A Quantum Treatment of Public Goods Economics

Kay-Yut Chen Tad Hogg Raymond Beausoleil

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Abstract

Quantum generalizations of conventional games broaden the range of available strategies, which can help improve outcomes for the participants. With many players, such quantum games can involve entanglement among many states which is difficult to implement, especially if the states must be communicated over some distance. This paper describes a quantum approach to the economically significant n-player public goods game that requires only two-particle entanglement and is thus much easier to implement than more general quantum mechanisms. In spite of the large temptation to free ride on the efforts of others in the original game, two-particle entanglement is sufficient to give near optimal expected payoff when players use a simple mixed strategy for which no player can benefit by making different choices. This mechanism can also address some heterogeneous preferences among the players.

1 Introduction

Quantum information processing provides a variety of new capabilities with potentially significant performance improvements over conventional techniques. One example is quantum computation with its ability to rapidly solve problems, such as factoring [1], which appear to be otherwise intractable. However, implementing machines with enough bits and coherence time to solve problems difficult enough to be of practical interest is a major challenge. Another application, quantum cryptography, is feasible today for exchanging keys over distances of tens of kilometers. A third area, which potentially can come into practical use soon, is quantum economic mechanisms and games. Extending classical games into the quantum realm broadens the range of strategies [2], and has been examined in the context of the Prisoner's dilemma [3, 4, 5, 6] and the n-player minority game [7]. Quantum games do not require long sequences of coherent operations and hence are likely to be easier to realize than large-scale quantum computations.

In this paper, we present a quantum version of an important social dilemma: public goods. Provisioning for public goods is a well-studied social choice problem. A typical example is a group deciding whether to provide a common good, such as a park, in the face of potential free riders. The free rider problem [8] cannot be solved with traditional means without either a third party to enforce agreements or a repeated game scenario in which participants can self-police. Government is one typical solution. While government is a good solution to public goods involving a large population such as national defense, it is inefficient for public goods in smaller groups such as neighborhood watch. Provision of these smaller scale public goods often relies on altruism and other weaker incentives. Invariably, contributions to these public goods are not at efficient levels.

Quantum mechanics offers the ability to address the free-rider problem in the absence of a third party enforcer in a single shot game, i.e., without repetition. With suitable design, simple mixed strategies almost entirely avoid the free rider problem in the context of the public goods game and give expected performance close to the Pareto efficient value when the size of the group is large. In our case, the power of the quantum mechanism comes from entanglement. Quantum entanglement allows individuals to pre-commit to agreements where otherwise it would be individually rational to renege.

Three different quantum mechanisms with different degrees of entanglement are reported in this paper. These differ in their performance characteristics and ease of implementation. The results provide information about how one can best design a quantum mechanism to achieve an efficient outcome in the public good game.

Beyond their economic properties, an important issue for proposed quantum mechanisms is whether they can be built. In particular, creating and communicating highly-entangled states among n players poses significant implementation challenges. We address this issue by restricting attention to quantum systems that can be practically implemented by technologies likely to soon be available, namely those that only require entanglement among pairs of states. Thus an interesting practical question for applying quantum information processing to economically-relevant games is whether quantum mechanisms limited to using two-particle entangled states can significantly improve performance compared to the original game. For a specific example of the public goods game, we present a such mechanism and evaluate its performance. This mechanism could be feasible to implement in the near future even for players at different locations.

The paper is organized as follows. Sec. 2 describes the general approach to "quantize" a classical game. Sec. 3 discusses the economics of the public goods game. Sec. 4 describes the quantum version of the public goods game and its solutions. Sec. 5 concludes with possible extensions to our mechanism.

2 Quantum Games

This section describes one approach to generalize conventional games to make use of entangled quantum states. We then discuss issues involved in their implementation, particularly the significant benefit for games restricted to use only two-particle entanglement.

2.1 Creating Quantum Games

A game consists of a set S of choices available to the players and an associated payoff to each player based on those choices. With $s_k \in S$ denoting the choice made by player k and $s = (s_1, \ldots, s_n)$. A game is defined by the payoffs to the players depending on these choices, i.e., $P_k(s)$.

One approach [3, 7] for generalizing these games to quantum operators considers superpositions of all possible choices $\sum_s \psi_s |s\rangle$ summing over all choices in S for each player. The quantum version of the game starts by creating an initial superposition. Subsequently each player is allowed to operate only on their corresponding part of the state. In the last stage, the final superposition is used to produce a definite choice for each player via a further joint operation followed by a measurement. The initial and final operations, acting on the full superposition, are fixed and known to the players.

To give a direct generalization of the original game, the player's operations should include choices that correspond to the original choices. That is, the initial and final operations on the state should reproduce the payoff structure for the original game if all players restrict their individual operations to just those corresponding to the actions allowed in the original game.

More precisely, the game proceeds as follows:

- Starting with a particular initial superposition v, create the entangled state Jv, where J is an entanglement operator that commutes with the classical single-player operators.

- Players select an operation to apply to their part of the superposition, giving $v' = (U_1 \otimes \ldots \otimes U_n) Jv$ where U_k is operator used by player k.

- Finally undo the initial entanglement, giving $\psi = J^\dagger v'$. For a given game, i.e., choices for v and J, the final superposition is a function of the players' choices, i.e., $\psi(U_1, \ldots, U_n)$.
- Measure the state, giving a specific value for each player's choice. The probability to produce choices s (i.e., a particular assignment, 0 or 1, to each bit) is $|\psi_s|^2$.

The choice of J determines the type and amount of entanglement among the players. The commutation condition on J ensures that if each player selects the operator corresponding to one of the choices in the original game, the final result of the quantum game will, with probability 1, reproduce those choices.

2.2 Implementing Entanglement for Many Players

Ideally, we would like our scheme to rely on the distribution of entangled states between distant players, implying that the qubits are encoded in the polarization states of photons transmitted throughout a fiber-optic network. Given a bright source of polarization-entangled photon pairs [9], these qubits can be delivered by propagation through optical fibers, and purified using high-quality linear optical elements [10]. In principle, maximally entangled n-photon states can be constructed from entangled two-photon states [11, 12], and these states can be further manipulated using linear optical elements to perform universal gate operations [13].

However, scaling a fully entangled game from 2 to n players can be nontrivial even when linear optics is used. Suppose a trial between any two players succeeds with probability β (incorporating the net efficiency with which entanglement can be created, distributed, purified, manipulated, and detected), so the mean number of trials needed to successfully register a mutual choice between two players is $1/\beta$. Because an accidental (or deliberately disruptive) measurement of a single qubit in the n-particle maximally entangled state destroys the entire state, we expect the number of trials needed to complete a maximally-entangled game for n players will scale no better than β^{-n}. Suppose instead we implement the game by distributing entangled two-particle states between either all enumerated pairs of players or nearest neighbors, as described in Sec. 4.3 and Sec. 4.4, respectively. In these cases, we expect that the mean number of trials needed to complete the game will scale as either $n(n-1)/2\beta$ or n/β, and are therefore relatively easier to implement for games with a large number of players.

For example, in the simplest near-term implementation, a single game system can be constructed at a central location, and players can travel to the game and individually specify the operators to be applied to their qubits. As the technology evolves, the necessary hardware for specification of qubit operations can be distributed to distant players, who then can apply their operators to photonic qubits transported to them over an optical network. In either case, entangled pairs can be generated and distributed consecutively until all players have successfully registered a choice for each pair in which they are a member.

Although great strides continue to be made in multi-particle experiments [11, 12], it is clear that — until $\beta \longrightarrow 1$ — two-particle games are far more feasible, and could allow tests of quantum game theory to be performed in the near future.

Given some single-trial success probability β, the number of trials is limited by the rate at which two-particle entangled qubits can be provided. A bright source of entangled photon pairs has been constructed using an argon-ion laser and parametric down-conversion in BBO crystals, capable of producing 140 detected two-photon coincidences per second per milliwatt of Ar^+ pump power [9]. In principle, given an electrically-driven source of single photons [14], entangled photon pairs also could be generated in a compact all-solid-state system using down-conversion in periodically-poled lithium niobate waveguides [15, 16]. However, in the future it is possible that up to 10^9 pairs per second could be produced using a single quantum dot embedded in a p-i-n junction surrounded by a microcavity [17].

3 Public Goods Economics

3.1 Overview

A pure market economy fails to provide efficient levels of public goods for two key reasons. By definition, a public good is non-excludable. Once the good is provided, there is no means of charging for it or restricting access to it. This creates the free rider problem in which people are tempted to use the public good without paying for it. The prisoner's dilemma is a perfect illustration of this free-rider problem. In this two-person game, each player has the choice to "cooperate" and "defect". Payoffs for both players are higher when both of them choose to cooperate instead of defect. However, each individual is better off by defecting.

Furthermore, even if there exists a third party (usually the government) to enforce contribution to the public goods, individuals have the incentive to hide their preferences on how much they value the public good. This information asymmetry makes it difficult to determine the efficiency of public goods distribution.

Some of these issues have been addressed in economics literature. For example, if the public good can be provisioned through a government, there exist mechanisms to reveal preferences of individuals [18]. Also, experimental work on public goods [19] compares people's actual behaviors to the predictions of game theory.

The free rider aspect of the problem, however, is more difficult to overcome. In the absence of a benevolent dictator, self-motivation becomes the dominant factor. Luckily, two phenomena mitigate the effects of free riding. The first is the folk theorem [20, 21]. If the game is played repeatedly within a relatively small group, the folk theorem suggests an efficient outcome may be enforceable through the strategy of punishing a defector. It is arguable whether this will work in practice because game theory rationality places a strong burden on the individuals to determine the correct strategies. The second phenomenon is individuals' motivations may not be completely selfish: experimental evidence suggests people may be altruistic, at least in relatively small groups [19]. The same empirical evidence also suggest that although these two phenomena helps raise the levels of contributions, they are far from optimal.

3.2 A General Framework

For simplicity in discussing the public goods game, we assume there is only one public good and one private good which players can use to contribute to producing the public good. It is possible to generalize to multiple goods.

There are n players indexed by k. We make the following definitions:

x amount of public good

y_k initial endowment of private good of player k, i.e., player k's initial wealth

c_k contribution of player k

$Q_k(x,y)$ utility of player k when consuming x units of public good and y units of private good. Q is assumed to be continuously differentiable, concave and non-decreasing in both x and y.

$g(C)$ production function of the public good as a function of total contributions $C = \sum_k c_k$. g is assumed to be continuously differentiable and strictly increasing.

If contribution is voluntary and continuous, each individual would want to choose a contribution to maximize:

$$\max_{c_k} Q_k(g(C), y_k - c_k) \tag{1}$$

which leads to

$$\frac{1}{dg/dC} = \frac{dQ_k/dx}{dQ_k/dy} \tag{2}$$

for all k when evaluated at the maximizing choices with $x = g(C)$ and $y = y_k - c_k$. These give n equations for the n contribution values $\{c_k\}$.

This condition says each person will contribute up to the point where the marginal rate of substitution is equal to the marginal benefit of his contribution in providing the public good.

We use the standard economic efficiency measure of Pareto optimality [22]. That is, there exists no other allocation such that one player is strictly better off while all others are at least as well off as before. In our context, Pareto efficiency requires [22]

$$\frac{1}{dg/dC} = \sum_k \frac{dQ_k/dx}{dQ_k/dy} \tag{3}$$

This criterion can be generalized to multiple public and private goods [22].

Let C be the Pareto efficient level of total contribution and C' be the equilibrium level of total contribution. The above two conditions mean that $\frac{dg(C)}{dC} < \frac{dg(C')}{dC}$.

Therefore, the equilibrium level of total contribution does not always guarantee a Pareto efficient outcome. For g with diminishing rate of return, $C > C'$. Thus the equilibrium level of contribution is less than the efficient level. For other g, there may be multiple equilibria as well as contribution levels at the efficient levels.

3.3 A Public Goods Game

For simplicity, we consider the special case when g is linear. In this case, both the efficient allocation and the equilibrium have contributions that are either all or nothing. This choice illustrates the core issue of the free-rider problem. Specifically, we take

$$\begin{aligned} Q_k(x, y) &= x + y \\ g(C) &= aC/n \end{aligned} \tag{4}$$

for all k. Here a is a parameter characterizing the public good production and $C = \sum_k c_k$ is the total contribution level.

A direct comparison of a player's utility for all and no contribution shows the unique Nash equilibrium is given as follows:

- If $a \leq 1$, $C = 0$ and this is the Pareto efficient outcome. Note, when $a = 1$ total contribution is equivalent to no contribution. Thus, either case is efficient.
- If $1 < a < n$, $C = 0$, but is an inefficient outcome. One efficient outcome in this case is $c_k = y_k$ but this is not an equilibrium since each player increases payoff by defecting, i.e., switching to $c_k = 0$.
- If $n \leq a$, $c_k = y_k$ is the efficient outcome and also the equilibrium. When $n = a$, total contribution is a weak equilibrium in which each player is indifferent between contributing or not.

This analysis can be interpreted as follows. The production function g multiplies the total contribution by a. The result is then equally divided back to the players. If a is less than 1, there is no gain to produce the public good and so the efficient outcome is *not* to produce any. If a is greater than n, then for each unit the player receives back more than the contribution, thus it is advantageous to contribute, no matter what other players do, and the equilibrium will be efficient.

The interesting case, giving a social dilemma, is when a is between 1 and n. In this case, the public good per person increases with contribution. However, the marginal benefit of each contribution is still smaller than 1. Thus a player receives only a/n in benefit for a unit of additional contribution, which is a net loss. Therefore, it is rational *not* to contribute. However, failure to contribute is an inefficient outcome. Thus we have a social dilemma in that the group as a whole is better off if all contribute, but each person prefers not to contribute and hence their rational choices lead to no public good production. Moreover, this case has multiple Pareto efficient outcomes. For example, both total contribution and total contribution from all but one person are efficient outcomes.

3.4 Heterogeneity and Asymmetric Information

When individuals have substantial wealth differences, it is possible for total contribution from all players to make some of them worse off than if there were no contributions at all. In such a situation, the efficient outcome of total contribution may not be desirable. More generally, in addition to achieving an efficient outcome, a mechanism should ideally also satisfy voluntary participation constraints (i.e., all players should want to play the game since they recognize the outcome will be better for them than if they do not play at all).

To address this issue, we focus our attention on a smaller set of efficient outcomes that also satisfy the voluntary participation constraints. Thus, in addition to Pareto efficiency, we also require $Q_k(g(C), y_k - c_k) \geq Q_k(g(0), y_k)$ for all k, i.e., each person is better off in this efficient outcome then they would be if none contribute. For our example, this implies

$$\frac{a}{n}\sum_{j=1}^{n} c_j \geq c_k \tag{5}$$

for all k. An outcome that is both Pareto efficient and satisfies Eq. (5) is defined as an *individually rational Pareto efficient outcome*.

The following contribution profile is an individually rational Pareto efficient outcome:

$$c_k = \begin{cases} y_k & \text{if } y_k < C^* \\ C^* & \text{if } y_k \geq C^* \end{cases} \tag{6}$$

where

$$C^* = \frac{a}{n - an + am}\sum_{j=1}^{m} y_j \tag{7}$$

the y_k are taken to be sorted in ascending order and m is the largest integer less than n for which $C^* \geq y_k$ holds for all $k = 1 \ldots m$.

Under this additional constraint, if the distribution of wealth is narrow (specifically, $a\bar{y} \geq y_k$ for all k where $\bar{y} = \frac{1}{n}\sum_k y_k$ is the mean value of the private goods), then everyone should contribute everything. If there is a wider distribution of wealth, then there is a cut-off point C^*. Everyone should contribute everything if their wealth $y_k \leq C^*$ and contribute only up to C^* if their wealth is more than C^*. Thus to maintain voluntary participation the rich should contribute more in absolute terms than the poor, but less in percentage terms.

If wealth is distributed narrowly, (satisfying Eq. (5) if individual contributes everything) then there is no need for asymmetric contribution. Therefore, it is sufficient to treat the problem as if wealth is equal.

However, if condition Eq. (5) is not satisfied, a new incentive issue arises. To be able to solicit the "correct" amount of contribution from every individual, we not only need to solve the free-rider problem, but also correctly identify the wealth level of every individual. Furthermore, individuals have incentives to pretend to be poorer than they are to minimize their contributions.

4 A Quantum Mechanism for Public Goods Provisioning

We first characterize equilibria of the quantum game of the homogeneous version of the public goods game. This allows us to study several configurations, such as different entanglement and interpretation of the qubits, of the quantum game. Subsequently, the results in the simple homogeneous case will be extended to the heterogeneous case.

For the quantum mechanism, each player can choose either to contribute nothing ($c_k = 0$, "defect") or everything ($c_k = y$, "cooperate"). We can also consider an intermediate case in which players can select from a discrete range of contribution values, $0, y/K, 2y/K, \ldots, y$ for various choices of K, but in our case allowing such intermediate contributions gives lower average payoffs for the strategies we present below.

Here is an example of the intermediate case. For $n = 3$ players and using 3 bits to specify discrete choices: either contribute fully ($c_k = y$, "cooperate") or contribute nothing ($c_k = 0$, "defect"), there are 8 states. Suppose we let the value 0 correspond to "cooperate". Then the payoffs to the three players are (using $y = 1$)

$$\begin{array}{cccc}
000 & a & a & a \\
001 & 2a/3 & 2a/3 & 2a/3+1 \\
010 & 2a/3 & 2a/3+1 & 2a/3 \\
011 & a/3 & a/3+1 & a/3+1 \\
100 & 2a/3+1 & 2a/3 & 2a/3 \\
101 & a/3+1 & a/3 & a/3+1 \\
110 & a/3+1 & a/3+1 & a/3 \\
111 & 1 & 1 & 1
\end{array} \tag{8}$$

The quantum version of the game is set up as follows: first create entangled qubits (with 0 and 1 representing cooperate and defect, respectively), allow the individuals to operate on their individual qubit, then combine the result (by undoing the initial entangling operation). To preserve the correspondence with the original game, the entanglement operator should commute with those quantum operations corresponding to the classical choices. The final measurement gives a definite value for each qubit, which then corresponds to the individuals' choices.

In general, players are allowed to apply any operator to their qubit(s). We consider general single-qubit operators, given by

$$U(\theta, \phi, \alpha) = \begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} & e^{i\alpha}\sin\frac{\theta}{2} \\ -e^{-i\alpha}\sin\frac{\theta}{2} & e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix} \tag{9}$$

up to an irrelevant overall phase factor. (A further generalization would allow measurements on the single qubit. This gives no advantage in at least in some cases [7].)

For $n = 2$, this reduces to the Prisoner's dilemma, which has a nice interpretation in terms of conventional mechanisms. Entangled states allow player 1 to affect the final outcome produced by the action of player 2 and vice versa. In a way, it allows for pre-commitment. Consider the following argument. Player 1 would love to tell player 2 that if player 2 commits to cooperate, then he would also cooperate. However, without playing a repeated version of the game, the ability to punish the other player or without a 3rd party to enforce the commitment, both players will realize immediately they are better off reneging their commitments. Entanglement allows the parties to commit without a third party to enforce the commitments.

The expected payoffs can be viewed as functions of the players' choices and game definition: $P_k(U_1, \ldots, U_n; J, a)$ (where we take $y = 1$ without loss of generality since it just rescales the payoffs).

4.1 Equilibria for the Quantum Public Goods Game

In this section, we characterize equilibria for three schemes of entanglement of the public goods quantum mechanism. If players are allowed to use any single qubit operators given by Eq. (9), there is no single pure strategy equilibrium. However, we found mixed strategy equilibria, with expected payoffs depending on the degree of entanglement provided in the initial state. In each case, we find multiple equilibria. These payoffs are superior to that produced by the classical game in which all players defect so no public good is produced.

We assume all individuals are risk-neutral expected utility maximizers. Player k's expected payoff function is given by $P_k(\psi) = \sum_s P_k(s)|\psi(s)|^2$ where $P_k(s)$ is the payoff for player k given the choices specified by state s.

We use the Bayesian Nash equilibrium as the solution concept for the quantum game. Each individual will play a strategy (pure or mixed) such that they are mutually maximizing their expected payoff. None has the incentive to make a unilateral change to their strategy.

A single-player operator $\hat{u}$ forms a symmetric Nash equilibrium if for any other choice $u \neq \hat{u}$

$$P_k(\psi(\hat{u}, \ldots, \hat{u})) \geq P_k(\psi(\hat{u}, \ldots, \hat{u}, u, \hat{u}, \ldots, \hat{u})) \tag{10}$$

entanglement	expected performance
full	$(1+a)/2$
all pairs	$a - 2^{-(n-1)}(a-1)$
neighbor pairs	$(1+3a)/4$

Table 1: Expected payoffs of mixed strategy equilibrium for various arrangements of entanglement and using the all-or-none contribution rule. For comparison, the original game, corresponding to no use of entanglement, has payoff 1 at the equilibrium. In all cases, the efficient outcome has payoff $a > 1$, which is larger than the expected payoff of all these cases.

for all players k, with u substituted for the k^{th} player's choice on the right-hand side. For homogeneous preferences, it is sufficient that this hold for just one player. More generally, asymmetric equilibria involve possibly different operations for each player.

Whether such an equilibrium exists, and if so whether it is unique and gives the optimum payoffs for the players, depends on the set of allowed operations, the amount and type of entanglement (specified by the choice of J) and the nature of the payoffs.

Our analysis includes mixed strategy equilibria since in many cases, particularly with respect to the quantum version of the public goods game, there is no pure strategy equilibria. The strategic space for quantum games are infinite. We limit our attention to finite mixed strategies. That is, we only allow individuals to randomly (with any probabilities assignment) choose within a finite set of operators. We also make the standard assumption that individuals have access to a perfect randomization process.

In the next three subsections, we report three different schemes of entanglement and their corresponding mixed-strategy Nash equilibrium. Table 1 summaries the resulting performances.

4.2 Full Entanglement

A conceptually simple approach allows arbitrary entanglement among the players' qubits. As one example, consider fully entangled states. The initial entangled state is $(|00...0\rangle + i|11...1\rangle)/\sqrt{2}$, using the $2^n \times 2^n$ entanglement matrix

$$J_n = \frac{1}{\sqrt{2}}(I + i\sigma_x \otimes \ldots \otimes \sigma_x) \tag{11}$$

where the product in the second term consists of n factors of σ_x, the 2×2 Pauli matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Allowing general single-bit operators of Eq. (9), we find no pure strategy Nash equilibrium for the players. However, there are a variety of mixed strategy

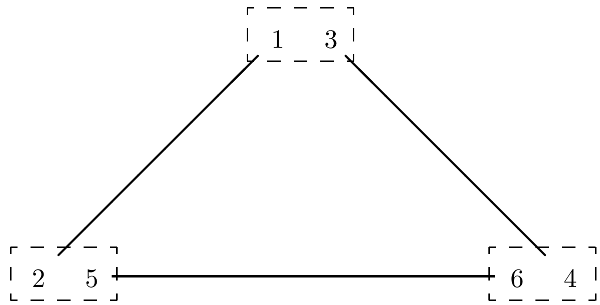

Figure 1: Six qubits giving two-particle entanglement among three players. The first player operates on bits 1 and 3, which are entangled with bits 2 and 4, respectively owned by the second and third players.

equilibria. As one example, let

$$\begin{aligned} u(0) &\equiv U(0,0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ u(1) &\equiv U(0,\pi/2,0) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \end{aligned} \tag{12}$$

Note $u(0)$ corresponds to the classical "cooperate" option. A mixed strategy consisting of each player randomly selecting $u(0)$ or $u(1)$, each with probability $1/2$, gives expected payoff of $(1+a)/2$. This is an equilibrium: if any one player switches to using a different operator, or different mixture of operators, the expected payoff for that player remains equal to $(1+a)/2$. While this payoff is less than the efficient outcome, it is substantially better than the classical outcome with payoff of 1 since all choose to defect.

Although this scheme is not practical with respect to implementation due to its use of highly entangled states, we include it as a comparison to other schemes.

4.3 Two-particle Entanglement

Full entanglement is difficult to implement as n increases, particularly for qubits communicated over long distances. Thus we consider restricting entanglement to only pairs of qubits. In this case, we suppose each pair of players has a maximally entangled pair, so each player has $n-1$ qubits.

The entanglement matrix for a case consisting of $N = \binom{n}{2}$ pairs is

$$J_{\text{pair}}(N) = J_2 \otimes \ldots \otimes J_2 \tag{13}$$

with the product consisting of N factors of the entanglement operator of Eq. (11) for the case of $n = 2$, i.e., full entanglement among two qubits.

With multiple bits per player, we also need to specify how the final measured state is to be interpreted. One approach is to allow various amounts of contribution rather than all or none. That is, if z of the $n-1$ bits for player k are 0, player k's contribution is $c_k = yz/(n-1)$, ranging from 0 to y. So instead of two choices, the player has a range of possible contributions. This choice gives the same result as the fully entangled case: the mixed strategies have expected payoff of $(1+a)/2$ and remain weak equilibria.

For example, $n = 3$ uses six qubits corresponding to the pairs of players $(1,2)$, $(1,3)$ and $(2,3)$, as shown in Fig. 1. Thus, for example, the first player operates on the first and third qubit in this ordering of the bits. The state $|0,0,0,1,1,1\rangle$ has $0,0$ for the first and third qubit, so the first player has $z = 2$ and contributes y. The second player, using the second and fifth bits, has $0,1$ with $z = 1$ and contributes $y/2$.

An alternate interpretation of the bits provides higher payoffs while maintaining the same mixed strategy equilibria. Specifically, we again suppose contributions are all or nothing but now consider the player to contribute if *any* of the $n-1$ bits equals 0. This simple change in the construction of the game gives expected payoff equal to

$$a - 2^{-(n-1)}(a-1) \tag{14}$$

which, since $1 < a < n$, is only slightly less than the highest possible payoff, a. As examples, the expected payoffs for $n = 3$ and 4 are, respectively, $(1+3a)/4$ and $(1+7a)/8$. As n increases, the expected payoff approaches the optimal value.

We could also consider other interpretations, e.g., full contribution if a majority of the bits are 0, and otherwise no contribution.

Significantly, the mixed strategy remains an equilibrium even if a player applies different operators to each of the $n-1$ bits.

4.4 Two-particle Entanglement with Neighbors

Two-particle entanglement among all possible pairs of players requires $n(n-1)/2$ entangled pairs. While significantly easier to implement than entanglement among n-players, we can also consider behavior with even less entanglement. Specifically, consider the players in some arbitrary order and only provide an entangled pair between successive players in that order (with an additional pair between the first and last). This entanglement requires only $2n$ qubits.

This case maintains the same equilibrium mixed strategies. If we interpret the two bits of each player as allowing partial contributions, the expected payoff remains $(1+a)/2$. Using the all-or-none method, where a player contributes everything if at least one of the two bits equals 0, the payoff is $(1+3a)/4$ for all n. Note this is the same as the payoff of the full two-particle case, Eq. (14), for $n = 3$ (as expected: for $n = 3$ the neighbor pairs are the same as a two-particle entanglement between all pairs of players).

Again, the payoff is superior to the classical game Nash equilibrium. Unlike entanglement among all pairs, the payoff does not improve with larger n. Thus this result illustrates a tradeoff: lower performance when using fewer pairs.

Based on the two cases of all pairs and just neighbors, one would expect intermediate numbers of pairs to give intermediate performance. A full investigation of this tradeoff remains for future work. In particular, to the extent that producing, storing and manipulating entangled states is costly, a key economic issue is how the benefit of higher expected payoff from using additional entanglement compares with its implementation cost.

4.5 Robustness of the Results

All our results are based on very strict assumptions on initial wealth levels (y_k), the public goods production function (g), and individuals' utility function (Q_k). A natural question is how far these results can be generalized with respect of these three assumptions.

As discussed in Sec. 3.4, when the distribution of wealth is narrow, efficient contribution levels requires no or total contributions. The incentive tradeoff also remains the same since the different levels of wealth only offset individuals' utility by constant amounts. Thus the quantum solution still applies.

However, the results are more sensitive to the assumptions on the public goods production function (g) and individuals' utility function (Q_k). In particular, the quantum mixed strategy equilibrium depends on a fine balance of the probabilities of the possible outcomes so that individuals will weakly preferred (or indifferent to) staying with the equilibrium strategy. This balance will be destroyed if either g or Q_k is changed. However, intuitively we expect similar quantum equilibria to exist that will be superior (in the Pareto sense) to equilibria of the classical game. Since the outcome space is probabilistic and hence continuous, a small change in either g or Q_k may induce equilibria that are close to the solution we reported here. This issue is one of the directions of our future work.

5 Conclusion

Quantum mechanics can be used to develop new formulations of classical economics games which give arise to new solutions. In this paper, we have shown how a quantum mechanism can be constructed to address the free-rider aspect of the public goods problem, without the need of third party enforcement nor repeated play. Implementation issues are also explored and addressed.

Most of the power of this new mechanism comes from entangled states, which in theory allow individuals to co-ordinate and commit in environments when classical means do not. Incidentally, entanglement is also the major issue determining whether a quantum mechanism is practical or not.

Three different schemes of entanglement are explored. We found that two-particle entanglement, which is feasible for the near future, can address the free-rider problem and achieve nearly efficient outcomes. Furthermore, we have also argued that the mechanism is robust with respect to a limited amount of heterogeneity in the system if there is no adverse selection. If the distribution of wealth is narrow (as defined in Sec. 3.4), an efficient quantum solution that assumes homogeneous wealth will also satisfy the voluntary participation/individually rational constraints making heterogeneity a non-issue.

Furthermore, if the issue of adverse selection (incentive to hide information) is addressed by some other method, then the quantum mechanism can be used in tandem to address the general case. Specifically, in the case of heterogenous wealth, if every individual's wealth is revealed to the mechanism, then the mechanism can be modified slightly to yield the desirable outcome as follows. First calculate the optimal contribution for every individual based on the revealed wealth levels as described in Sec. 3.4. Then the players play the quantum game with the knowledge that the final qubits are interpreted as follows: an individual contributes the optimal amount, not his total wealth, if one or more of his qubits are zero. Essentially, all the contribution levels are pre-determined and the issues reduce to just the free-rider problem.

The particular form of the equilibria we found depends on the simple functional forms for utilities and production functions of Eq. (4). Thus an interesting direction for further investigation is the effect of other economically-reasonable functional forms. Since we found a variety of weak equilibria, with identical expected payoffs, some changes in the utilities may shift these equilibria differently thereby distinguishing their payoffs.

Our quantum mechanism generalizes the conventional public goods game by providing additional options for the players. As such, it can also be simulated classically by allowing each player to send a choice of operator (i.e., the three parameters of Eq. (9)) to a central location. This observation, which also applies to other studies of quantum games [23], means the practical benefit of such quantum mechanisms depends on the context of the game, e.g., the differences in security and communication costs as well as the level of trust assumed for the central institution. For instance, the quantum version allows only a single measurement of the outcome rather than revealing individual operator choices, and hence can provide additional privacy. Such privacy can also be achieved via conventional cryptographic methods but with security based on the apparent difficulty of solving certain problems, e.g., factoring, rather than inherent in quantum physics.

Game theoretic solutions (such as the Bayesian Nash equilibrium we discuss in this paper) are at best approximations of real human behavior. In this case, rationality dictates that each individual has a full understanding of the quantum mechanical implications of his choices. How well this describes the actual behavior of people involved in quantum games is an interesting direction for future work with laborabory experiments involving human subjects.

There are many natural extensions of this research. First, people may use criteria other than expected payoff, e.g., to minimize variance in payoff if they are risk adverse. Second, the case of heterogeneous players and adverse selection requires further analysis. This work also suggests experimental research, exploring the issues of practicality of implementation and human behavior with respect to manipulating quantum states.

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A Derivation of Mixed Strategy Payoff

This appendix derives Eq. (14) and shows the mixed strategy is indeed an equilibrium: no single player can benefit from deviating from the mixture. A similar derivation applies to the other cases with different entanglement (i.e., full or two-particle only among neighbors) and interpreting multiple bits per player as indicating partial contributions. For simplicity, we take the private good value to be $y = 1$.

Suppose player 1 selects operators $u^{(2)}, \ldots, u^{(n)}$ while all others select either $u(0)$ or $u(1)$ of Eq. (12) with equal probability for all their bits. The initial state $J_{\text{pair}}(N)(1, 0, \ldots, 0)$ is

$$\bigotimes \frac{1}{\sqrt{2}}(1, 0, 0, i)$$

with one factor for each pair. Subsequent operations on each pair are independent. Consider the pair between players j, k, who use operators A and B respectively, so the final state of their pair is

$$\psi_{\text{pair}}(A, B) = J_2^{\dagger}(A \otimes B) \frac{1}{\sqrt{2}}(1, 0, 0, i) \tag{15}$$

Players other than the first use either $u(0)$ or $u(1)$. Evaluating Eq. (15) for these cases gives

$$\begin{aligned} \psi_{\text{pair}}(u(0), u(0)) &= (1, 0, 0, 0) \\ \psi_{\text{pair}}(u(0), u(1)) &= (0, 0, 0, 1) \\ \psi_{\text{pair}}(u(1), u(0)) &= (0, 0, 0, 1) \\ \psi_{\text{pair}}(u(1), u(1)) &= (-1, 0, 0, 0) \end{aligned} \tag{16}$$

so players making the same choice produce a pair equal to $\pm|00\rangle$ (i.e., both cooperate), while those making opposite choices give $|11\rangle$ (i.e., both defect).

For a given instance of this mixed strategy, let $t_k \in \{0, 1\}$ indicate the operator choice of player $k = 2, \ldots, n$: $u(t_k)$. Then the final state for the pair j, k is $\psi_{\text{pair}}(u(t_j), u(t_k))$. Thus the portion of the final state corresponding to pairs not involving player 1 is

$$\psi_{\text{other}} = \bigotimes_{j,k} \psi_{\text{pair}}(u(t_j), u(t_k))$$

with the tensor product over all pairs $2 \le j < k \le n$. The nonzero components of this vector are all ± 1, which have unit magnitude so do not affect the probabilities of the final measurement.

The pair involving players 1 and k gives $v^{(k)}(t_k) = \psi_{\text{pair}}(u^{(k)}, u(t_k))$. For any choice of operator $u^{(k)}$, evaluating Eq. (15) using Eq. (12) gives

$$\begin{aligned} v^{(k)}_{0,0}(1) &= -v^{(k)}_{1,1}(0) \\ v^{(k)}_{0,1}(1) &= v^{(k)}_{1,0}(0) \\ v^{(k)}_{1,0}(1) &= -v^{(k)}_{0,1}(0) \\ v^{(k)}_{1,1}(1) &= v^{(k)}_{0,0}(0) \end{aligned} \tag{17}$$

so, apart from some sign changes, player k switching from $u(0)$ to $u(1)$ simply reverses the result of the two-particle interaction between players 1 and k.

The overall final state vector is the product of these results for the individual pairs. The values for the bits involving player 1 determine its nonzero components. That is, the final state has the form

$$\bigotimes_{k=2}^{n} \left(\sum_{x_k, y_k} v^{(k)}_{x_k,y_k}(t_k) |x_k, y_k\rangle \right) \otimes \psi_{\text{other}}$$

with the x_k, y_k each summed over 0 and 1.

Measuring this final state produces definite values for the x_k, y_k, with probability $\Pr(x, y, t) = \prod_{k=2}^{n} |v^{(k)}_{x_k,y_k}(t_k)|^2$. For this state, we determine the payoff to player 1 as follows.

First, the all-or-none interpretation of the bits means player 1 contributes 1 if any of the $x_k = 0$. Defining the indicator function $\chi(p)$ to equal 1 when the proposition p is true and 0 otherwise, this contribution is $1 - \prod_k \chi(x_k = 1)$.

The contribution for player $k > 1$ is 1 if it has a 0 bit in its pair with player 1 (i.e., $y_k = 0$) or at least one player (other than players 1 or k) makes the same choice of operator as player k (since then Eq. (16) shows that pair of players will have value $|0, 0\rangle$ so, in particular, player k will have at least one of its bits equal to zero). Let n_b be the number of players $2, \ldots, n$ that select operator $u(b)$, for $b = 0, 1$. Note n_b is the number of values in $t_2, \ldots, t_n$ equal to b, and $n_0 + n_1 = n - 1$. With these definitions, the contribution of player k is $\chi(y_k = 0 \wedge n_{t_k} = 1) + \chi(n_{t_k} > 1)$.

Combining the contributions from all players, the payoff $P_1(x, y, t)$ to player 1 for this measured state has the form $\frac{a}{n}(1 + A) + (1 - \frac{a}{n})B$ with

$$\begin{aligned} A &= \sum_k \left(\chi(y_k = 0 \wedge n_{t_k} = 1) + \chi(n_{t_k} > 1)\right) \\ B &= \prod_k \chi(x_k = 1) \end{aligned}$$

In this expression, $\sum_k \chi(n_{t_k} > 1)$ can be written as $\sum_t n_t \chi(n_t > 1)$.

The expected payoff for player 1 for the choices of the other players (specified by the t_k values) is $\sum_{x,y} \Pr(x, y, t) P_1(x, y, t)$.

Finally, the mixed strategy used by the other players means each of the 2^{n-1} choices for the values of the t_k is equally likely, and must be summed over to get the expected payoff of player 1: $\langle P_1 \rangle = 2^{-(n-1)} \sum_{x,y,t} \Pr(x, y, t) P_1(x, y, t)$.

In the sum over x, y, only the factor $|v^{(k)}_{x_k,y_k}(t_k)|^2$ in $\Pr(x, y, t)$ depends on x_k, y_k. Thus for terms involving player k, the remaining factors in $\Pr(x, y, t)$ sum to 1 since the $v^{(k)}(t_k)$ are normalized vectors.

Thus $\langle P_1 \rangle$ is a sum of three terms. The first is

$$2^{-(n-1)} \frac{a}{n} \sum_t (1 + n_0 \chi(n_0 > 1) + n_1 \chi(n_1 > 1))$$

or

$$\frac{a}{n}(1 + (n-1)(1 - 2^{2-n}))$$

The second term is

$$2^{-(n-1)} \frac{a}{n} \sum_{k,t,x_k,y_k} |v^{(k)}_{x_k,y_k}(t_k)|^2 \chi(y_k = 0 \wedge n_{t_k} = 1)$$

For $n_{t_k} = 1$, the only terms contributing to the sum over t are those for which $t_j \neq t_k$, for all $j \neq k$, i.e., there are just two cases: $t_k = 0$ and the rest are 1, and vice versa. So this term becomes

$$2^{-(n-1)} \frac{a}{n} \sum_k \sum_{t=0}^{1} \sum_{x_k} |v^{(k)}_{x_k,0}(t)|^2$$

The inner two sums give $|v^{(k)}_{0,0}(0)|^2 + |v^{(k)}_{1,0}(0)|^2 + |v^{(k)}_{0,0}(1)|^2 + |v^{(k)}_{1,0}(1)|^2$ which, from Eq. (17), equals $\sum_{x,y} |v^{(k)}_{x,y}(0)|^2 = 1$ since the $v^{(k)}$ vectors are normalized. Thus this term is $2^{-(n-1)} \frac{a}{n}(n-1)$.

Similarly, Eq. (17) gives the third term equal to $2^{-(n-1)} \left(1 - \frac{a}{n}\right)$.

Combining these results, $\langle P_1 \rangle$ is $a - 2^{-(n-1)}(a-1)$. This result is independent of the operators selected by player 1, i.e., the values of the $v^{(k)}$.

Other choices for the mixed strategy operators $u(0), u(1)$ are possible as well. They need only satisfy Eq. (16) (up to an overall phase factor) and also compensate for any choices made by the first player via Eq. (17), again up to overall phase factors.

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