Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

The dynamics of reputations

Bernardo A Huberman and Fang Wu

HP Labs and Stanford University, Palo Alto, CA 94304, USA
E-mail: bernardo.huberman@hp.com and fangwu@stanford.edu

Received 1 April 2004
Accepted 1 April 2004
Published 27 April 2004
Online at stacks.iop.org/JSTAT/2004/P04006
DOI: 10.1088/1742-5468/2004/04/P04006

Abstract. We study the endogenous dynamics of reputations in a system consisting of firms with long horizons that provide goods or services with varying levels of quality, and large numbers of customers who assign to them reputations on the basis of the quality levels that they experience when interacting with them. We show that for given discounts of the past on the part of the customers, and of effort levels on the part of the firms, the dynamics can lead to either well defined equilibria or persistent nonlinear oscillations in the number of customers visiting a firm, implying unstable reputations. We establish the criteria under which equilibria are stable and also show the existence of large transients which can also render fixed points unattainable within reasonable times. Moreover we establish that the timescales for the buildup and decay of reputations in the case of private information are much longer that those involving public information. This provides a plausible explanation for the rather sudden increase and collapse of reputations in a number of much publicized cases.

Keywords: game-theory (experiment)

 PII: S1742-5468(04)78677-X 1742-5468/04/P04006+17$30.00

Contents

1. Introduction

The concept of reputation and its role in society and economics has been thoroughly discussed and documented over the years. Reputations play an important role in the private enforcement of contracts [3], in deciding the level of trust in commercial exchanges [5], in setting the value of particular brands, and in deciding whom to hire or consult for professional advice. More recently, the emergence of internet mediated interactions over vastly disperse geographic locations has made the enforcement of contracts through courts of law difficult enough so as to make reputations an important alternative mechanism for the enforcement of contracts [6]. A vivid example is found in auction sites such as eBay, where both buyers and sellers assign reputations to each other on a frequent and dynamic basis [7].

That reputations play a crucial role in deciding the fate of firms and individuals has been highlighted by a number of recent high profile corporate scandals, characterized by a misrepresentation of profits to shareholders on the part of firms and executives with high reputations. While those brand names effectively prevented the close scrutiny of firms by financial analysts and regulators over a period of several years, once rumours of their financial wrongdoing started to circulate the firm's reputations suffered such sudden and severe blows that they were forced into bankruptcy by their own creditors and shareholders.

In economics, reputation effects enter naturally through game theoretic arguments, since in any repeated games with imperfect information and different types of players, reputation effects are summarized by an opponent's beliefs about a player's type. Moreover, the notion of a firm as a bearer of reputation [11, 14] has led to a number of game theoretic arguments that show the existence of equilibria in markets for reputations [2, 4, 8, 10]. These are markets where the whole firm's asset consists of its brand name reputation. Tadelis has studied the case where transactions carried out in a market for names are hidden from the potential customers of given firms. In this adverse selection model the dynamics consists of a few synchronous time steps which lead to an equilibrium in which brand trades are active [15]. Moreover, Mailath and Samuelson [13] look at a different market for names in the context of Markov perfect equilibria and solve for the equilibrium without determining its stability or the time it takes to achieve it.

While these studies have thrown light on a number of issues surrounding the nature of reputations and its value as an economic asset for the firm or individuals, they do not accurately account for the actual evolution of reputation values, their persistence or decay. Moreover, they concentrate on finding equilibria under which reputations or brands can be traded, without determining their stability. This can be problematic if it turns out that no fixed points are stable for a range of realistic parameters, or equally troubling if relaxation to equilibria takes longer than any practical timescale.

In this paper we study the endogenous dynamics of reputations in a system consisting of firms with long horizons that provide goods or services with varying levels of quality, and large numbers of customers who asynchronously assign to them reputations on the basis of the quality levels that they experience when interacting with them. Based on the reputations that customers ascribe to firms, they decide to either continue to interact with a given one or go to another one with a higher level of perceived quality. Firms can in turn react to varying levels of customer loyalty by changing the quality levels they provide, but at a cost if they decide to increase it. Conversely, firms can decrease their costs by lowering the quality of their offerings. Crucially, the firm's decision to change the level of quality is not instantaneous, as it reflects the time lags involved in collecting information about customer purchases and decisions to change the quality of their offering. In addition, customers are allowed to have imperfect memories of their past interactions with the firm.

Furthermore, we consider two different factors that are operational in the real world. First, we study scenarios where customers have private information about the firm's offerings, which gets updated as the number of interactions with the firm increases. Second, we consider the case where search is costly, leading to situations where public information about a firm's reputation is used by customers to influence their decisions on which firm to interact with.

We show that for given discounts of the past on the part of the customers, and of effort levels on the part of the firms, the dynamics can lead to either well defined equilibria or persistent nonlinear oscillations in the number of customers visiting a firm, implying unstable reputations. We establish the criteria under which equilibria are stable, and also show the existence of large transients which can also render fixed points unattainable within reasonable times. Moreover, we establish that the timescales for the buildup and decay of reputations in the case of private information are much longer that those involving public information. This latter result provides a plausible explanation for the rather sudden increase and collapse of reputations in a number of much publicized recent cases. We also determine optimal strategies that maximize given utilities of the firms.

We first consider the dynamics of reputation buildup, persistence and decay when firms have a fixed level of quality offering. We show that if customers have only private information, such buildup and decay is slow when compared to the times with which they repeatedly interact with the firms. When public information is also considered, and herding effects are included, the buildup and decay of reputations is much faster than the case of private information.

We then study the full dynamics of the system by allowing the firms to react to varying levels of customer visits by changing their quality levels, along with the increased costs incurred when improving the quality of their offerings. An important component of the dynamics is brought about by the fact that firms cannot instantaneously vary their quality when noticing a change in customer visits. The consequent delays produce unstable reputations which in some cases decay back to equilibrium in times that are very long compared to characteristic response times of the firms.

The next section sets up the model and solves for the buildup and decay in reputation in the case where firms have fixed levels of quality offerings. We show the actual dynamics of reputation growth in this rather constrained scenario. We then allow for the firms to adapt by varying their quality offerings and solve for the ensuing dynamics. Next we consider the more realistic case of delays in the firm's reactions to customer responses, and solve for the dynamics in order to establish the existence and stability of equilibria. In a further improvement of the theory, we also allow customers to have finite memories of their past interactions with the firms, and show that this can lead to oscillatory behaviour. We determine the parameter values for which equilibria can exist, and finally consider trend-following situations and study their dynamics. A final section summarizes our results and discusses implications.

2. Reputation growth and decay

In order to derive the dynamics of reputation buildup and decay, we consider a market composed of two firms, 1 and 2, and a large number of customers, who interact asynchronously with the two firms at a given rate α. We define each firm's reputation, p_i ($i = 1, 2$), as the consumer's posterior expectation that the firm provides a good quality or service product. This is no more than the probability that a customer's interaction with the firm is successful; i.e., each time a customer has access to firm i, it either succeeds (S_i) with a probability p_i or fails (F_i) with a probability $1 - p_i$ at obtaining a satisfactory result.

A series of interactions between customers and forms can then be characterized by a sequence $\{S_1, F_1, S_2, F_2\}$, where the indices 1 and 2 label the firms. Within this framework the dynamics of the customer's assessment of the level of quality of the firms is determined in part by the probability of occurrence for a given sequence.

In order to derive the dynamical equations, we assume first that the customers make their decisions independently of each other, according to their past experiences, and not relying on each other's opinions. Each time customers update their choice, they estimate the probability distributions of the two qualities, p_1, p_2, by looking at their past experience and then calculate from the distributions the probability, ρ, that p_1 is greater than p_2, which determines the probability that they will choose firm 1 over firm 2 at this time. The probability that they will choose firm 2 is $1 - \rho$. We assume that each choice is memorized so that it can be accessed later in the future.

Since initially customers have no idea about the qualities of each firm, it is natural for them to first choose evenly between them, so at $t = 0$ we have $p_1, p_2 = 1/2$. As time passes customers accumulate more experiences, which gradually guides them towards more biased decisions. Consider a given customer that has experienced a satisfactory quality m_i times, and for n_i times has had unsatisfactory experiences. We will denote this sequence of

JSTAT (2004) P04006

experiences by the notation $S_1^{m_1} F_1^{n_1} S_2^{m_2} F_2^{n_2}$. The probability for a given sequence $S_i^{m_i} F_i^{n_i}$ to happen when p_i is given can be easily calculated:

$$\Pr(S^m F^n | p) = p^m (1-p)^n. \tag{1}$$

The posterior probabilities can then be obtained from the Bayesian formula

$$\Pr(p|S^m F^n) = \frac{\Pr(S^m F^n|p)\Pr(p)}{\int \Pr(S^m F^n|p)\Pr(p)\,\mathrm{d}p}, \tag{2}$$

where $\Pr(p)$ is the prior probability. A reasonable assumption is to take the uniform distribution over $[0,1]$, i.e., $\Pr(p_i) = 1$ for both $i = 1, 2$. We thus obtain, from (1) and (2),

$$\Pr(p|S^m F^n) = \frac{p^m(1-p)^n}{B(m+1, n+1)}, \tag{3}$$

which is the standard β-distribution with parameters $m+1$ and $n+1$. This equation provides an estimate of the qualities p_i after a customer experiences m successes and n failures.

As stated before, customers makes their choice by comparing the two distributions of p_1 and p_2. The probability that they will choose firm 1 after having a given sequence of experiences $S_1^{m_1} F_1^{n_1} S_2^{m_2} F_2^{n_2}$ is determined by

$$\begin{aligned}\rho(m_1, n_1, m_2, n_2) &\equiv \Pr(p_1 > p_2)\\ &= \int_0^1 \mathrm{d}p_1 \,\frac{p_1^{m_1}(1-p_1)^{n_1}}{B(m_1+1, n_1+1)} \int_0^{p_1} \mathrm{d}p_2\, \frac{p_2^{m_2}(1-p_2)^{n_2}}{B(m_2+1, n_2+1)}\\ &= \frac{1}{B(m_1+1, n_1+1)} \int_0^1 \mathrm{d}p_1\, p_1^{m_1}(1-p_1)^{n_1} I_{p_1}(m_2+1, n_2+1),\end{aligned} \tag{4}$$

where $I_z(\alpha, \beta)$ is the *regularized incomplete beta function* [1].

We can now derive the equations for the dynamics of the customer reactions to the firm's quality offerings. Because of the probabilistic nature of this problem, the parameters m_i, n_i are in general different for each customer. However, by making a *mean field approximation*, one can replace the value of the individual parameters by their average values. It is therefore possible to speak of 'm_i and n_i as properties of the market', and they no longer need to be integers. Thus, in a time interval $\mathrm{d}t$, customers have a probability $\alpha\,\mathrm{d}t$ of updating their choice, and with probability ρ they will choose firm 1 and with probability $1-\rho$ will choose firm 2. Furthermore, among the fraction of customers $\alpha\,\mathrm{d}t\,\rho$ that choose firm 1, p_1 customers will have a satisfactory experience. This gives the average increment of m_1 as

$$\mathrm{d}m_1 = \alpha\rho p_1\,\mathrm{d}t, \tag{5}$$

and similarly for n_1, m_2 and n_2, which can be easily turned into differential equations. We thus have

$$\frac{\mathrm{d}m_1}{\mathrm{d}t} = \alpha\rho p_1, \tag{6}$$

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = \alpha\rho(1 - p_1), \tag{7}$$

$$\frac{\mathrm{d}m_2}{\mathrm{d}t} = \alpha(1 - \rho)p_2, \tag{8}$$

$$\frac{\mathrm{d}n_2}{\mathrm{d}t} = \alpha(1 - \rho)(1 - p_2), \tag{9}$$

where $\rho = \rho(m_1, n_1, m_2, n_2)$ and $p_i = p_i(t)$ are in general time dependent.

The fraction f of customers that choose firm 1 at any given time can also be determined. In the time interval $\mathrm{d}t$,

$$\begin{aligned}\mathrm{d}f &= (\text{customers shifting from 2 to 1}) - (\text{customers shifting from 1 to 2}) \\ &= \alpha\,\mathrm{d}t\,(1 - f)\rho - \alpha\,\mathrm{d}t\,f(1 - \rho) \\ &= \alpha\,\mathrm{d}t\,(\rho - f).\end{aligned} \tag{10}$$

Thus the dynamics of f is governed by the Huberman–Hogg equation [9]

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \alpha(\rho - f), \tag{11}$$

which along with equations (4), (6)–(9), fully describes the customer dynamics.

The firm dynamics, i.e., how the quantities $p_i(t)$ vary with time, will be discussed in the next section. If we first assume that their quality values are constant, the above equations can be numerically solved under given initial conditions. For example, if at $t = 0$ there is no prior customer experience of the firm's qualities, one has $m_1 = n_1 = m_2 = n_2 = 0$, and $f = 1/2$; the solution for constant firm qualities $p_1 = 0.7$, $p_2 = 0.3$ evolves, as shown in figure 1. As can be seen, f builds up gradually in units of the time it takes for customers to update their choices. Equally importantly, the variance of the distribution decreases with the number of experiences. The same behaviour applies to the way reputations dissipate in time.

While the evolution of reputation building that we just displayed assumed that the prior probabilities $\Pr(p)$ are uniform over $[0, 1]$, the theory can easily incorporate other prior probability distributions. For example, if the quality is unlikely to be extremely high or low (not likely to be near 0 or 1), a normal distribution around some centre value might be a more suitable approximation to the prior probabilities, which also yields the same slow buildup and decay.

Before closing it is important to stress that an essential aspect of this model is the posterior distribution interpretation of reputation [13]. The fact that qualities are described by a distribution rather than a single number means that customers choose on the basis of the perceived mean and variance of a reputation rather than an absolute number that they use to compare several firms. Thus the choice of an older firm might be due to its having a long lived satisfactory mean value of its quality and, perhaps more importantly, a small variance associated with it.

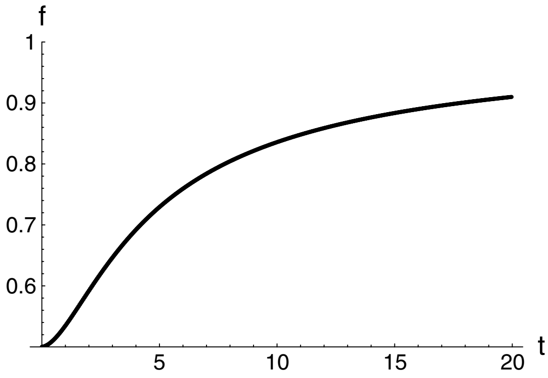

Figure 1. The time evolution of f, the fraction of customers accessing firm 1, for $p_1 = 0.7, p_2 = 0.3$. Time is in units of α^{-1}. The dynamics was solved by a C program using a fourth-order Runge–Kutta method.

2.1. Memory

In the formulation given above, the past experiences of consumers are weighted equally, regardless of whether they took place yesterday or one year ago. In most instances, however, memory effects are such that recent experiences have greater importance in determining a customer's sense of the quality of the firm than past experiences. Accordingly, we now modify our model so that interactions with firms that took place at earlier times are discounted at a faster rate than those that took place recently. At a given time $t < \tau$ the increment of m_1 is still given by

$$\mathrm{d}m_1 = \alpha\rho p_1 \,\mathrm{d}t, \tag{12}$$

but in an interval $\mathrm{d}t$ at time $t > \tau$, while customers gain new experiences, they also forget their former experiences beyond a past time $t - \tau$:

$$\mathrm{d}m_1 = (\alpha\rho p_1)(t)\,\mathrm{d}t - (\alpha\rho p_1)(t-\tau)\,\mathrm{d}t. \tag{13}$$

Dividing by $\mathrm{d}t$ on both sides gives the differential-delay equation

$$\frac{\mathrm{d}m_1}{\mathrm{d}t} = \alpha\left[\rho(t)p_1(t) - \rho(t-\tau)p_1(t-\tau)\right] = \alpha\left[\rho p_1\right]_{t-\tau}^{t}. \tag{14}$$

The equations for the other history variables n_1, m_2, n_2 can be derived in the same way, thus yielding

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = \alpha[\rho(1-p_1)]_{t-\tau}^{t}, \tag{15}$$

$$\frac{\mathrm{d}m_2}{\mathrm{d}t} = \alpha[(1-\rho)p_2]_{t-\tau}^{t}, \tag{16}$$

$$\frac{\mathrm{d}n_2}{\mathrm{d}t} = \alpha[(1-\rho)(1-p_2)]_{t-\tau}^{t}, \tag{17}$$

while the customer dynamics remains unchanged:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \alpha[\rho(m_1, n_1, m_2, n_2) - f], \tag{18}$$

since all the history variables m_i, n_i take their values at time t.

Notice that in equilibrium all variables are constant. For example, we have

$$m_1 = \int_0^\tau \alpha\rho p_1 \,\mathrm{d}t = \alpha\tau\rho p_1 = N\rho p_1, \tag{19}$$

where $N = \alpha\tau$ is the number of customer experiences in a period τ. Similarly, we have for other history variables that

$$n_1 = N\rho(1 - p_1), \tag{20}$$

$$m_2 = N(1 - \rho)p_2, \tag{21}$$

$$n_2 = N(1 - \rho)(1 - p_2). \tag{22}$$

3. Adaptive dynamics of the firm

In the previous section we derived the dynamics that govern reputation buildup and decay assuming that the firm's strategies are static, i.e., the quality of their offerings does not change with time. Since this is a rather unrealistic assumption, we now remedy this shortcoming by incorporating the firm's reactions to customer responses as they experience different levels of quality.

It is often the case that firms tend to decrease the quality of their offerings as their earnings decrease. Since a firm's income grows with the number of its customers, and an increase in quality tends to imply added costs, a reasonable choice for the firm's utility function is

$$G = \text{income} - \text{cost} = C(f - rp), \tag{23}$$

where f is the fraction of customers choosing a specific firm, p is the firm's quality, r is the effort incurred in achieving a given quality level, and C is a constant. The value of r indicates how easily a firm can achieve a high quality offering. If r is large, the firm has to invest a large amount, whereas a small value of r implies small effort to achieve high quality. We can thus speak of a firm as competent (in the sense of Mailath and others) when it has a lower value of the parameter r.

The utility function given above determines the strategies that maximize the firm's utility. Since the only variable a firm can adjust is its quality p, an increase in its value will eventually lead to an increase in the number of customers purchasing services from the firm. But that increase will be offset by the extra cost incurred in increasing quality levels. On the other hand, decreasing costs by decreasing quality will also lower its utility, as customers that experience a degradation in quality stop purchasing from the firm. Given these two competing tendencies, the strategy dynamics can be easily derived under the following assumptions.

(1) The firm will not increase its quality if it already dominates the market, i.e., $f = 1$ for firm 1 or $f = 2$ for firm 2.

(2) The firm will not decrease its quality if it is already at its lowest value, i.e., $p = 0$.

The dynamics is then determined by

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = \beta_1(s_1(1-f) - p_1), \tag{24}$$

$$\frac{\mathrm{d}p_2}{\mathrm{d}t} = \beta_2(s_2 f - p_2), \tag{25}$$

where $s_{1,2}$ are two weight factors describing whether the firms are more inclined to improve their qualities or to cut down their expenses. These weight factors determine the 'strategy' parameters and also the gains, when in equilibrium. The other two parameters, β_1 and β_2, describe the rate and magnitude of quality adjustment.

Because the probability of success can never exceed 1, it is necessary to introduce a cutoff when $p > 1$. The cutoff version of equations (39) and (40) together with the customer dynamical equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \alpha(\rho - f) \tag{26}$$

fully describe the dynamics of the problem.

In order to study the evolution of the firms and their interactions with the customers, we first note that the standard equilibrium can be obtained by setting all derivatives to zero:

$$p_1^0 = s_1(1 - f^0), \tag{27}$$

$$p_2^0 = s_2 f^0, \tag{28}$$

$$f^0 = \rho(m_1^0, n_1^0, m_2^0, n_2^0), \tag{29}$$

where the superscript 0 indicates values at equilibrium that do not change with time. We thus obtain an equilibrium given by

$$m_1^0 = N(1 - f^0)p_1^0, \tag{30}$$

$$n_1^0 = N(1 - f^0)(1 - p_1^0), \tag{31}$$

$$m_2^0 = N f^0 p_2^0, \tag{32}$$

$$n_2^0 = N f^0 (1 - p_2^0), \tag{33}$$

where $N = \alpha\tau$ is the number of measurements in a delay period. Thus equation (29) can be written in the form

$$f^0 = \rho^0(f^0, p_1^0, p_2^0). \tag{34}$$

It is clear that f^0 can be solved from equations (27), (28), and (34). We then obtain the formal result

$$f^0 = f^0(s_1, s_2). \tag{35}$$

The gains at equilibrium are given by

$$G_1^0 = f^0 - r_1 p_1^0 = 1 - (1 + r_1 s_1)(1 - f^0(s_1, s_2)), \tag{36}$$

$$G_2^0 = 1 - f^0 - r_2 p_2^0 = 1 - (1 + r_2 s_2) f^0(s_1, s_2). \tag{37}$$

Thus for fixed $r_{1,2}$, $G_{1,2}$ are also functions of $s_{1,2}$. The values $s_i(r_1, r_2)$ that maximize G_i determine the best strategy for the ith customer, which in principle can be solved from the equations

$$\frac{\partial G_1^0}{\partial s_1} = 0, \qquad \frac{\partial G_2^0}{\partial s_2} = 0. \tag{38}$$

As a specific example we now study two firms: one that produces good quality products with $r = 0.5$ and one that produces bad quality products with $r = 0.9$. For the case in which two firms have close strategies $s_1 = 1.0$ and $s_2 = 0.75$, the corresponding gains are $G_1 = 0.32$ and $G_2 = 0.08$ (figure 2(a)). As the good firm adjusts its strategy to a value of $s_1 = 2.0$, it dominates the market and beats the bad firm (figure 2(b)). If on the other hand the good firm sets its strategy value to $s_1 = 0.8$, the bad firm gains more than in the case $s_1 = 1.0$ (figure 2(c)). We thus see that a good firm tends to increase its strategy parameter, or it cares about its reputation. On the other hand, if the bad firm sets its strategy to $s_2 = 0.3$, its gain rises significantly, as seen in figure 2(d). In other words, it does not value its reputation as much as the good firm, a conclusion consistent with that of Tadelis [15]–[17] and Mailath and Samuelson [12, 13].

4. Unstable reputations

We have previously shown that history discounting on the part of the customers implies a time delay in their dynamics. In fact, time delays also arise naturally in the dynamics of the firms. When a firm adjusts its quality according to its market position (as determined by the fraction, f, of customers that access it), the measured value of f is always that of an earlier time $t - \tau$. This is because it is seldom possible for a firm to obtain aggregate data instantaneously. Replacing $f(t)$ with $f(t-\tau)$ in the equations for the firm strategies, we thus have

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = \beta_1[s_1(1 - f(t-\tau)) - p_1], \tag{39}$$

$$\frac{\mathrm{d}p_2}{\mathrm{d}t} = \beta_2[s_2 f(t-\tau) - p_2]. \tag{40}$$

The behaviour generated by these equations is shown in figures 3(a) and (b), where the delay τ varies from 10.0 to 20.0, and $r_1 = 0.2, r_2 = 0.5$. As can be seen, as the information gathering delay τ gets larger, oscillations become prominent, while in those situations where equilibria exist it takes longer for the system to relax back to the fixed point. For sufficiently large values of τ oscillations can grow in amplitude, thus leading to an unstable fixed point. In such situations equilibrium can never be reached, making it harder for a firm to find an optimum strategy.

Since in practice no firm desires to function in a fluctuating market, it is helpful to find a way of controlling these nonlinear oscillations. We have already seen that while the

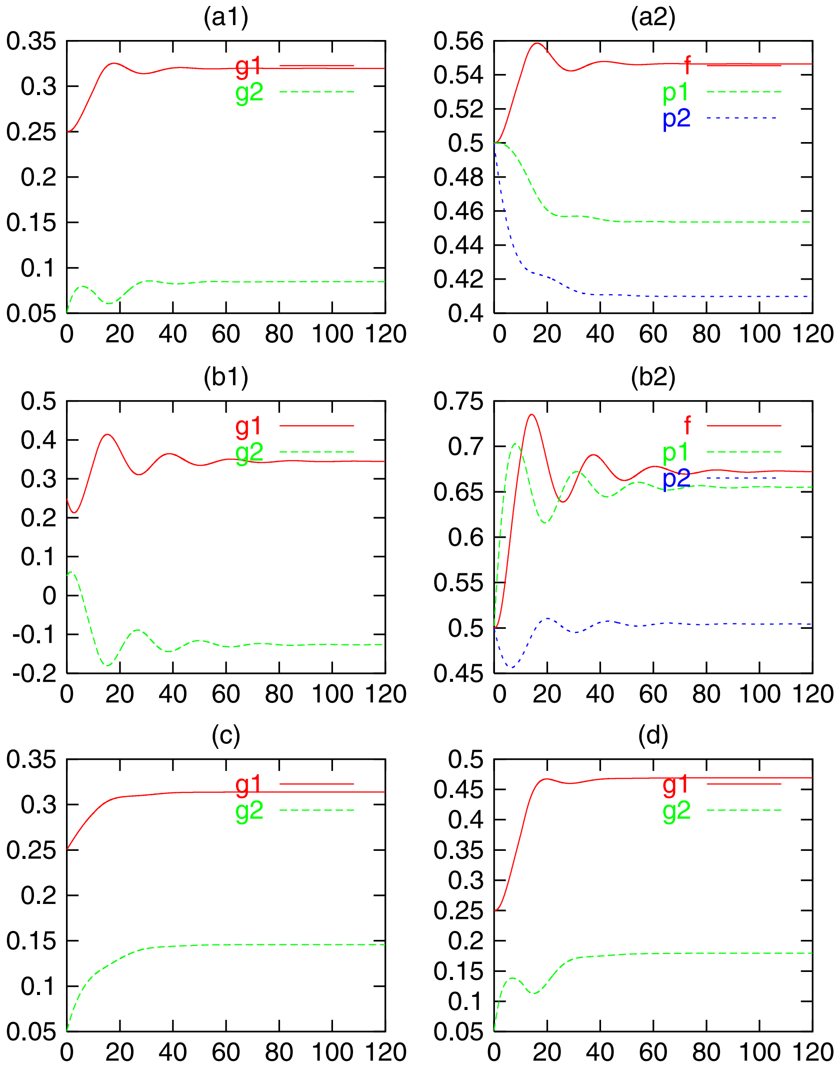

Figure 2. The gain and quality of firms' offerings as a function of time for a number of different parameter values. In all the figures $\beta_1 = \beta_2 = 0.1, \tau = 10.0$. (a) $s_1 = 1.0, s_2 = 0.75$; (b) $s_1 = 2.0, s_2 = 0.75$; (c) $s_1 = 0.8, s_2 = 0.75$; (d) $s_1 = 1.0, s_2 = 0.3$. Notice how the gains of the firms change as the parameter values change.

strategy parameter determines the equilibrium values (or central values of the oscillations if equilibrium cannot be attained) it does not regulate the amplitudes or the periods. This implies that adjustments to the strategy parameter, which do cause a shift in the central value of the oscillations, will not make them disappear.

These considerations led us to study the effect of a reduced rate parameter, β_1, on the dynamics of the system. As shown in figure 3(c), we see that when β_1 is reduced by a factor of 10 from 0.1 to 0.01, the oscillations do indeed disappear. In other words, the rate parameter provides the firms with a mechanism to dampen the unwanted nonlinear oscillations. In spite of this positive effect, it is important to point out that a firm cannot reduce the rate parameter arbitrarily, for at low values of r the system reaches equilibrium after a long transient, thus preventing a firm from quickly dominating the market.

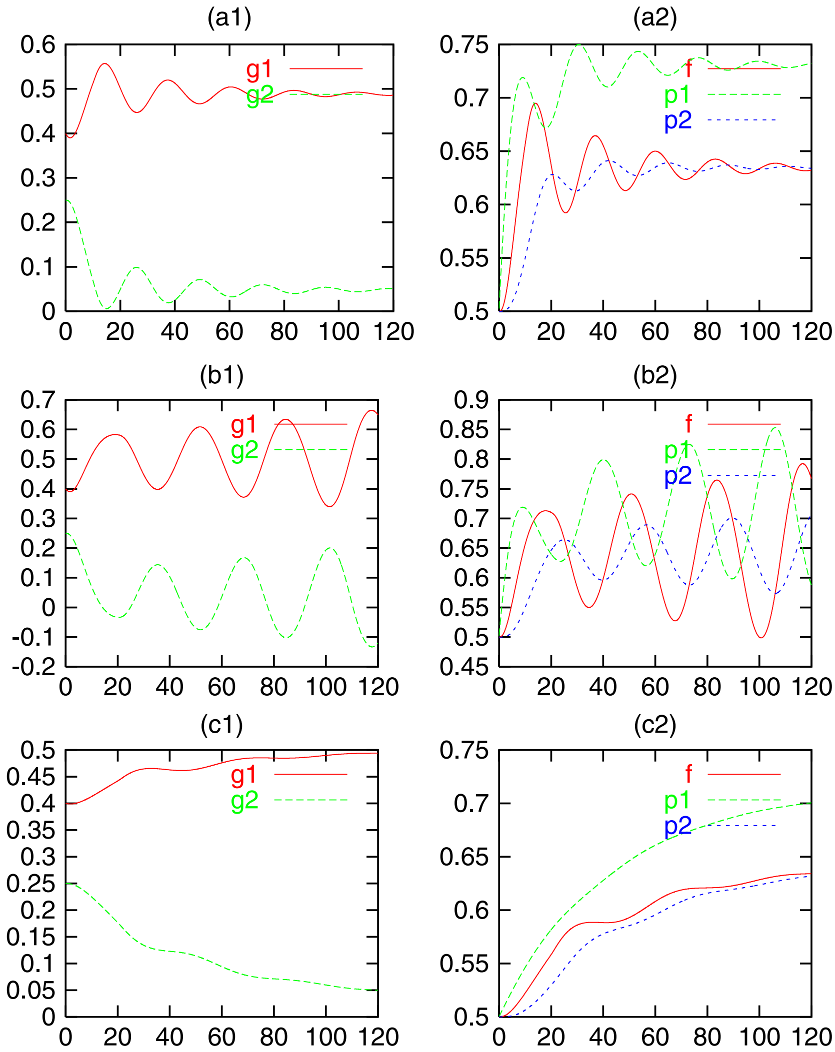

Figure 3. The gain and quality for different parameter values. In all the figures $s_1 = 2.0, s_2 = 1.0$. (a) $\beta_1 = \beta_2 = 0.1, \tau = 10.0$. (b) $\beta_1 = \beta_2 = 0.1, \tau = 20.0$. (c) $\beta_1 = 0.01, \beta_2 = 0.1, \tau = 20.0$. As can be seen, unstable oscillations can occur, which can be stabilized by adjusting the rate parameters.

5. Equilibria and instability

In the previous section we showed that for particular values of the parameters, the dynamics of adaptive firms interacting with their customers may undergo significant nonlinear oscillations. This implies that the equilibrium fixed point found under the assumption that all time derivatives are set to zero is no longer stable. Given the fact that delays in changing quality levels in response to customer visits are unavoidable, and that memory effects are often present, it is important to find out the range of parameter values for which stable equilibria do exist.

Before doing so, it is important to point out that while the notion of a stable equilibrium implies that a system perturbed away from its initial values relaxes back to the fixed point, it says nothing about the time that it takes for the transient change to die away. From a practical point of view, the relaxation time back to equilibrium needs to be shorter than a characteristic lifetime cycle of the firm, since otherwise the notion of equilibrium would make little sense. While transients are usually discarded in the study of equilibria, they can play an important role if they are long enough so as to change the resulting dynamics on a short timescale.

Based on those considerations, we examined the stability of equilibria in our theory by proceeding in the following fashion. For a given set of parameters $(\beta_1, \beta_2, s_1, s_2, \tau)$, we added a small perturbation to the equilibrium fixed point at $t = 0$ and then observed the time evolution of the system. If the system converged within a 'lifetime' t_L in the sense that its oscillation amplitude attenuates below a threshold value (e.g., one half of the original magnitude of the perturbation), we decided that the system is stable. Otherwise it was considered unstable.

Since the oscillations of nonlinear dynamical systems are often caused by time delays, one expects the system dynamics to become unstable for sufficiently large values of τ. We thus denote the critical time delay above which the system becomes unstable as τ_{C1}. On the other hand, the bigger the time delay, the larger are the reputation parameters (recall that $m_1^0, n_1^0, m_2^0, n_2^0$ are all proportional to $N = \alpha\tau$) and the stronger is the reputation decay.

Since reputations can dissipate gradually as this mechanism becomes strong enough, the system becomes increasingly immune to perturbations, which suggests the existence of another critical time delay, τ_{C2}, above which the system becomes stable again. This is indeed what our stability analysis reveals.

As a specific example we take $\beta_1 = \beta_2 = 0.1, s_2 = 1$, and examine the system stability for various s_1 and τ. A perturbation of magnitude $f_1 = 0.05$ is added to the equilibrium fixed point at $t = 0$. The system is then allowed to evolve for a time $t_L = 200$ (in time units of α^{-1}). If the magnitude of the oscillations, f, around t_L attenuates to less than 0.01 (20% of initial f_1) we regard the system as stable.

The results are shown in figure 4. As we can see from the figure, the two curves τ_{C1} and τ_{C2} meet approximately at $s_1 = 1.05$. For values of $s_1 < 1.05$ the system is stable regardless of the value of τ. It is also seen that the larger the value of s_1, the more likely it is for the system to be unstable. This fact can be easily explained by noticing that s_1 measures how much firm 1 values its customer base, f. If s_1 is large, firm 1 makes great efforts to adjust its quality, thus leading to oscillations and thus its unstable reputation.

6. Reputations and trend following

So far we have studied the dynamics of reputations based on the assumption that individuals independently access a given resource and assess the quality of their offering. Within that private information model, the number of prior positive or negative experiences determines the individuals' future preferences, which then affect the overall dynamics of the firms and their customers. As we saw, the buildup and decay of reputations takes place over long times compared to the times at which individuals interact with firms, and in many cases reputations become unstable because of delays or memory discounting on the part of the customers.

There is yet another mechanism that contributes to the dynamics of customer access to firms, and which relies not only in the individuals' private experiences but also on

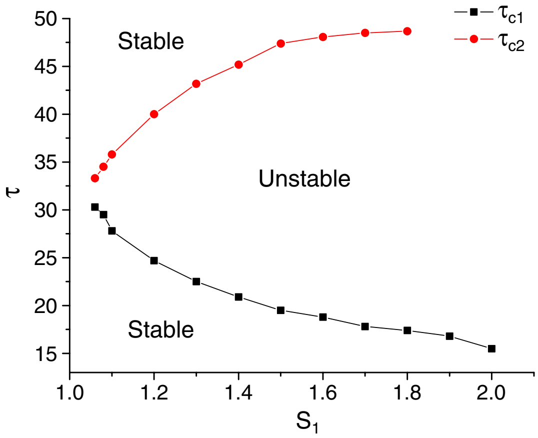

Figure 4. The critical time delay for changing s_1.

interactions among customers who already ascribe a reputation to a given firm. When a search for particular services or goods is costly, recommendations and trend following can lead to the effective choice of a firm or services, and at very small cost. For example, if someone wants to buy a car they might first consult their friends or determine a popular brand before making a decision on which brand to choose. Perhaps because of their costless nature, it is evident that other people's actions and opinions may exert a great influence on a customer, thus contributing to the overall dynamics of reputations in the market.

In order to study the dynamics of reputations with the inclusion of trend-following effects, we once again consider the case of two firms and a number of customers, of which a fraction f choose one firm or the other at any given time. We assume again that the customers reevaluate their choices at a given rate α and also determine their private assessment of the quality of a firm. Moreover, because of imperfect information, their assessments may differ from their actual values. In what follows, rather than assuming a uniform distribution of *a priori* quality values, we take the perceived qualities to be normally distributed, with standard deviation σ, around their actual values p. In terms of the qualities and uncertainty σ, the probability that a customer will prefer firm 1 over firm 2 when they make a choice is given by

$$\rho = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{p_1 - p_2}{2\sigma}\right)\right), \tag{41}$$

while the dynamics is again governed by the Huberman–Hogg equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \alpha(\rho - f). \tag{42}$$

If trend following is also taken into account, an additional term must be added to the perceived quality, and which we take to be proportional to the fraction of customers who choose firm i $(i = 1, 2)$ at any given time. Notice that this simple assumption captures the requirement that a customer is more likely to prefer a product that is preferred by the majority of other customers. The probability ρ thus becomes

$$\rho = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{(1-r)(p_1 - p_2) + r(f_1 - f_2)}{2\sigma}\right)\right), \tag{43}$$

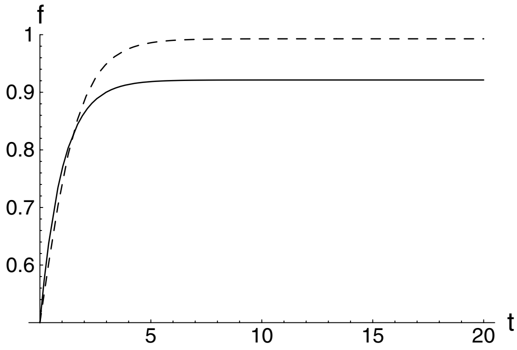

Figure 5. The fraction of customers accessing a given firm as a function of time for several scenarios. The solution to the trend-following model is plotted here as a thin curve, for which we keep the values $p_1 = 0.7, p_2 = 0.3$ and set $\sigma = 0.2$, with no time delay ($\tau = 0$) and no reputation effect ($r = 0$). The behaviour of a trend-following market with $r = 0.5$ is plotted as the dashed curve. It can be seen that f in the latter case is built up much faster.

where r is a weight factor that denotes the significance of the reputation effect. If $r = 0$ the market is recommendation independent, while if $r = 1$ it is recommendation dominated. The dynamics of the customers (42) remains the same as above. Finally, in order to include time delays in information, the qualities and fraction that enter into ρ at time t should be the corresponding values at a delayed time $t - \tau$.

In the limit where the firms do not make any effort to adjust their qualities, the market dynamics is fully described by (42) and (43).

In more realistic scenarios, however, the firms adjust their qualities according to their actual gains. This can be achieved in two ways: either by increasing their income or by decreasing their costs. Assuming again that a firm's utility is proportional to the fraction of customers they have, and that the costs are proportional to their product's quality, we can write the dynamical equations of the two firms as

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = \beta_1[(1 - s_1)(1 - f) - s_1 p_1], \tag{44}$$

$$\frac{\mathrm{d}p_2}{\mathrm{d}t} = \beta_2[(1 - s_2)f - s_2 p_2], \tag{45}$$

where β_i $(i = 1, 2)$ are the rates at which the firms adjust their qualities, and s_i $(i = 1, 2)$ are two weight factors describing whether the firms are more inclined to improve their qualities or to cut down their costs. For example, if $s_1 = 0$ firm 1 tends to improve its quality whenever the fraction of clients choosing its product is less than 1.

The dynamics generated by this model has the same qualitative features as the one based on private information alone, but with reputation growth and decay changing over much faster scales. This is because the trend-following dynamics is proportional to f, whereas reputation building due to private information leads is based on the record of the agent's past performance, which is independent of f.

When the delays and uncertainty on the part of the customers are fairly small, the system converges to an equilibrium point, as was shown before. As the information available to the customers becomes more corrupted (increasing the value of σ), the equilibrium point moves away from its optimal value. With increasing delays, this equilibrium eventually becomes unstable, leading to oscillatory behaviour similar to the one exhibited by the system when only private information was available. In these cases, the number of customers accessing a given firm continues to vary so that the system spends relatively little time near the optimal value, with a consequent drop in its overall performance and unstable reputations dominating the dynamics.

This behaviour provides an explanation for the very sudden loss of reputations that very large corporations have suffered recently, and which in light of the earlier theories one would have expected to decay very slowly. As rumours spread about the lack of confidence that customers are expressing about a firm, trend-following effects can dominate and lead to a collapse of the firm's reputation as measured by the number of customers doing business with it.

Finally, we point out that an interesting consequence of this dynamics is that follow-the-trend mechanisms are such that a sudden finite change of p_i will induce a sudden change in ρ. This is because in this case p_i enter the expression of ρ directly:

$$\rho = \frac{1}{2}\Big(1 + \mathrm{erf}\Big(\frac{p_1 - p_2}{2\sigma}\Big)\Big), \tag{46}$$

whereas in the scenario discussed in section 2, if the qualities p_i undergo a sudden finite change, ρ will not change suddenly, since the p_i affect ρ indirectly via the parameters m_i, n_i, which themselves have smooth changes.

7. Conclusions

In this paper we have presented an endogenous dynamical theory of reputations in a system consisting of firms that provide goods or services with varying levels of quality, and large numbers of customers who assign to them reputations on the basis of the quality levels that they experience when interacting with them. Based on the reputations that customers ascribe to firms, they decide to either continue to interact with them or go to another one with a higher level of perceived quality. Firms can in turn react to varying levels of customer loyalty by changing the quality levels they provide, but at a cost if they decide to increase it. Conversely, firms can decrease their costs by lowering the quality of their offerings. Crucially, the firm's decision to change the level of quality is not instantaneous, as it reflects the time lags involved in collecting information about customer purchases and decisions to change the quality of their offering. In addition, customers are allowed to have imperfect memories of their past interactions with the firm.

Furthermore, we considered situations where customers have private information about the firm's offerings, and which gets updated as the number of interactions with the firm increases, as well as trend-following situations. In the latter, public information about a firm's reputation is used by customers to influence their own decisions of which firms to interact with.

We showed that for given memory horizons on the part of the customers, and of effort levels on the part of the firms, the dynamics can lead to either well defined equilibria or persistent nonlinear oscillations in the number of customers visiting a firm, implying unstable reputations. We established the criteria under which equilibria are stable and also showed the existence of large transients which can also render fixed points unattainable within reasonable times. Moreover, we showed that the timescales for the buildup and decay of reputations in the case of private information are much longer that those involving public information. This latter result provides a plausible explanation for the rather sudden increase and collapse of reputations in a number of much publicized cases.

Acknowledgments

Fang Wu thanks Chenyang Wang for sharing his ideas on the probability description of reputations. This material is based upon work supported by the National Science Foundation under Grant No 9986651.

References

[1] Abramowitz A and Stegun I A, 1970 *Handbook of Mathematical Functions* (New York: Dover)

[2] Bergemann D and Valimaki J, *Entry and innovation in vertically differentiated markets*, 1999 *Yale Cowles Foundation Discussion Paper* 1226 (June 1999) pp 45ff

[3] Bernstein L, *Opting out of the legal systems: extralegal contractual relations in the diamond industry*, 1992 *J. Legal Stud.* **21** 115

[4] Diamond D W, *Reputation acquisition in debt markets*, 1989 *J. Polit. Econ.* **97** 828

[5] Doney P M and Cannon J P, *An examination of the nature of trust in buyer–seller relationships*, 1997 *J. Mark.* **61** 35

[6] Friedman D, *Contracts in cyberspace*, 2000 *American Law and Economics Association Mtg (May 2000)*

[7] Friedman E and Resnick P, *The social cost of cheap pseudonyms*, 2001 *J. Econ. Manag. Strateg.* **10** 173

[8] Fudenberg D and Tirole J, 2000 *Game Theory* (Cambridge, MA: MIT Press) ch 9

[9] Huberman B A and Hogg T, *The behavior of computational ecologies*, 1988 *The Ecology of Computation* ed B A Huberman (Amsterdam: North-Holland) pp 77–115

[10] Kreps D and Wilson R, *Reputation and imperfect information*, 1982 *J. Econ. Theory* **27** 253

[11] Kreps D, *Corporate culture and economic theory*, 1990 *Perspectives on Positive Political Economy* ed J Alt and K Shepsle (Cambridge: Cambridge University Press) pp 90–143

[12] Mailath G J and Samuelson L, *Your reputation is who you're not, not who you'd like to be*, 1998 *Working Paper* University of Pennsylvania and University of Wisconsin

[13] Mailath G J and Samuelson L, *Who wants a good reputation?*, 2001 *Rev. Econ. Stud.* **68** 415

[14] Milgrom P and Roberts J, 1992 *Economics, Organization and Management* (Englewood Cliffs, NJ: Prentice-Hall)

[15] Tadelis S, *What's in a name? Reputation as a tradeable asset*, 1999 *Am. Econ. Rev.* **89** 548

[16] Tadelis S, *The market for reputations as an incentive mechanism*, 2002 *J. Polit. Econ.* **92** 854

[17] Tadelis S, *Firm reputation with hidden information*, 2003 *Economic Theory* **21** 635–51