LECTURE 4 MONDAY 21st JUNE 2004.

ALGEBRAIC CYCLES & HODGE THEORY.

0. Motivation: To generalise to higher dimension all the curve stuff.
1. Case of curves (div, div₀, Pic, Pic°, Jac, Abel-Jac map, Abel's Thm, Jac Inversion)
2. Sections & chern classes (from Nag)
3. Alg cycles, Chow grps,
4. Chow for codim 1 (div)
5. Chow on zero cycles
6. ~~Hodge theory~~
7. Connections to cohomology. (incl. Hodge conjecture)

— 4pm.

8. Results on Cohom. of moduli spaces.

Recall line bundles: Recall holomorphic

(1) line bundles. Recall meromorphic sections of hol. line bundles. [Recall $H^1(M, \mathcal{M}^*) = 0$ $\Rightarrow$ for every $\xi \in H^1(M, \mathcal{O}^*)$, $\Gamma(M, \mathcal{M}(\xi)) \neq 0$]

(2) Check: Ratio of any 2 mer. crs of the same line bundle is a meromorphic function.

(3) Corollary: deg of section is ind. of which

MODEL: CASE OF CURVES

(4) Section is chosen! ∴ deg of line bundle is well defined!

$\mathfrak{H}$ is first chern class!!

Relate to: if line bundle is top trivial $\iff$ flat representation $\iff$ deg of divisor $= 0 \iff$ chern class $= 0$

(5) Now ~~a~~ a line bundle may not have hol. sections. denote $\mathrm{Hol}(\varphi)$ V.S of hol. sections of a line bundle.

(6) Clearly if $\deg \varphi < 0$; $\mathrm{Hol}(\varphi) = \phi$

dim Hol(φ) OTOH dim (mer φ) $= \infty$ since you get one from another by mult w/ a mer function

(7) Riemann Roch

$$\dim \mathrm{hol}(\varphi) - \dim \mathrm{hol}(K \otimes \varphi^{-1}) = \deg(\varphi) - g + 1$$

(8) Q Ex: let $\varphi = K^2 = K \otimes K$

$$\deg(\varphi) = 2g - 2$$

$$\therefore \deg(K \times \varphi^{-1}) = 2 - 2g < 0$$

$$\therefore \dim \mathrm{hol}(K \times \varphi^{-1}) = 0$$

$\therefore \dim hol(K^2) = \deg(K^2) - g + 1$
$= 3g - 3.$

$\therefore$ Space of holom quadratic differentials is ind of which structure is on surface!

Exercise: $\dim hol(K) = g$

$\dim hol(K) - \dim hol(K \otimes K^{-1}) = 2g - 2 - g + 1$

$\dim hol(K) - 1 = g - 1$

(9) Define $J(X)$

Period matrix of X $\Pi = \begin{bmatrix} \int_{L_1} \omega_1 & \cdots & \int_{L_{2g}} \omega_1 \\ \vdots & & \vdots \\ \int_{L_1} \omega_g & \cdots & \int_{L_{2g}} \omega_g \end{bmatrix}$
$g \times 2g$

Columns of Π form $2g$ vectors in $\mathbb{C}^g$. lin ind over $\mathbb{R}$.

$J(X) = \mathbb{C}^g / L(\Pi)$!

Torelli's thm: period matrix determines complex structure of X up to bihol. equiv.

(10) Given p, q, associate vector $\int_p^q \omega_1 \ldots \int_p^q \omega_g \in \mathbb{C}^g$

vector modulo $L(\Pi)$ choice of path is immaterial!

11) ∴ map X into $J(X)$. Choose b.p p^0
hol.

$$A: X \longrightarrow J(X)$$

$$A(p) \to \left(\int_{p_0}^{p} \omega_1 \cdots \int_{p_0}^{p} \omega_g \right) \bmod L(\Pi)$$

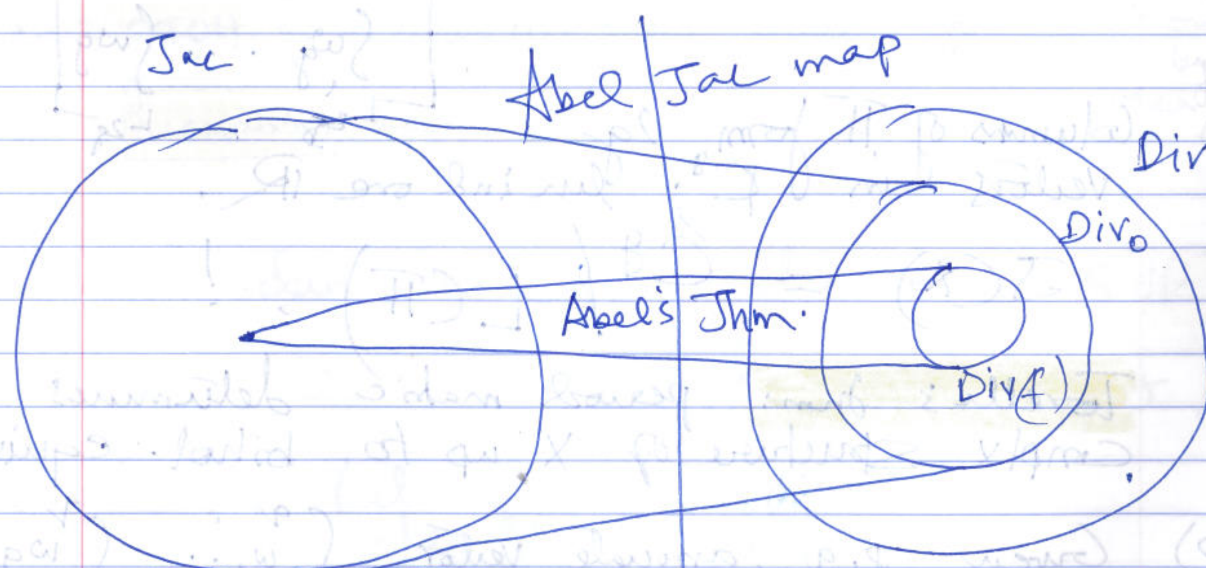
(12) Extend to divisors of degree 0 on $X \longrightarrow J(X)$

Abel: D is divisor of a mer. function on X
iff $\deg D = 0$ & $A(D) = 0$ in $J(X)$.

∴ Kernel of Abel Jac. map is precisely Div of mero functions.

∴ $A(p)$ is injective on X : embeds X into $J(X)$

(13) Jacobi inversion :—
It is surjective ie every pt in $J(X)$ is image of some degree 0 divisor.



$$Pic_0 = Jac(X)$$

$$\frac{Div}{Div\,f} = Pic$$

$$\frac{Div_0}{Div\,f} = Pic_0$$

1. Coll. of all eff div of deg d form proj variety $Sym^d(X)$.
2. Ker $Cl^0(X)$ of deg: $Cl(X) \to \mathbb{Z}$ is $\cong$ Jacobian.
3. fixing a pt p on curve, get a morphism
4. fiber of Φ_d thru D $\Phi_d: Sym^d(X) \to Jac$ sending D to $[D - dp_0]$ consists of everything in its div. class
5. Φ_d is surjective for $d \geq g$ (from R.R)

ALGEBRAIC CYCLES

Defn: Let X be a nonsingular variety over k. An algebraic cycle of codimension p is an element of the free Abelian grp on irreducible subvarieties of X of codimension p.

Defn: The group of these cycles is denoted $Z^p(X)$.

Defn. Effective cycles $Z^p(X) \geq 0$ (nonneg linear combinations)

Defn: $R^p(X) \subset Z^p(X)$ is subgroup generated by all cycles $\mathrm{div}(f)_W$ where W ranges over irr. subvarieties of codim $p-1$ in X, & $f \in K(W)^{*}$.

Defn $CH^p(X) = Z^p(X)/R^p(X)$ quotient called Chow group of codim p cycles modulo rational equivalence.

Ex: for $p=1$, X smooth proj curve,
$CH^1(X) = Cl(X) = Pic(X)$

Q. A little bit of Schubert Calculus:

Q: How many lines meet 4 given lines in 3space

An: Intersection Theory on Grassmanian $G(2,4) \cong$ proj lines in proj 3 space. dim = 4.
Given fixed L_i in $\mathbb{P}^3$, locus of lines meeting L_i is codim 1. Let $D \in H^2(G(2,4))$ be this coh. class. Compute D^4. Answer = 2.

3. Generalization of Schubert Calculus on Grassmanian is 'int. product

$$CH^p(X) \otimes_{\mathbb{Z}} CH^q(X) \to CH^{p+q}(X)$$

making $CH^*(X)$ into comm, graded, associative ring.

$$CH^0(X) = \mathbb{Z}$$

$$CH^p(X) = 0 \text{ for } p > \dim X.$$

∴ Chow is analog of even cohomology $\oplus_{i=0}^{n} H^{2i}(X, \mathbb{Z})$ in topology.

4. Following 'Cohomology like properties':

(1) $X \longmapsto \bigoplus_p CH^p(X)$ is contravariant functor from smooth varieties to graded rings.

(2) If X is projective, $n = \dim X$; well defined degree hom

$$\deg : CH^n(X) \to \mathbb{Z}.$$

$$\deg(\Sigma n_i P_i) \to \Sigma n_i$$

5. so you can define intersection Nos for cycles of codim 1. When $k = \mathbb{C}$, $\exists$ cycle class homs

$$CH^p(X) \to H^{2p}(X, \mathbb{Z})$$

So intersection product in Chow $\Leftrightarrow$ cup product in cohomology & cycle class of a vector bundle is p^{th} chern class of that vector bundle.

3. Gysin maps: If $f: X \to Y$ is proper morphism of smooth varieties

$d = \dim Y - \dim X$.

$$\exists\ f_*: CH^p(X) \longrightarrow CH^{p+d}(Y)\ \forall p.$$

Q: ~~clos~~ Can you compose them?

4. If V is a vector bundle (ie locally free sheaf) of rank r, $\exists$ chern classes $c_p(V) \in CH^p(X)$

s.t.: (a) $c_0(V) = 1$
(b) $c_p(V) = 0$ for $p > r$
(c) for any exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

$c(V_2) = c(V_1)\, c(V_3)$.
Whitney sum

where $c(\ ) = \sum c_p(\ )$ total chern class.

5. If $f: \mathbb{P}(V) \to X$ is the proj bundle ass. to V of rank r.

$CH^*(\mathbb{P}(V))$ is $CH^*(X)$ algebra generated by $\xi = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$, first Chern class of taut line bundle, ξ

$$\xi^r - c_1(f_*V)\,\xi^{r-1} + c_2(f_*V)\,\xi^{r-2} - \dots + (-1)^r c_r(V) = 0$$

Ex: for case of curves; $c_1: \mathrm{Pic}(X) \to CH^1(X)$ is isomorphism.

Grothendieck Riemann Roch:

Def: $K_0(X)$ = groth ring of vector bundles on X

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

modulo $[E_2] = [E_1] + [E_3]$

Note: Whitney sum ⇒ chern class well defined in $K_0(X)$

mult given by tensor product,

the chern character $ch(E) = \sum \exp(a_i)$, where a_i are roots of chern polynomial $\sum c_i(E) T^{r-i} = 0$

Todd class of $E = td(E) = \prod \frac{a_i}{1 - \exp(-a_i)}$

$$ch: K_0(X) \otimes \mathbb{Q} \xrightarrow{\sim} CH^*(X) \otimes \mathbb{Q}$$

Can extend notion of Chern classes to coh sheaves by identifying $K_0(X)$ with $G_0(X)$
Q: is $ch \circ f_* = f_* \circ ch$ in proper

Now GRR says that if $f: X \to Y$ & $\mathcal{F}$ coh sheaf on X. ($\in K_0(X)$)

$$f_*\left(ch(\mathcal{F})\, td(T_X)\right) = ch(f_*\mathcal{F})\, td(T_Y).$$

Divisors on varieties of Higher Dimension

3 types of equiv relations

for $CH^1(X) \cong Pic(X)$ so picture unchanged
$=$ Class group!!

If X is proj, define equiv rel on $CH^1(X)$ as follows:

Let C be any smooth curve & $D \subset X \times C$ be a divisor that does not contain fiber $X \times C \to C$. For any pair of pts p, q on C, divisor $D \cap X \times \{p\} - D \cap X \times \{q\}$ is a divisor on X. set it alg. equiv to 0.

Quotient of $CH_1(X)$ by this is called Neron Severi group $NS(X)$.

This generalizes degree hom:

$$cl: CH^1(X) \to NS(X).$$

Can also be defined using int. theory up to torsion.

alg =hom

Num.

In other words define divisor D to be numerically equiv to 0 if int No$(D\cdot C)=0$ $\forall$ curves C in X.

Then: in fact some multiple of D is actually alg equiv to 0.

$\therefore$ alg zero $\subset$ Num zero.

Lastly if $k=\mathbb{C}$, define homological equiv to zero as kernel of

$$CH^1(X) \to H^2(X,\mathbb{Z}).$$

(this comes from chow prop)

Alg = hom!!

Collection of effective divisors on X corresp to a fixed class c in $NS(X)$ forms a proj scheme $\mathrm{Hilb}_c(X)$.

Analogous to curves Kernel of $A^1(X)$

$$CH^1(X) \to NS(X)$$

is an abelian variety called Picard variety called $\mathrm{Pic}^0(X)$. If fix one divisor c we get

$$\mathrm{Hilb}_c(X) \to \mathrm{Pic}^0(X)$$

Abel Jacobi morphism

fibers of this corresp to effective divisors of fixed class in $CH^1(X)$.

Sufficiently large multiple of an ample class c, it surjects.

4. **Zero Cycles on Varieties of high dim:**

divisors = $CH_\circ(X)$.

In other words $CH_0(X)$, $\dim X = n$.

$$CH_0(X)_0 = \text{Ker}\left(\deg : CH^n(X) \to \mathbb{Z}\right)$$

This is zero cycles of degree 0 mod rational equiv

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alg =num =hom. cycles numerically equiv to 0

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" algebraically " "

Similary, surjective hom [Alb defined as dual to Pic univ bundle on $X \times \text{Pic}^\circ(X)$ induces morp $X \to \text{Alb}(X)$

$$CH_0(X)_0 \longrightarrow \text{Alb}(X) \quad /K$$

$$\begin{array}{ccc} d \Downarrow & & \uparrow \\ \text{Sym}(X) & \to & \text{Alb}(X) \text{ albanese} \end{array}$$

In general $CH_0(X)_0 \longmapsto \text{Alb}(X)$ is not an isomorphism (though it is on torsion)

Bloch conjectured that if $H^{2,0}(X) = 0$ then injective. $CH_0(X)_0 \cong \text{Alb}(X)$.

Bloch Beilinson $CH_0(X)_0 \underset{\text{over } \mathbb{Q}}{\cong} \text{Alb}(X)$.

5. Equivalence Relations:

3 notions: alg, num, homological

($\exists$ curve C, $\beta \in CH^p(X \times C)$, $x_1, x_2 \in C$ s.t. $i_1^*\beta - i_2^*\beta = \alpha$; $i_1 : X \hookrightarrow X \times C$) [int prod w/ $\beta \in CH^{n-p}(X) = 0$] [Ker $CH^p(X) \to H^{2p}(X, \mathbb{Z})$]

$$CH^p_{alg}(X) \subset CH^p_{hom}(X) \subset CH^p_{num}(X) \subset CH^p(X)$$

Grothendieck Std conjecture:

$$CH^p_{num}(X) = \{ x \in CH^p(X) \mid nx \in CH^p_{hom}(X) \text{ for some } n \}$$

iow

$$CH^p_{hom}(X) \otimes \mathbb{Q} = CH^p_{num}(X) \otimes \mathbb{Q}$$

To show this can show

$$CH^p_{\cancel{hom}\ alg}(X) \otimes \mathbb{Q} = CH^p_{num}(X) \otimes \mathbb{Q},$$

True for div & 0 cycles, but in general?

$$CH^p_{hom}(X) / CH^p_{alg}(X) = \mathrm{Griff}^p(X)$$

$$\mathrm{Griff}^n = \mathrm{Griff}^1 = 0$$

Bloch: Is griff divisible over alg closed fields?
Esnault: No

Non triv. of $\mathrm{Griff}^2(X) \neq 0$ for general hyper surface of deg 5 in $\mathbb{P}^4_{\mathbb{C}}$.

6. Chow & Abelian Varieties:

Q: Is there a fin gen. group $N(X)$ $(=\mathbb{Z})$

Surj map $\xi : CH^P(X) \to N(X)$

Let $C(X) = \operatorname{Ker} \xi$

map from $C(X)$ to pts on an Abelian variety A (Abel Jacobi)

so any map from $C(X)$ to an Abelian variety factors through A ?

No such general construction.

There is a complex torus associated to p cycles generalising Picard & Albanese varieties, called p^{th} intermediate Jacobian

$$J^P(X) = \frac{H^{2p-1}(X, \mathbb{C})}{F^P H^{2p-1}(X, \mathbb{C}) + \operatorname{im} H^{2p-1}(X, \mathbb{Z})}$$

p^{th} piece of Hodge filtration of $H^{2p-1}(X, \mathbb{C})$.

take $\xi : CH^p(X) \to H^{2p}(X, \mathbb{C})$

let $C(X) = \ker \xi = CH^p_{hom}(X)$

Abel-Jacobi map:

$$CH^p_{hom}(X) \to J^p(X)$$

Not Surjective

Conjectured injective on torsion elts.

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6 Relation with Cohomology!

Chow groups analog for even cohomology

To make more precise, look at cycle class map

$$CH^p(X) \to H^{2p}(X, \mathbb{Z})$$

Lefschetz thm on $(1,1)$ classes says image of

$CH^1(X) \to H^2(X, \mathbb{Z})$ is kernel of $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}) / F^1(H^2(X, \mathbb{C}))$

Equiv. w.r.t Hodge decomposition

$$H^m(X, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(X)$$

into spaces of harmonic forms $H^{p,q}(X)$

the classes of divisors in $H^2(X, \mathbb{Z})$ are precisely those s.t $\alpha_{\mathbb{C}} \in H^2(X, \mathbb{C})$ lies in $H^{1,1}(X)$.

Similar assertion for $CH^n(X) \to H^{2n}(X, \mathbb{Z})$ is obvious since this is surjective hom.

$\therefore$ Conjecture: image of $CH^p(X)$
$$\downarrow$$
$$H^{2p}(X, \mathbb{Z})$$

consists precisely of subgrp of Hodge classes

$$Hg^p(X, \mathbb{Z}) = \ker\left[H^{2p}(X, \mathbb{Z}) \to \frac{H^{2p}(X, \mathbb{C})}{F^p H^{2p}(X, \mathbb{C})} \right]$$

Known to be false unless we tensor w/ $\mathbb{Q}$

$\therefore$ $CH^p(X) \otimes \mathbb{Q} \to Hg^p(X, \mathbb{Q})$ is surjective is famous Hodge Conjecture

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