Phase Space Reconstruction using Input-Output Time Series Data

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In this letter we suggest that a method recently proposed by Wayland et al. [1] for recognising determinism in a time series can also be used as a diagnostic to determine the minimum embedding dimension required for reconstructing phase space. Furthermore we suggest a simple extension to the Wayland et al. method which enables phase space to be reconstructed from input–output time series data. We compare the results of this extension to the results given by the extensions to the method of false nearest neighbours put forward by Rhodes and Morari [2] and the method of averaged false nearest neighbours by Cao et al. [3].

I. INTRODUCTION

In this letter we suggest that a method recently proposed by Wayland et al. [1] for recognising determinism in a time series can also be used as a diagnostic to determine the minimum embedding dimension required for reconstructing phase space. Furthermore we suggest a simple extension to the Wayland method which enables phase space to be reconstructed from input–output time series. We compare the results of this new diagnostic with the results of two other diagnostics recently proposed in the literature [2,3].

The paper by Casdagli [4] is a common starting point for many researchers when faced with the problem of applying nonlinear dynamics techniques to the modelling of systems using input–output time series data. The main idea drawn from this paper is that an extended phase space can be reconstructed from the input and output time series. If we denote the output time series by $y(t)$ and the input time series by $u(t)$ then Casdagli's results say an extended reconstructed phase space can be formed with vectors

$$\begin{aligned} z(t) = \big[&y(t-(k-1)s), \ldots, y(t-s), y(t), \\ &u(t-(l-1)s), \ldots, u(t-s), u(t)\big], \end{aligned} \tag{1}$$

where k is the embedding dimension of the output time series and l is the embedding dimension of the input time series. We have assumed that the time delay s is the same for the input and the output time series although this need not be the case. In the following we assume that an appropriate time delay has been found. In addition we will normalise all time series to lie in the range -2 and 2.

For a given time delay the problem is to design a diagnostic to find appropriate values for k and l from time series data. Rhodes and Morari [2] have extended the false nearest neighbour algorithm of Kennel et al. [5] to determine k and l. More recently Cao et al. [3] have suggested an alternative method by extending the work of [6]. In this letter we argue that the method of Wayland et al. can similary be extended to provide such a diagnostic.

The outline of this letter is as follows: In the next section we describe the Wayland method for detecting determinism in a time series, and explain why it can be used as a diagnostic for determining an appropriate embeddign dimension for phase space reconstruction. We re-inforce our contention with an example using data from the chaotic Lorenz system. We then introduce our scheme for reconstruction using input–output time series data by extending the Wayland method. To illustrate our technique we apply it to data from Duffing's equation and to data obtained from a model of a Bi–Polar Junction Transistor (BJT). We compare our results with those obtained from an implementation of the methods of [2] and [3].

II. THE WAYLAND METHOD

According to Wayland et al. [1] a time series is said to be deterministic if the reconstructed vectors

$$x(t) = \big[y(t-(k-1)s), \ldots, y(t-s), y(t)\big]$$

can be modelled as the iteration of a continuous function f. A test for continuity can be developed based on the fact that points close together will map to points close together under a single iteration of the map f.

Let x_0 be a reference vector chosen from $x(t), t = 1, 2, \ldots, N$, and let $x_1, x_2, \ldots, x_m$ be the m–nearest neighbours of x_0 chosen from $x(t), t = 1, 2, \ldots, N$. In addition, we ensure that none of these points are temporally correlated. Let $y_0, y_1, \ldots, y_m$ be the images of the vector x_0 and its neighbours respectively. If the data is deterministic *and* correctly embedded we expect the translation vectors

$$v_j = y_j - x_j$$

to be nearly equal provided the near neighbours are within a small region of phase space. Wayland et al. quantify this insight by computing the translation error

$$E_{trans} = \frac{1}{m+1}\sum_{j=0}^{m}\frac{\|v_j - <v>\|^2}{\|<v>\|^2},$$

where

$$<v> = \frac{1}{m+1}\sum_{j=0}^{m} v_j.$$

This local translation error is extended to a more global measure of translation error by choosing N_r random reference vectors from $x(t), t = 1, 2, \ldots, N$. For each reference vector we compute an associated E_{trans} and then calculate the global translation error $E = \text{median}(E_{trans})$.

In addition to the embedding dimension k and the time delay s there are two other free parameters. These are N_r the number of reference vectors and m the number of near neighbours. We will remove the parameter N_r by using all embedded data points as reference vectors just like the extensions in [2] and [3]. In so doing we will take E to be the average of E_{trans} rather than the median. E is thus a function of embedding dimension so $E(k)$. Following Cao [6] we will calculate the quantity

$$\epsilon(k) = \frac{E(k+1)}{E(k)}.$$

The translation error $E(k)$ will generally decrease with increasing embedding dimension. As k increases $\epsilon(k)$ will typically rise, however, there will be a marked change in the rate of increase of $\epsilon(k)$ when a suitable embedding dimension is attained. This change is distinctive and the k at which it occurs is what we will choose as the embedding dimension. There is the possibility of an increase in $E(k)$ for large k due to de-correlations in the embedded data. This will cause a decrease in $\epsilon(k)$ and so as an additional indicator we suggest choosing the first maximum of $\epsilon(k)$ if the distinctive rise has not yet occured. We will study how robust this prescription is to the observational noise level in the data and the number m of neighbours chosen in the calculations.

III. INPUT–OUTPUT DIAGNOSTIC

The above diagnostic can be extended to input–output time series data in a simple manner. The scheme is essentially the same but the method of determining nearest neighbours is modified. The near neighbours of a vector x_0 are instead determined in the extended reconstructed space from the vectors $z(t)$ (see 1). The vector x_0 is associated with a vector z_0. Let $z_1, z_2, \ldots, z_m$ be the (de-correlated) near neighbours of z_0. We denote by w_j the images of these near neighbours. For each $z_j, j = 1, 2, \ldots, m$ we project down to the "x" subspace, i.e., $x_j = Cz_j$ and $y_j = Cw_j$ where C picks out the parts of $z(t)$ constructed using the output time series. We calculate the translation error as before with these vectors.

We note that the inputs can possibly be deterministic or stochastic. The above scheme is based on the fact that vectors close in reconstructed phase space subject to similar inputs should end up in the same place.

We observe that the choice of the near neighbours in this extended reconstructed space may be dominated by closeness in reconstructed phase space or closeness in the reconstructed input phase space. For example two vectors may be deemed close in the extended phase space because their distance apart in phase space masks the difference in the inputs. The two vectors although close in phase space could be subject to vastly different inputs thus compromising the translation errors. To avoid this eventuality we suggest normalising the output and input time series. (As mentioned above we normalise all time series to lie in the range -2 and 2.)

Once again following the route of Cao et al. [3] we will calculate

$$\epsilon(k,l) = \frac{E(k+1,l)}{E(k,l)}.$$

We will consider the first maximum of this quantity as a suitable choice for the embedding dimension k. To distinguish between different values of l we will choose the $\epsilon(k,l)$ for which $k+l$ is a minimum. We will also favour values of k and l where $l > k$. For example if the first maxima of $\epsilon(k,l)$ give two choices $(k,l) = (2,2)$ and $(k,l) = (1,3)$ say, we will choose the latter. A reason for this choice comes from our interest in modelling electronic device components [7] for the purpose of simulations. We believe a model with as little feedback as possible, i.e., small k, should be more stable under iteration than a model with large k.

IV. EXAMPLES

We present an example illustrating the effectiveness of the method when applied to output time series data. The output data is obtained by integrating the chaotic Lorenz equations. For this example appropriate embedding dimensions are known from studies elsewhere (see for example Abarbanel [8]). We will study how robust our prescription is to observational noise in the data and the number of near neighbours in the diagnostic.

We present two examples to illustrate the effectiveness of our extension to the Wayland scheme to accomodate input–output time series data. In the first example we will consider data from Duffing's equation. Once again we will study the robustness of our method to observational noise on the input and output time series, and the number of near neighbours on the diagnostic. We also compare the results of our method to the results produced by using the methods suggested by Rhodes and Morari [2] and Cao et al. [3]. The second data set we study using our method is obtained from simulating a model of a BJT – the Ebers–Moll model [9].

A. Output time series

The Lorenz differential equations are

$$\dot{u} = \sigma(-u+v)$$
$$\dot{v} = ru - v - uw$$
$$\dot{w} = -bw + uv,$$

where for $\sigma = 10$, $r = 28$ and $b = \frac{8}{3}$ chaotic solutions are generated. We generate time series data by integrating the Lorenz equations using a variable step-size Runge-Kutta method – matlab's `ode23` routine – and output the u coordinate every 0.01 time units after transients have diminished. We obtain a $10,000$ point time series, and determine a lag $s = 35$ by choosing the first minimum of the average mutual information function.

We apply the Wayland et al. diagnostic for different numbers of neighbours $m = 5, 10,$ and20 and with a decorrelation interval of 10. The results are shown in Figure 1. We see that an embedding dimension of 4 is suggested consistent with known values. We also notice that this value appears to persist with respect to the number of near neighbours used in the diagnostic.

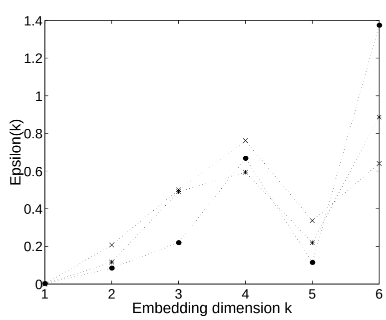

FIG. 1. A plot of $\epsilon(k)$ for various numbers of near neighbours, $\bullet$–$m = 5$, $*$–$m = 10$ and x–$m = 20$. We notice that there is no distinctive increase in the slope of $\epsilon(k)$ but a maximum occurs at $k = 4$. We thus conclude that $k = 4$ is a suitable embedding dimension which is consistent with known results.

To see how robust the diagnostic is to noise in the data we add observational noise at various levels. The noise added is zero mean Gaussian with standard deviations of $5\%, 10\%, 20\%$and40% the standard deviation of the clean Lorenz signal. In Figure 2 we show how $\epsilon(k)$ varies for $m = 10$ on each of the noisy data sets. We see that as the noise level in the data increases it becomes harder to distinguish a marked change in the slope of $\epsilon(k)$ as is expected but the drop-of in performance is quite graceful.

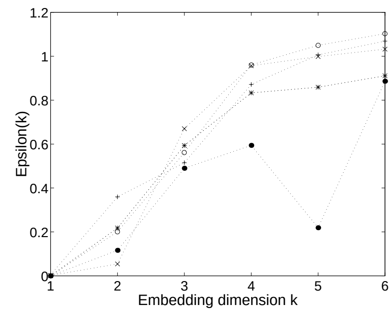

FIG. 2. A plot of $\epsilon(k)$ with $m = 10$ for data sets corrupted with observational noise. $\bullet = 0\%$, $* = 5\%$, $x = 10\%$, $o = 20\%$ and $+ = 40\%$. We see that the plateau at $k = 4$ persists for noise levels upto 10% but then degrades gracefully thereafter.

B. Input–Output time series

The first example we use to study our method is Duffing's differential equation. This equation is given by

$$\dot{u} = v$$
$$\dot{v} = u - u^3 - \epsilon v + \gamma\cos(\omega t).$$

We use parameter values which generate chaotic solutions, i.e., $\epsilon = 0.25$, $\gamma = 0.3$ and $\omega = 1.0$. We consider the system as a driven system with the input $g(t) = \cos(\omega t)$. We generate a $10,000$ point output time series by integrating the differential equations and outputing the u component every 0.05 time units after transients have diminished. The input time series is obtained by evaluating $g(t)$ every 0.05 time unit. We use a lag of $s = 26$ by locating the first minimum of the average mutual information function applied to the output time series.

In Figure 3 we ahow the result of applying our diagnostic with $m = 10$ to clean input and output data. We see that embedding the output data in two dimensions ($k = 2$) and using one input ($l = 1$) is suggested. To see how this answer persists in the presence of noise we show in Figure 4 the results of applying our diagnostic with $m = 10$ to data corrupted with 10% observational noise at *both* the inputs and outputs. We notice that the effect of the noise has caused the suggested embedding dimension to increase but even with such noisy data it was still possible to detect a suitable embedding strategy. For comparison we show in Figures 5 and 6 the results of applying the Rhodes and Morari scheme and the Cao et al. scheme to the clean data respectively. We see that both schemes suggest embedding with $(k,l) = (2,1)$ which is consistent with the values suggested by our diagnostic.

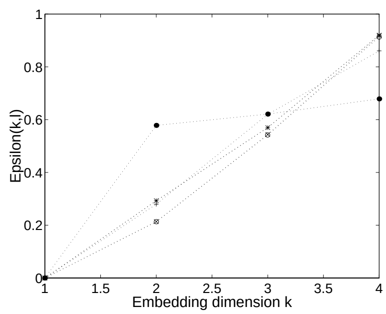

FIG. 3. A plot of $\epsilon(k,l)$ for $l = 1, \ldots, 5$. $\bullet$–$l = 1$, $*$–$l = 2$, x–$l = 3$, o–$l = 4$ and $+$–$l = 5$. We see that choosing $k = 2$ and $l = 1$ is a suitable and minimal choice for embedding the Duffing input–output time series data.

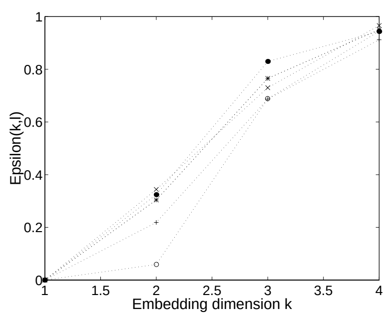

FIG. 4. A plot of $\epsilon(k,l)$ for $l = 1, \ldots, 5$. $\bullet$–$l = 1$, $*$–$l = 2$, x–$l = 3$, o–$l = 4$ and $+$–$l = 5$. We see that the effect of the noise has been to increase the dimension suggested by the diagnostic. Examining the figure we see that two strategies could be followed (i) $(k,l) = (3,1)$ or (ii) $(k,l) = (3,4)$. Since the first strategy has lower total dimension ths is the one we would use.

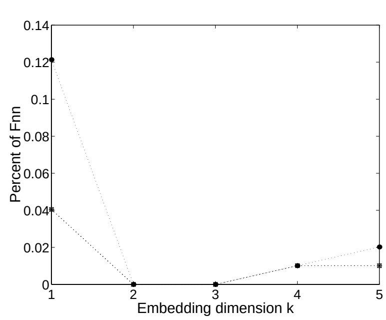

FIG. 5. A plot of the percentage of false nearest neighbours for $l = 1, \ldots, 5$. $\bullet$–$l = 1$, $*$–$l = 2$, x–$l = 3$, o–$l = 4$ and $+$–$l = 5$. We see that consistent with our diagnostic the method suggests embedding with $(k,l) = (2,1)$.

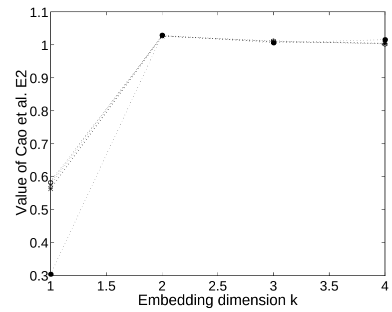

FIG. 6. A plot of Cao et al. $E2$ statistic for $l = 1, \ldots, 5$. $\bullet$–$l = 1$, $*$–$l = 2$, x–$l = 3$, o–$l = 4$ and $+$–$l = 5$. We see that consistent with our diagnostic this method suggests embedding with $(k,l) = (2,1)$.

The second example we study to compare our method uses input–ouput data obtained from a nonlinear transistor. We consider the Ebers–Moll model [9] for a BJT shown schematically in Figure 7. We obtain time series data by applying voltages across the base and emitter, and across the collector and emitter. We integrate the circuit equations and obtain the currents at I_c and I_b. For the purposes of this study we will consider the current I_c as the output data and the voltage across the collector and emitter, V_{ce} as the input data.

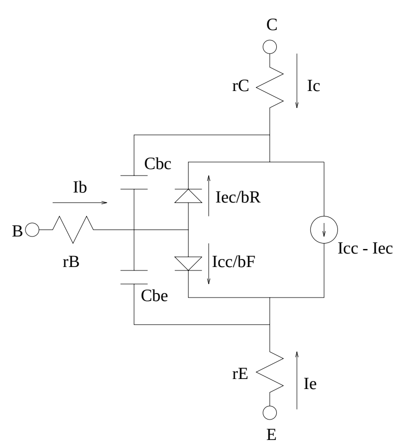

FIG. 7. Ebers-Moll transistor model.

We integrate from time zero to time $1e-6$ outputing every $1e-10$ steps. This generates approximately $10,000$ input–output data points. The voltage V_{be} consists of a fixed dc-offset plus an amplitude modulated signal given by $(1 + m\sin(\omega_m t))V_c \sin(\omega_c t)$, where $m = 4/5$, $V_c = 5V$, $\omega_m = 50MHz$ and $\omega_c = 5GHz$. The voltage V_{ce} (our input sequence) consists of a fixed dc-offset and a one–tone signal $f = 20\sin(50\pi t/T)$ where $T = 1e-6$.

In Figure 8 we show the results of applying our diagnostic with number of near–neighbours $m = 10$. Studying the figure we see that a suitable embedding strategy is to choose $k = 3$ and $l = 1$. In Figures 9 and 10 we show the results obtained by using the diagnostics of Rhodes and Morari and Cao et al. We see that the embedding strategies suggested by these two diagnostics are consistent with the results we obtained use our method.

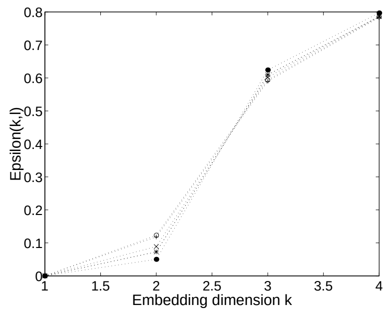

FIG. 8. A plot of $\epsilon(k,l)$ for $l=1,\ldots,5$. $\bullet$–$l=1$, $*$–$l=2$, x–$l=3$, o–$l=4$ and $+$–$l=5$. We see that choosing $k=3$ and $l=1$ is a suitable and minimal choice for embedding the BJT input–output time series data.

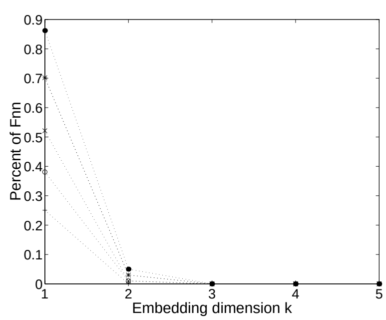

FIG. 9. A plot of the percentage of false nearest neighbours for $l=1,\ldots,5$. $\bullet$–$l=1$, $*$–$l=2$, x–$l=3$, o–$l=4$ and $+$–$l=5$. We see that consistent with our diagnostic the method suggests embedding with $(k,l)=(3,1)$.

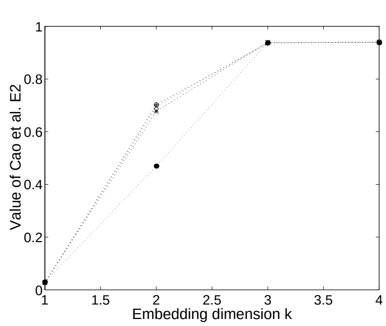

FIG. 10. A plot of Cao et al. $E2$ statistic for $l=1,\ldots,5$. $\bullet$–$l=1$, $*$–$l=2$, x–$l=3$, o–$l=4$ and $+$–$l=5$. We see that consistent with our diagnostic this method suggests embedding with $(k,l)=(3,1)$.

V. CONCLUSION

We have demonstrated that a method originally proposed by Wayland et al. to recognise determinism in a time series can also be used as a diagnostic for determining an appropriate embedding dimension for phase space reconstruction. Our contribution has extended the Wayland et al. scheme to produce a diagnostic for determining the embedding dimension for input-output time series. We have shown that the diagnostic degrades gracefully with noise and produces results consistent with those of other diagnostics.

Acknowledgements

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[1] R. Wayland, D. Bromley, D. Pickett, and A. Passamente, Recognizing determinism in a time series, *Physical Review Letters*, 70(5):580–582, 1993.

[2] C. Rhodes and M. Morari, Determining the model order of nonlinear input–output systems, *Proceedings of the American Control Conference*, Seattle, 2190–2195, 1995.

[3] L. Cao, A. Mees, K. Judd and G. Froyland, Determining the minimum embedding dimension of input–output time series data, *Int. J. Bif. Chaos*, 8(7):1491–1504, 1998.

[4] M. Casdagli, *A dynamical systems approach to modeling input-output systems.* In M. Casdagli and S. Eubank, editors, *Nonlinear Modeling and Forecasting, SFI Studies in the Sciences of Complexity*, pages 265–281, Addison-Wesley, 1992.

[5] M. B. Kennel, R. Brown and H. D. I. Abarbanel, Determining embedding dimension for phase–space reconstruction using a geometrical construction, *Physical Review A*, 45(6):3403–3411, 1992.

[6] L. Cao, Practical method for determining the minimum embedding dimension of a scalar time series, *Physica D*, 110:43–50, 1997.

[7] D. M. Walker, R. Brown and N. B. Tufillaro, Constructing transportable behavioural models for nonlinear electronic devices, *Physics Letters A*, submitted.

[8] H. D. I. Abarbanel, R. Brown, J. J. Sidorowich, and L. S. Tsimring, *Rev. Mod. Phys* **65**, 1331 1993.

[9] I. Getreu, *Modeling the Bipolar Transistor* (Tektronix Inc, Oregon, 1967).