be seen as an extrapolation of that theory to posets.

Actually, feature (or pattern) detection consists of a particular case of parameter extraction, where the parameter element ("vector") returned by an erosion ε includes information about the position of features that are "similar enough" to some given template. The adjoint dilation δ composes a superposition of versions of the given template, which are "placed at" the positions indicated by the eroded (detected) image. The resulting morphological opening $\delta\varepsilon$ is actually the feature extraction operation. The whole process is illustrated in Figure 1.

5.7 Enhancement

As seen above, image simplification can usually be described by an erosion between posets. The adjoint dilation of such erosion could be considered as enhancement.

If a signal x has its resolution reduced by some erosion ε, than the enhancement of $\varepsilon(x)$ by the adjoint dilation δ produces the opening of x (a filtered version of x). Notice that usually x can not be reconstructed from its eroded version; instead, the filtered version $\delta\varepsilon(x)$ is selected.

Sometimes, we wish to perform the enhancement of some signal x that is not necessarily the result of some previous simplification. Notice however that, according to Proposition 19, only the signals in the set $\hat{\mathcal{B}}^{\varepsilon}$, defined in (54)-(53), belong to the domain of the adjoint dilation δ. Therefore, theoretically, only these signals can be properly enhanced. In practice, one would artifitially extend the domain of δ to all signals, losing some of the properties of adjunctions.

On the other hand, one can certify that for traditional linear resolution-reduction operators (e.g., decimation, non-ideal low-pass filtering), the associated domain $\hat{\mathcal{B}}^{\varepsilon}$ is identical to the whole signal space, i.e., any signal can be enhanced by the corresponding adjoint dilations (e.g., interpolation, high-pass filtering).

5.8 Coding

Coding is a particular case of representation, where the encoding procedure is an erosion ε, and the corresponding decoding procedure is its adjoint dilation δ.

If the associated opening $\gamma = \delta\varepsilon$ is the identity, then we say that this coding scheme is lossless.

If γ is not the identity, we have a lossy scheme, and the signal obtained by decoding of an encoded file is a "filtered" version $\gamma(x)$ of the original signal x. This filtered version

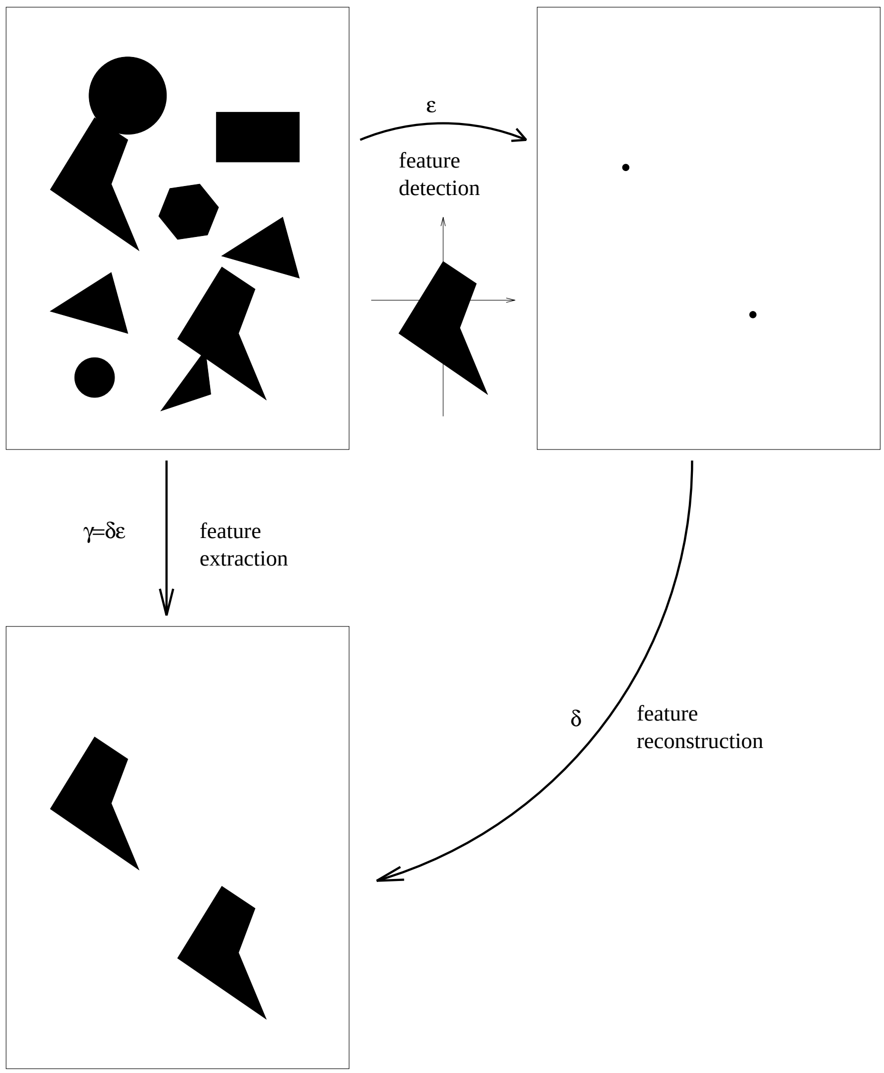

Figure 1: Feature detection and extraction by template matching, from the morphological point of view. The feature detection operator ε and the corresponding reconstruction δ form an adjunction. The corresponding morphological opening represents the feature extraction operation.

is therefore "smaller" than the original signal, according to a underlying partial ordering, induced by the adjunction (ε, δ).

5.9 Segmentation

Segmentation is a particular case of quantization. To segment a signal $f : D \mapsto R$ is to generate an erosion $\varepsilon : D \mapsto L$ that associates each domain point x in D to a label in the label set L. Usually, ε is not invertible, so more than one domain point are associated to the same label. The set of points associated to the same label is called *a segment*.

In the remaining of this subsection, we discuss segmentation by watersheds [16], as an example.

Usually, there is an underlying measure μ associated to a segmentation scheme. For watershed segmentation, this measure is characterized by the *topographic distance*, defined by Meyer in [16], which returns the smallest amount of grayscale variation among all paths that link two points.

More specifically, $\mu(x)$ is given in this case by:

$$\mu(x) = \inf_{m \in \mathcal{M}} \mathcal{D}(x, m), \tag{42}$$

where $\mathcal{D}$ is the topographic distance[6], and $\mathcal{M}$ is the set of local minima of the gradient function associated to f.

A segmentation function ε can be then defined as follows:

$$\varepsilon(x) = \arg \inf_{m \in \mathcal{M}} \mathcal{D}(x, m), \tag{43}$$

which can be obtained by a gradient descent operation on the function μ. The set $\mathcal{M}$ serves then as the label set, i.e., $\mathcal{M} \equiv L$.

If we define the partial ordering: $x \leq y \Leftrightarrow$

$$\mathcal{D}[y, \varepsilon(y)] = \mathcal{D}[x, \varepsilon(x)] + \mathcal{D}(x, y), \tag{44}$$

then ε is an erosion, and also an opening, between the domain D and the set of local minima $\mathcal{M}$. In this case, the adjoint dilation is the identity.

6 Discussion and Conclusion

In this work, some traditional signal processing tasks are shown to be morphological operations in specific posets. An investigation of signal processing fundaments from the point of

[6] We assume that the topographic distance is complemented by the Euclidean distance inside *plateaux*, as suggested in [16].

view of MM is presented, and it seems to indicate that MM could serve as a mathematical framework for signal processing in general.

What are the benefits of the above results? Will we be able to design better signal processing algorithms or improve existing ones, based on this work? Will we get new approaches or applications? The answers to these questions are still unclear, and further research is required in order to fully address them. However, we consider the following points as pontential benefits:

1. This work expands the understanding of morphological fundaments and their relation to other signal processing approaches. It also provides a different point-of-view for further understanding of well-known signal processing tasks and approaches.

2. The results presented here suggest that an appropriate approach for designing signal processing operations/algorithms/systems could be: First, define a partial ordering that matches one's intuition of what "simple version" is, regarding a specific application. Then, select/design operators making sure that they satisfy morphological properties on the resulting poset (i.e., they are erosions, openings, etc.). In summary, first select a criterion, and then search for an operator that matches that criterion. This approach may ensure that the processing results correspond to one's expectations.

3. By unifying different approaches, one could interchange methods and concepts between the approaches. For instance, suppose the well-known *median filter* is considered an erosion in some specific poset, it would be interesting to discover what is its adjoint dilation, and its corresponding morphological opening (which is an idempotent filter, as stressed before). The same would be true for other operators, like the *curvature flow*. As for now, the above questions stand as open problems.

7 Acknowledgment

The author would like to thank Doron Shaked and Ron Kimmel reviewing this work and providing suggestions for improvement. In particular, we thank Doron for important discussions regarding this work, and specifically on drawing the analogy to information theory. He also acknowledges discussions with H. Heijmans and C. Ronse regarding MM on complete semilattices and posets, which helped to shape this article.

8 Appendix A – Erosions, Dilations, and Skeletons on Posets

8.1 Erosions and Dilations

Definition 6 (Erosion): *An operator* $\varepsilon : \mathcal{A} \mapsto \mathcal{B}$ *is called an* erosion *between the posets* $\left(\mathcal{A}, \overset{\mathcal{A}}{\leq}\right)$ *and* $\left(\mathcal{B}, \overset{\mathcal{B}}{\leq}\right)$ *iff for all* $\{Y_i\} \subseteq \mathcal{A}$*:*

- $\exists \overset{\mathcal{B}}{\bigwedge}_i \varepsilon(Y_i) \in \mathcal{B} \Leftrightarrow \exists \overset{\mathcal{A}}{\bigwedge}_i Y_i \in \mathcal{A}$,
- *If* $\exists \overset{\mathcal{A}}{\bigwedge}_i Y_i \in \mathcal{A}$*, then:*

$$\overset{\mathcal{B}}{\bigwedge_i} \varepsilon(Y_i) = \varepsilon\left(\overset{\mathcal{A}}{\bigwedge_i} Y_i\right). \tag{45}$$

Definition 7 (Dilation): *An operator* $\delta : \mathcal{B} \mapsto \mathcal{A}$ *is called a* dilation *between* $\left(\mathcal{B}, \overset{\mathcal{B}}{\leq}\right)$ *and* $\left(\mathcal{A}, \overset{\mathcal{A}}{\leq}\right)$ *iff for all* $\{X_i\} \subseteq \mathcal{B}$*:*

- $\exists \overset{\mathcal{A}}{\bigvee}_i \delta(X_i) \in \mathcal{A} \Leftrightarrow \exists \overset{\mathcal{B}}{\bigvee}_i X_i \in \mathcal{B}$,
- *If* $\exists \overset{\mathcal{B}}{\bigvee}_i X_i \in \mathcal{B}$*, then:*

$$\overset{\mathcal{A}}{\bigvee_i} \delta(X_i) = \delta\left(\overset{\mathcal{B}}{\bigvee_i} X_i\right). \tag{46}$$

Notice that, if $\mathcal{A}$ and $\mathcal{B}$ are complete inf semilattices, then the infima $\overset{\mathcal{A}}{\bigwedge}$ and $\overset{\mathcal{B}}{\bigwedge}$ exist for any subset of $\mathcal{A}$ and $\mathcal{B}$, respectively. Therefore, if $\mathcal{A}$ and $\mathcal{B}$ are complete inf semilattices, then $\varepsilon : \mathcal{A} \mapsto \mathcal{B}$ is an erosion iff $\overset{\mathcal{B}}{\bigwedge}_i \varepsilon(Y_i) = \varepsilon\left(\overset{\mathcal{A}}{\bigwedge}_i Y_i\right)$. Similarly, if $\mathcal{A}$ and $\mathcal{B}$ are complete sup semilattices, then $\delta : \mathcal{B} \mapsto \mathcal{A}$ is a dilation iff $\overset{\mathcal{A}}{\bigvee}_i \delta(X_i) = \delta\left(\overset{\mathcal{B}}{\bigvee}_i X_i\right)$.

Proposition 17 *Erosions and dilations are increasing, i.e., if* $\varepsilon : \mathcal{A} \mapsto \mathcal{B}$ *is an erosion and* $\delta : \mathcal{B} \mapsto \mathcal{A}$ *is a dilation, then*

$$Y_1 \overset{\mathcal{A}}{\leq} Y_2 \;\Rightarrow\; \varepsilon(Y_1) \overset{\mathcal{B}}{\leq} \varepsilon(Y_2), \tag{47}$$

$$X_1 \overset{\mathcal{B}}{\leq} X_2 \;\Rightarrow\; \delta(X_1) \overset{\mathcal{A}}{\leq} \delta(X_2). \tag{48}$$

Proof $Y_1 \overset{\mathcal{A}}{\leq} Y_2 \Rightarrow Y_1 \overset{\mathcal{A}}{\wedge} Y_2 = Y_1 \Rightarrow \varepsilon\left(Y_1 \overset{\mathcal{A}}{\wedge} Y_2\right) = \varepsilon(Y_1) \Rightarrow \varepsilon(Y_1) \overset{\mathcal{B}}{\wedge} \varepsilon(Y_2) = \varepsilon(Y_1) \Rightarrow \varepsilon(Y_1) \overset{\mathcal{B}}{\leq} \varepsilon(Y_2)$. *Similarly,* $X_1 \overset{\mathcal{B}}{\leq} X_2 \Rightarrow X_1 \overset{\mathcal{B}}{\vee} X_2 = X_2 \Rightarrow \delta\left(X_1 \overset{\mathcal{B}}{\vee} X_2\right) = \delta(X_2) \Rightarrow \delta(X_1) \overset{\mathcal{A}}{\vee} \delta(X_2) = \delta(X_2) \Rightarrow \delta(X_1) \overset{\mathcal{A}}{\leq} \delta(X_2)$. □

Proposition 18 *If* (ρ, σ) *is an adjunction between* $\left(\mathcal{A}, \overset{\mathcal{A}}{\leq}\right)$ *and* $\left(\mathcal{B}, \overset{\mathcal{B}}{\leq}\right)$, *then* ρ *is an erosion between* $\left(\mathcal{A}, \overset{\mathcal{A}}{\leq}\right)$ *and* $\left(\mathcal{B}, \overset{\mathcal{B}}{\leq}\right)$, *and* σ *is a dilation between* $\left(\mathcal{B}, \overset{\mathcal{B}}{\leq}\right)$ *and* $\left(\mathcal{A}, \overset{\mathcal{A}}{\leq}\right)$.

Proof *Let us first show that, if* $\overset{\mathcal{B}}{\bigwedge_i} \varepsilon(Y_i)$ *exists, then* $\overset{\mathcal{A}}{\bigwedge_i} Y_i$ *also exists. It holds:* $\overset{\mathcal{B}}{\bigwedge_i} \varepsilon(Y_i) \overset{\mathcal{B}}{\leq} \varepsilon(Y_i)$, *for all* i, *then*

$$\delta\left[\overset{\mathcal{B}}{\bigwedge_i} \varepsilon(Y_i)\right] \overset{\mathcal{A}}{\leq} Y_i, \quad \forall i. \tag{49}$$

On the other hand, for any $Z \in \mathcal{A}$ *such that* $Z \overset{\mathcal{A}}{\leq} Y_i$, $\forall i$, *we get:* $\varepsilon(Z) \overset{\mathcal{B}}{\leq} \varepsilon(Y_i) \Rightarrow \varepsilon(Z) \overset{\mathcal{B}}{\leq} \overset{\mathcal{B}}{\bigwedge_i} \varepsilon(Y_i)$, *which leads to:*

$$Z \overset{\mathcal{A}}{\leq} \delta\left[\overset{\mathcal{B}}{\bigwedge_i} \varepsilon(Y_i)\right], \quad \forall Z \overset{\mathcal{A}}{\leq} Y_i, \quad \forall i. \tag{50}$$

From (49) and (50), we conclude that $\overset{\mathcal{A}}{\bigwedge_i} Y_i$ *exists (it is equal to* $\delta\left[\overset{\mathcal{B}}{\bigwedge_i} \varepsilon(Y_i)\right]$*).*

Now, we show that, if $\overset{\mathcal{A}}{\bigwedge_i} Y_i$ *exists, then* $\overset{\mathcal{B}}{\bigwedge_i} \varepsilon(Y_i)$ *exists, and is equal to* $\varepsilon\left(\overset{\mathcal{A}}{\bigwedge_i} Y_i\right)$. *We have:* $\overset{\mathcal{A}}{\bigwedge_i} Y_i \overset{\mathcal{A}}{\leq} Y_i$, *for all* i. *Therefore:*

$$\varepsilon\left(\overset{\mathcal{A}}{\bigwedge_i} Y_i\right) \overset{\mathcal{B}}{\leq} \varepsilon(Y_i). \tag{51}$$

On the other hand, for any $W \in \mathcal{B}$ *such that* $W \overset{\mathcal{B}}{\leq} \varepsilon(Y_i)$, $\forall i$, *we get:* $\delta(W) \overset{\mathcal{A}}{\leq} Y_i \Rightarrow \delta(W) \overset{\mathcal{A}}{\leq} \overset{\mathcal{A}}{\bigwedge_i} Y_i$, *which leads to:*

$$W \overset{\mathcal{B}}{\leq} \varepsilon\left(\overset{\mathcal{A}}{\bigwedge_i} Y_i\right), \quad \forall Z \overset{\mathcal{B}}{\leq} \varepsilon(Y_i), \quad \forall i. \tag{52}$$

From (51) and (52), we conclude that $\overset{\mathcal{B}}{\bigwedge_i} \varepsilon(Y_i)$ *exists, and is equal to* $\varepsilon\left(\overset{\mathcal{A}}{\bigwedge_i} Y_i\right)$.

We have proven, therefore, that ε *is an erosion. The proof that* δ *is a dilation is dual.* □

For an adjunction (ρ, σ), we call σ the *adjoint dilation* of ρ, and ρ the *adjoint erosion* of σ.

We show next that every erosion has an adjoint dilation. In order to state that with precision, we define first the sets:

$$\mathcal{A}_X^\varepsilon \triangleq \{Y \in \mathcal{A} \mid\ X \overset{\mathcal{B}}{\leq} \varepsilon(Y)\}, \tag{53}$$

and

$$\hat{\mathcal{B}}^\varepsilon \triangleq \left\{X \in \mathcal{B} \mid\ \exists \overset{\mathcal{A}}{\wedge} \mathcal{A}_X^\varepsilon\right\}. \tag{54}$$

Proposition 19 *If ε is an erosion between $\left(\mathcal{A}, \overset{\mathcal{A}}{\leq}\right)$ and $\left(\mathcal{B}, \overset{\mathcal{B}}{\leq}\right)$, then the operator*

$$\delta_\varepsilon(X) \triangleq \bigwedge^{\mathcal{A}} \mathcal{A}_X^\varepsilon, \qquad \forall X \in \hat{\mathcal{B}}^\varepsilon, \tag{55}$$

is such that $(\varepsilon, \delta_\varepsilon)$ is an adjunction between $\left(\mathcal{A}, \overset{\mathcal{A}}{\leq}\right)$ and $\left(\hat{\mathcal{B}}^\varepsilon, \overset{\mathcal{B}}{\leq}\right)$.

Proof *First, $X \overset{\mathcal{B}}{\leq} \varepsilon(Y) \Rightarrow Y \in \mathcal{A}_X^\varepsilon \Rightarrow \overset{\mathcal{A}}{\wedge} \mathcal{A}_X^\varepsilon \overset{\mathcal{A}}{\leq} Y \Rightarrow \delta_\varepsilon(X) \overset{\mathcal{A}}{\leq} Y$.*

On the other hand, $\delta_\varepsilon(X) \overset{\mathcal{A}}{\leq} Y \Rightarrow \overset{\mathcal{A}}{\wedge} \mathcal{A}_X^\varepsilon \overset{\mathcal{A}}{\leq} Y \Rightarrow \overset{\mathcal{B}}{\wedge} \varepsilon\left(\mathcal{A}_X^\varepsilon\right) \overset{\mathcal{B}}{\leq} \varepsilon(Y)$. Therefore, $\overset{\mathcal{B}}{\wedge} \left\{\varepsilon(Y), Y \in \mathcal{A} \mid\ X \overset{\mathcal{B}}{\leq} \varepsilon(Y)\right\} \overset{\mathcal{B}}{\leq} \varepsilon(Y)$. But, notice that $X \overset{\mathcal{B}}{\leq} \overset{\mathcal{B}}{\wedge} \left\{\varepsilon(Y), Y \in \mathcal{A} \mid\ X \overset{\mathcal{B}}{\leq} \varepsilon(Y)\right\}$. Therefore, $X \overset{\mathcal{B}}{\leq} \varepsilon(Y)$. □

Corollary 4 *If $\mathcal{A}$ is a complete inf semilattice, then:*

$$\hat{\mathcal{B}}^\varepsilon = \{X \in \mathcal{B} \mid\ \mathcal{A}_X^\varepsilon \neq \emptyset\}. \tag{56}$$

Corollary 5 *If $\mathcal{A}$ and $\mathcal{B}$ are complete lattices, and $\varepsilon(U_\mathcal{A}) = U_\mathcal{B}$, where $U_\mathcal{A}$ and $U_\mathcal{B}$ are the universes of $\mathcal{A}$ and $\mathcal{B}$, respectively, then $\hat{\mathcal{B}}^\varepsilon = \mathcal{B}$.*

As usual, a dual proposition and dual corollaries involving dilation instead of erosions exist.

8.2 Skeleton Representation

Let $\left\{\left(\mathcal{A}_i, \overset{i}{\leq}\right)\right\}$, $i = 0, 1, \ldots$, be a series of posets, with suprema denoted by $\overset{i}{\vee}$. Assume that there exists a subtraction operation in $\mathcal{A}_i$, denoted $\overset{i}{-}$, satisfying, for all $a, b \in \mathcal{A}_i$, $b \overset{i}{\leq} a$: $\left(a \overset{i}{-} b\right) \overset{i}{\vee} b = a$.

Consider also the series of erosions $\{\varepsilon_i : \mathcal{A}_i \mapsto \mathcal{A}_{i+1}\}$ defined on the series of posets $\left(\mathcal{A}_i, \overset{i}{\leq}\right)$.

The skeleton representation of an element $X \in \mathcal{A}_0$ is given by the series of elements $S_i(X)$, defined as follows:

$$S_i(X) \triangleq X_i \overset{i}{-} \gamma_i(X_i), \tag{57}$$

where γ_i is the morphological opening associated to ε_i, and

$$X_i \triangleq \begin{cases} \varepsilon_{i-1}(X_{i-1}), & i > 0, \\ X, & i = 0. \end{cases} \tag{58}$$

The set of elements $\{S_i(X)\}$ are regarded as a decomposition/representation of the element X.

Let us define the erosions $\hat{\varepsilon}_i : \mathcal{A}_0 \mapsto \mathcal{A}_{i+1}$ as follows:

$$\hat{\varepsilon}_i \triangleq \begin{cases} \varepsilon_0 \varepsilon_1 \cdots \varepsilon_{i-1}, & i > 0 \\ \text{identity}, & \text{otherwise.} \end{cases} \tag{59}$$

One can notice that (57) can be rewritten in terms of $\{\hat{\varepsilon}_i\}$ as follows:

$$S_i(X) = \hat{\varepsilon}_i(X) \overset{i}{-} \gamma_i[\hat{\varepsilon}_i(X)]. \tag{60}$$

The reconstruction of the original signal X from its skeleton representation is given by:

$$X = \overset{0}{\bigvee} \left\{ \hat{\delta}_i[S_i(X)] \right\}, \tag{61}$$

where $\hat{\delta}_i$ is the adjoint dilation of $\hat{\varepsilon}_i$.

9 Appendix B – Proofs

Proof (Proposition 1)

1.

$$\begin{aligned} \rho(Y) = \rho(Y) \quad &\Rightarrow \quad \rho(Y) \overset{\mathcal{B}}{\leq} \rho(Y) \Rightarrow \sigma\rho(Y) \overset{\mathcal{A}}{\leq} Y. \\ \sigma(X) = \sigma(X) \quad &\Rightarrow \quad \sigma(X) \overset{\mathcal{A}}{\leq} \sigma(X) \Rightarrow X \overset{\mathcal{B}}{\leq} \rho\sigma(X). \end{aligned}$$

2.

$$\left. \begin{array}{r} \sigma\rho(Y) \overset{\mathcal{A}}{\leq} \sigma\rho(Y) \Rightarrow \rho(Y) \overset{\mathcal{B}}{\leq} \rho\sigma\rho(Y) \\ \sigma\rho\sigma\rho(Y) \overset{\mathcal{A}}{\leq} \sigma\rho(Y) \overset{\mathcal{A}}{\leq} Y \Rightarrow \rho\sigma\rho(Y) \overset{\mathcal{B}}{\leq} \rho(Y) \end{array} \right\} \Rightarrow \rho(Y) = \rho\sigma\rho(Y). \tag{62}$$

$\sigma\rho\sigma = \sigma$ is proved dually.

3. $X_1 \overset{\mathcal{B}}{\leq} X_2 \Rightarrow X_1 \overset{\mathcal{B}}{\leq} \rho\sigma(X_2) \Rightarrow \sigma(X_1) \overset{\mathcal{A}}{\leq} \sigma(X_2)$. *Similarly,* $Y_1 \overset{\mathcal{A}}{\leq} Y_2 \Rightarrow \sigma\rho(Y_1) \overset{\mathcal{A}}{\leq} Y_2 \Rightarrow \rho(Y_1) \overset{\mathcal{B}}{\leq} \rho(Y_2)$.

4. *Notice that* $\sigma(X) \in \mathcal{A}^{\rho}_{X}$, *since* $X \overset{\mathcal{B}}{\leq} \rho\sigma(X)$. *Moreover,* $Y \in \mathcal{A}^{\rho}_{X} \Rightarrow X \overset{\mathcal{B}}{\leq} \rho(Y) \Rightarrow \sigma(X) \overset{\mathcal{A}}{\leq} Y$, *which proves that* $\sigma(X)$ *is the infimum of* $\mathcal{A}^{\rho}_{X}$.

5. *This item is proved dually to the previous one.*

□

Proof (Proposition 2) *Throughout this proof we simplify the notation by writting* γ *instead of* γ_ε, *and* δ *instead of* δ_ε.

Let us show first that the opening exists and is unique.

One can observe that $X \in \left\{Y \in \mathcal{A} \mid \varepsilon(X) \overset{\mathcal{B}}{\leq} \varepsilon(Y)\right\}$. *For this reason, and because* ε *is increasing, one can write:* $\gamma(X) = \overset{\mathcal{A}}{\bigwedge} \{Y \in \mathcal{A} \mid \varepsilon(X) = \varepsilon(Y)\}$. *Consider then the set* $C_X \triangleq \{Y \in \mathcal{A} \mid \varepsilon(X) = \varepsilon(Y)\}$. *For all* $X \in C_X$, $\varepsilon(X)$ *is a constant and, therefore, if we write* $\{X_i\} = C_X$, *then* $\overset{\mathcal{B}}{\wedge} \varepsilon(X_i)$ *exists (and is equal to that constant). Therefore, since* ε *is an erosion,* $\overset{\mathcal{A}}{\wedge} X_i$ *exists and is unique in* $\mathcal{A}$, *which proves the proposition.*

Let us show now that $\varepsilon\gamma = \varepsilon$. *If* $\varepsilon : \mathcal{A} \mapsto \mathcal{B}$, *then for all* X *in* $\mathcal{A}$:

$$
\begin{aligned}
\varepsilon\gamma(X) &= \varepsilon\left(\overset{\mathcal{A}}{\bigwedge}\left\{Y \in \mathcal{A} \mid \varepsilon(X) \overset{\mathcal{B}}{\leq} \varepsilon(Y)\right\}\right) \\
&= \overset{\mathcal{A}}{\bigwedge}\left\{\varepsilon(Y) \mid \varepsilon(X) \overset{\mathcal{B}}{\leq} \varepsilon(Y)\right\} = \varepsilon(X).
\end{aligned}
$$

We proceed now to show that the morphological opening is an algebraic opening. We have to prove that γ *is idempotent, increasing, and anti-extensive.*

Idempotent:

$$
\begin{aligned}
\gamma\gamma(X) &= \bigwedge\{Y \mid \varepsilon\gamma(X) \leq \varepsilon(Y)\} \\
&= \bigwedge\{Y \mid \varepsilon(X) \leq \varepsilon(Y)\} = \gamma(X).
\end{aligned}
$$

Increasing:

$$
\begin{aligned}
X \leq Y &\Rightarrow \varepsilon(X) \leq \varepsilon(Y) \\
&\Rightarrow \{Z \mid \varepsilon(X) \leq \varepsilon(Z)\} \supseteq \{Z \mid \varepsilon(Y) \leq \varepsilon(Z)\} \\
&\Rightarrow \bigwedge\{Z \mid \varepsilon(X) \leq \varepsilon(Z)\} \leq \bigwedge\{Z \mid \varepsilon(Y) \leq \varepsilon(Z)\} \\
&\Rightarrow \gamma(X) \leq \gamma(Y).
\end{aligned}
$$

Anti-extensive:

$$\begin{array}{rc} & \{Z \mid \varepsilon(X) \leq \varepsilon(Z)\} \supseteq \{X\} \\ \Rightarrow & \bigwedge\{Z \mid \varepsilon(X) \leq \varepsilon(Z)\} \leq \bigwedge\{X\} \\ \Rightarrow & \gamma(X) \leq X. \end{array}$$

Proof of property 2: $\delta[\varepsilon(X)] = \overset{\mathcal{A}}{\bigwedge} \{Z \mid \varepsilon(X) \overset{\mathcal{B}}{\leq} \varepsilon(Z)\} = \gamma(X)$. □

Proof (Proposition 5)

$$Q_q^{(T)}(x) \overset{\mathrm{T}}{\wedge} Q_q^{(T)}(y) = median\{a, b, 0\}, \tag{63}$$

where

$$a \triangleq \begin{cases} \lfloor x/q \rfloor, & x \geq 0, \\ \lceil x/q \rceil, & x < 0. \end{cases} \qquad b \triangleq \begin{cases} \lfloor y/q \rfloor, & y \geq 0, \\ \lceil y/q \rceil, & y < 0. \end{cases} \tag{64}$$

Hence,

$$Q_q^{(T)}(x) \overset{\mathrm{T}}{\wedge} Q_q^{(T)}(y) = \begin{cases} median\{\lfloor x/q \rfloor, \lfloor y/q \rfloor, 0\}, & (x \geq 0) \text{ and } (y \geq 0), \\ median\{\lceil x/q \rceil, \lceil y/q \rceil, 0\}, & (x < 0) \text{ and } (y < 0), \\ median\{\lfloor x/q \rfloor, \lceil y/q \rceil, 0\}, & (x \geq 0) \text{ and } (y < 0), \\ median\{\lceil x/q \rceil, \lfloor y/q \rfloor, 0\}, & (x < 0) \text{ and } (y \geq 0). \end{cases} \tag{65}$$

$$= \begin{cases} \lfloor median\{x, y, 0\} \rfloor /q, & (x \geq 0) \text{ and } (y \geq 0), \\ \lceil median\{x, y, 0\} \rceil /q, & (x < 0) \text{ and } (y < 0), \\ 0, & \text{otherwise}. \end{cases} \tag{66}$$

$$= \begin{cases} \lfloor median\{x, y, 0\} \rfloor /q, & median\{x, y, 0\} \geq 0, \\ \lceil median\{x, y, 0\} \rceil /q, & median\{x, y, 0\} < 0. \end{cases} \tag{67}$$

$$= Q_q^{(T)}(x \overset{\mathrm{T}}{\wedge} y). \tag{68}$$

It remains to prove that $D_q^{(T)}$ is the adjoint dilation of $Q_q^{(T)}$. The steps of the proof are identical to those of Proposition 4:

1. *Define the set*

$$\mathcal{A}_x^{Q_q^{(T)}} \triangleq \left\{ y \in I\!R \mid x \overset{\mathrm{T}}{\leq} Q_q^{(T)}(y) \right\}. \tag{69}$$

2. *Verify that $D_q^{(T)}$ is increasing in $\left(I\!R, \overset{\mathrm{T}}{\leq}\right)$.*

3. *Notice that $D_q^{(T)}(x) \in \mathcal{A}_x^{Q_q^{(T)}}$ for all $x \in I\!R$, because $x \overset{\mathrm{T}}{\leq} Q_q^{(T)} D_q^{(T)}(x)$.*

4. *Show that* $D_q^{(T)}(x)$ *is the infimum (according to* $\overset{\mathrm{T}}{\leq}$*) of* $\mathcal{A}_x^{Q_q^{(T)}}$*, because for all* $y \in \mathcal{A}_x^{Q_q^{(T)}}$*:* $x \overset{\mathrm{T}}{\leq} Q_q^{(T)}(y) \Rightarrow D_q^{(T)}(x) \overset{\mathrm{T}}{\leq} D_q^{(T)} Q_q^{(T)}(y) \overset{\mathrm{T}}{\leq} y$.

□

Proof (Proposition 7) *Let us make use of the following lemma:*

Lemma 1 *If* $A : \mathcal{A} \mapsto \mathcal{B}$*,* $B : \mathcal{B} \mapsto \mathcal{C}$*, and* $C : \mathcal{C} \mapsto \mathcal{D}$ *are three erosions, then* $CBA : \mathcal{A} \mapsto \mathcal{D}$ *is also an erosion.*

The proof is as follows:

$$\begin{aligned} CBA(x \wedge y) &= CB\,[A(x) \wedge A(y)] \\ &= C\,[BA(x) \wedge BA(y)] = CBA(x) \wedge CBA(y). \end{aligned} \tag{70}$$

Now, JPEG encoding can be divided in three steps: DCT transform (ϕ_t)*, quantization* (ϕ_q)*, and zigzag-Hufman coding* (ϕ_h)*. The first and the third steps are invertible and increasing, and, therefore, they are erosions. The second step* (ϕ_q) *can be shown to be also an erosion almost directly from Proposition 6. Therefore, according to the above lemma, the whole JPEG encoding algorithm is an erosion.*

To prove the second part of the proposition, consider the following lemma:

Lemma 2 *If* A*,* B*, and* C *are erosions, then the adjoint dilation* $(CBA)^*$ *of* CBA *is equal to* $A^*B^*C^*$*, where* A^**,* B^**, and* C^* *are the adjoint dilations of* A*,* B*, and* C*, respectively.*

This lemma is proved as follows:

$$\begin{aligned} x \leq CBA(y) &\Rightarrow C^*(x) \leq C^*CBA(y) \leq BA(y) \\ &\Rightarrow B^*C^*(x) \leq B^*BA(y) \leq A(y) \\ &\Rightarrow A^*B^*C^*(x) \leq A^*A(y) \leq y. \end{aligned} \tag{71}$$

$$\begin{aligned} y \geq A^*B^*C^*(x) &\Rightarrow A(y) \geq AA^*B^*C^*(x) \geq B^*C^*(x) \\ &\Rightarrow BA(y) \geq BB^*C^*(x) \geq C^*(x) \\ &\Rightarrow CBA(y) \geq CC^*(x) \geq x. \end{aligned} \tag{72}$$

The second part is then proved, once one notice that the components of the JPEG decoding algorithm are the corresponding adjoint dilations of those of the JPEG encoding algorithm. Proposition 6 states that dequantization is indeed the adjoint dilation of quantization, whereas the adjoint dilation of the other two (invertible) components are their respective inverse operators. □

Proof (Proposition 12) *Let us construct an example of such pair of partial orderings, where we consider the two cases:* $\mathcal{B} \not\subseteq \mathcal{A}$ *and* $\mathcal{B} \subseteq \mathcal{A}$.

First, for every $b \in \mathcal{B}$, *consider the set* $\mathcal{A}_b^{\varphi} \stackrel{\triangle}{=} \{a \in \mathcal{A} \mid \ b = \varphi(a)\}$. *Construct an operator* φ^*, *by assigning an arbitrary element of* $\mathcal{A}_b^{\varphi}$ *to* $\varphi^*(b)$, *for each* b. *Notice that any such construction satisfies* $\varphi\varphi^*\varphi = \varphi$.

Assume first that $\mathcal{B} \not\subseteq \mathcal{A}$. *In this case, define* $\stackrel{\mathcal{A}}{\preceq}$ *by:* $a_1 \stackrel{\mathcal{A}}{\preceq} a_2 \Leftrightarrow a_1 = a_2$ *or* $a_1 = \varphi^*\varphi(a_2)$. *Define* $\stackrel{\mathcal{B}}{\preceq}$ *as the trivial partial ordering:* $b_1 \stackrel{\mathcal{B}}{\preceq} b_2 \Leftrightarrow b_1 = b_2$.

In this case, we have $\varphi\varphi^*(b) = b, \forall b \in \mathcal{B}$, *by definition of* φ^*. *And* $\varphi^*\varphi(a) \stackrel{\mathcal{A}}{\preceq} a, \forall a \in \mathcal{A}$, *by definition of* $\stackrel{\mathcal{A}}{\preceq}$. *Therefore,* φ^* *is the generalized-inverse of* φ. *Moreover,* φ *is increasing, since* $a_1 \stackrel{\mathcal{A}}{\preceq} a_2 \Rightarrow \varphi(a_1) = \varphi(a_2) \Rightarrow \varphi(a_1) \stackrel{\mathcal{B}}{\preceq} \varphi(a_2)$. *And* φ^* *is increasing, since* $b_1 \stackrel{\mathcal{B}}{\preceq} b_2 \Rightarrow b_1 = b_2 \Rightarrow \varphi^*(b_1) = \varphi^*(b_2) \Rightarrow \varphi^*(b_1) \stackrel{\mathcal{A}}{\preceq} \varphi^*(b_2)$. *Therefore, according to Proposition 11,* (φ, φ^*) *is an adjunction.*

Assume now that $\mathcal{B} \subseteq \mathcal{A}$. *In this case, define a single partial ordering* $\preceq \equiv \stackrel{\mathcal{A}}{\preceq} \equiv \stackrel{\mathcal{B}}{\preceq}$ *as follows:*

$$a_1 \preceq a_2 \Leftrightarrow \exists n \geq 0 \mid \ a_1 = (\varphi^*)^n \varphi^n(a_2). \tag{73}$$

where φ^n *and* $(\varphi^*)^n$ *denote the n-fold operation by* φ *and* φ^*, *respectively. For* $n = 0$, *we have* $\varphi^0 = (\varphi^*)^0 = identity$.

Now, we still have that φ^* *is the generalized-inverse of* φ. *It remains only to check increasingness.*

By definition of $\preceq$, $a_1 \preceq a_2 \Rightarrow a_1 = (\varphi^*)^n \varphi^n(a_2)$ *for some* n. *Then, using that* $\varphi\varphi^*$ *is identity,* $a_1 \preceq a_2 \Rightarrow \varphi(a_1) = (\varphi^*)^{n-1} \varphi^{n-1}[\varphi(a_2)] \Rightarrow \varphi(a_1) \preceq \varphi(a_2)$, *and therefore* φ *is increasing. And finally,* $a_1 \preceq a_2 \Rightarrow a_1 = (\varphi^*)^n \varphi^n(a_2) \Rightarrow \varphi^*(a_1) = \varphi^* (\varphi^*)^n \varphi^n(a_2) = \varphi^* (\varphi^*)^n \varphi^n \varphi[\varphi^*(a_2)] = (\varphi^*)^{n+1} \varphi^{n+1}[\varphi^*(a_2)] \Rightarrow \varphi^*(a_1) \preceq \varphi^*(a_2)$, *which means that* φ^* *is also increasing.* □

Proof (Proposition 13) *Define* $\preceq$ *as follows:* $a_1 \preceq a_2 \Leftrightarrow a_1 = a_2$ *or* $a_1 = \varphi(a_2)$.

In this case, φ *is increasing, since* $a_1 \preceq a_2 \Rightarrow \varphi(a_1) = \varphi(a_2) \Rightarrow \varphi(a_1) \preceq \varphi(a_2)$. *Moreover,* φ *is idempotent by definition, and anti-extensive, by definition of* $\preceq$. *Therefore,* φ *is an opening.* □

Proof (Proposition 16) *Define* $\mathcal{A}_f^{\gamma} \stackrel{\triangle}{=} \{g \in \gamma(\mathcal{S}) \mid \ g \preceq f\}$. *Obviously,* $\gamma(f) \in \mathcal{A}_f^{\gamma}$. *Moreover, for all* $g \in \mathcal{A}_f^{\gamma}$: $g \preceq f \Rightarrow \gamma(g) \preceq \gamma(f)$. *And since* $\gamma(g) = g$, $g \preceq \gamma(f)$, *which makes* $\gamma(f)$ *the supremum of* $\mathcal{A}_f^{\gamma}$. □

10 References

[1] A.V. Oppenheimer and R.W. Schafer, *Digital Signal Processing*, Prentice Hall, 1975.

[2] W.K. Pratt, *Digital Image Processing*, Wiley, New York, 1978.

[3] J. Serra, editor, *Image Analysis and Mathematical Morphology, Volume 2: Theoretical Advances.* New York: Academic Press, 1988.

[4] P. Maragos, "Morphological Systems: Slope Transform and Max-min Difference and Differential Equations," *Signal Processing*, Vol. 38, September 1994.

[5] R. Kresch, *Morphological Image Representation for Coding Applications*, D.Sc. Thesis (in English), Dept. of Electrical Engineering, Technion, Israel, June 1995. Available at `http://www-sipl.technion.ac.il/publications/thesis/renato/renato.html`.

[6] J. Goutsias and H.J.A.M Heijmans, "An Axiomatic Approach to Multiresolution Signal Decomposition," ICIP-98.

[7] R. Kresch, "Extension of Morphological Operations to Complete Semilattices and Its Applications to Image and Video Processing," Mathematical Morphology and its Applications to Image and Signal Processing (Proc. of ISMM'98), H.J.A.M. Heijmans and J.B.T.M. Roerdink (eds.), pp. 35-42, June 1998.

[8] P. Maragos and R.W. Schafer, "Morphological Filters II. Their relations to median, order-statistics, and stack filters," *IEEE Trans. Acoust. Speech Signal Processi.* **ASSP-35**, pp.1153-1184, 1987.

[9] G. Sapiro, R. Kimmel, D. Shaked, and A.M. Bruckstein, "Implementing Continuous-Scale Morphology," internal report no.CIS9208, Technion–Israel Institute of Technology, Haifa, Israel, 1992.

[10] W. Brockett and P. Maragos, "Evolution Equations for Continuous-Scale Morphology," *ICASSP*, San Francisco, pp.23-26, 1992.

[11] H.J.A.M. Heijmans and C. Ronse, "The Algebraic Basis of Mathematical Morphology. I. Dilations and Erosions," *Comput. Vision Graphics Image Process.*, Vol. 50, pp. 245-295, 1990.

[12] J. Serra, *Image Analysis and Mathematical Morphology.* London: Academic Press, 1982.

[13] J. Serra, "Morphological Filtering: An Overview", *Proc. First Int. Workshop on Mathematical Morphology and its Applications to Signal Processing*, J. Serra and Ph. Salembier (eds.), Barcelona, Spain, pp.1-10, May 1993.

[14] C. Ronse, "A Lattice-Theoretical Morphological View on Template Extraction in Images", *J. Visual Comm. and Image Representation*, Vol. 7, No. 3, pp.273-295, September 1996.

[15] R. Jones and I. Svalbe, "Morphological Filtering as Template Matching", *IEEE Trans. PAMI*, Vol. 16, No. 4, pp.438-448, April 1994.

[16] F. Meyer, "Topographic Distance and Watershed Lines," *Signal Processing* 38 (1994), pp.113-125.

[17] T.H. Cover and J.A. Thomas, *Elements of Information Theory.* New York: John Wiley & Sons, Inc., 1938.

[18] W.B. Pennebaker and J.L. Mitchell, *JPEG Still Image Data Compression Standard*, Van Nostrand Reinhold, 1993.