LECTURE 2
Friday 4th June 2004.

1. Orbifolds
2. Teichmuller Space
3. Mapping Class groups
4. Moduli Space
5. Hyperbolic Geometry
6. Fenchel Nielsen coordinates.

1. Orbifolds

Q. Why is M_g not a manifold if it is X_g / Γ_g?

A. Though Γ_g acts on X_g properly discontinuously, it does not do so freely. If it did, then M_g would be a manifold. But action not free because of presence of R.S. w/ automorphisms. Recall that in example of $g=1$, $\mathrm{aut}(R.S) \cong$ ~~isotropy~~ Stab(moduli pt corresponding to that R.S).

Def 2.1 An orbifold Q of dimension n is a Hausdorff topological space X_Q w/ an open covering $X_Q = \bigcup U_i$

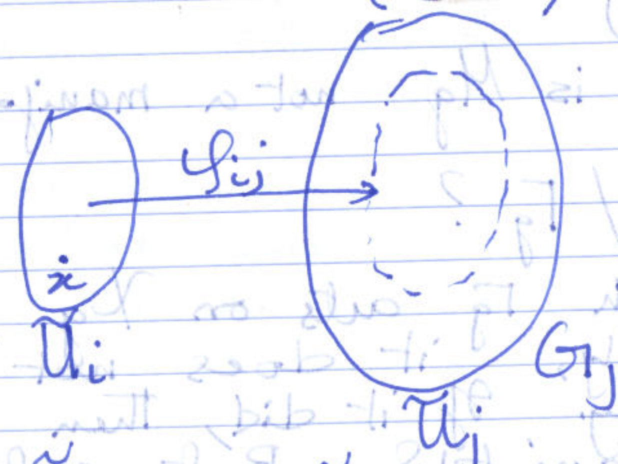
closed under finite intersections; along with homeomorphisms $\varphi_i : U_i \approx \tilde{U}_i / G_i$ where $\tilde{U}_i$ is an open domain in $\mathbb{R}^n$ & G_i is a finite group acting on $\tilde{U}_i$.

Compatibility requirements.

1. Whenever $U_i \subset U_j$, $\exists$ a monomorphism $f_{ij} : G_i \to G_j$ & an embedding $\tilde{\varphi}_{ij} : \tilde{U}_i \to \tilde{U}_j$ s.t.

$$\tilde{\varphi}_{ij}(\gamma x) = f_{ij}(\gamma)\, \tilde{\varphi}_{ij}(x)$$

$$\forall\, \gamma \in G_i,\ x \in \tilde{U}_i.$$



2.
$$\begin{tikzcd}
\tilde{U}_i \arrow[r] \arrow[d] & \tilde{U}_j \arrow[d] \\
\tilde{U}_i/G_i \arrow[r] \arrow[d, equal] & \tilde{U}_j / f_{ij}(G_i) \arrow[d] \\
\tilde{U}_i/G_i \arrow[r] & \tilde{U}_j/G_j \\
U_i \arrow[r] \arrow[u] & U_j \arrow[u]
\end{tikzcd}$$

Collection $\{\tilde{U}_i, G_i, \varphi_i\}$ is called a uniformising system for U_i.

To each x, associated G_x isotropy group. If G_x is trivial, x is nonsing. Else singular. Orbifold cell decomposition.

Euler char
Example
} GOTO previous lecture notes.

Orbifold fundamental group.

Composition of 2 paths $\alpha\beta$ is defined when $\alpha(1) = \beta(0)$. product traverses α first & β second.

To make sense; fix some conditions.

Note: $p: X \longrightarrow \Gamma \backslash X$ projection. Then if $x, y \in p^{-1}(x_0)$, Γ_y & Γ_x are conjugate in Γ.

Teichmuller Space

Intro: of genus g RS
Moduli space is space of isom. classes of complex structures on a genus g R.S.
In other words it is set of complex structures modulo orientation preserving homeomorphisms / diffeomorphism on R

$X_g \triangleq$ Teichmuller Suggested studying a finer space
Set of complex structures modulo homeomorphisms isotopic to identity.
Tricky since identity is different on different CRS's.

Formally: Fix a riemann surface R of genus g.
Marking is a homotopy class $[f]$ of orientation preserving diff $f: S \to R$

$X_g \triangleq$ Set of all marked Riemann surfaces of genus g upto equivalence

$$= \left\{ (S, f) \text{ s.t } S \text{ is a R.S. \& } f: S \to R \text{ is a marking} \right\} / \sim$$

$(S_1 f_1) \sim (S_2 f_2)$ if $\exists \varphi : S_1 \xrightarrow{\text{biholomorphic}} S_2$ such that $f_1 \circ \varphi$ is homotopic to f_2

$\Gamma_g \triangleq$ group of homotopy classes of orientation preserving diffeomorphisms $h: R \to R$

$\triangleq$ orientation preserving diffeo. mod those isotopic / homotopic to identity

$\equiv$ orientation preserving homeo mod those homotopic / isotopic to identity.

The forgetful mapping from $X_g \to M_g$ that forgets marking is actually the quotient by Γ_g.

Indeed we will put topology on M_g as quotient topology.

Theorem: X_g is homeomorphic to $\mathbb{R}^{6g-6}$ & Γ_g acts on X_g properly discontinuously

We will show that later.

~~Riemann~~ Uniformization theorem

(1) bihol to $\mathbb{C}$, $\hat{\mathbb{C}}$ or $\mathbb{H}$
(2) $\mathbb{C}$ & $\mathbb{H}$ homeo, but not same complex structure
(3) $\mathbb{C}, \mathbb{H} \subset \hat{\mathbb{C}}$, $\pi_1(S)$ is a disc sub of $PSL_2(\mathbb{C}) = PGL_2(\mathbb{C}) = Aut(\hat{\mathbb{C}})$
(4) Acts freely
(5) $\mathbb{C}$ covers tori, $\hat{\mathbb{C}}$ covers itself, $\mathbb{H}$ $g \geq 2$

Hyperbolic Geometry

Best way to study Teichmuller space. Link between Complex Analysis & hyperbolic geometry is that upper half plane $\mathbb{H}$ has a complete metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \left[\frac{\|z\|}{\operatorname{Im} z}\right]^2$$

of constant curvature -1 whose grp of orientation preserving isometries is $PSL_2(\mathbb{R})$. $(\mathbb{H}, ds^2)$ often called Poincaré upper half plane.

Fact that group of biholomorphisms of $\mathbb{H}$ = group of orientation preserving isometries of $\mathbb{H}$ is fundamental.

Because if X is a hyperbolic R.S (univ cover is $\mathbb{H}$), then it comes w/ a representation as a quotient of $\mathbb{H}$ by a discrete subgroup Γ of $PSL_2(\mathbb{R})$ $\cong$ isom to $\pi_1(X)$. (Fuchsian group)

Now the Poincare metric ds^2 is invariant under this subgroup & so descends down to $X = \Gamma \backslash \mathbb{H}$. It is complete.

$\therefore$ {Complex structures}/ isomorphism

$\Updownarrow$ correspond

{hyperbolic structures of constant -ve curvature}/isometries

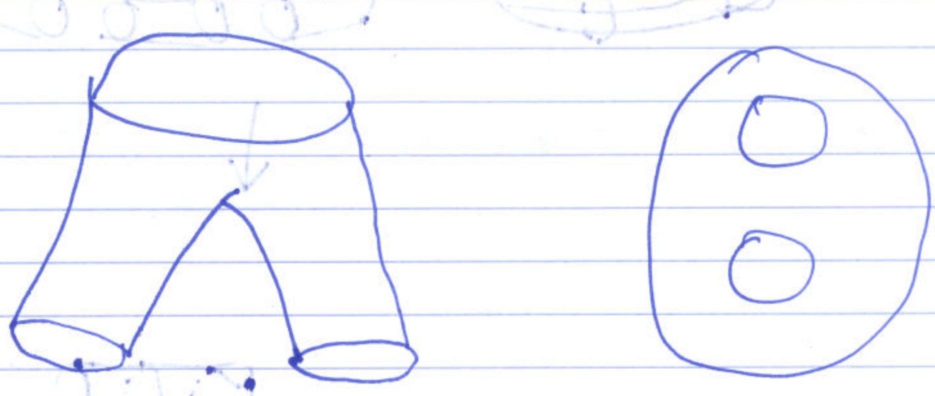
$\therefore g \geq 2$

M_g = isometry classes of compact oriented surfaces of genus g w/ hyperbolic metric.

Fenchel Nielsen Coordinates

Interpretation of Teichmuller space in terms of hyperbolic geometry allows us to define coordinates on X_g.

Pairs of pants



Prop. The complex structure on a pair of pants is uniquely determined by the hyperbolic lengths of its ordered bndry components.

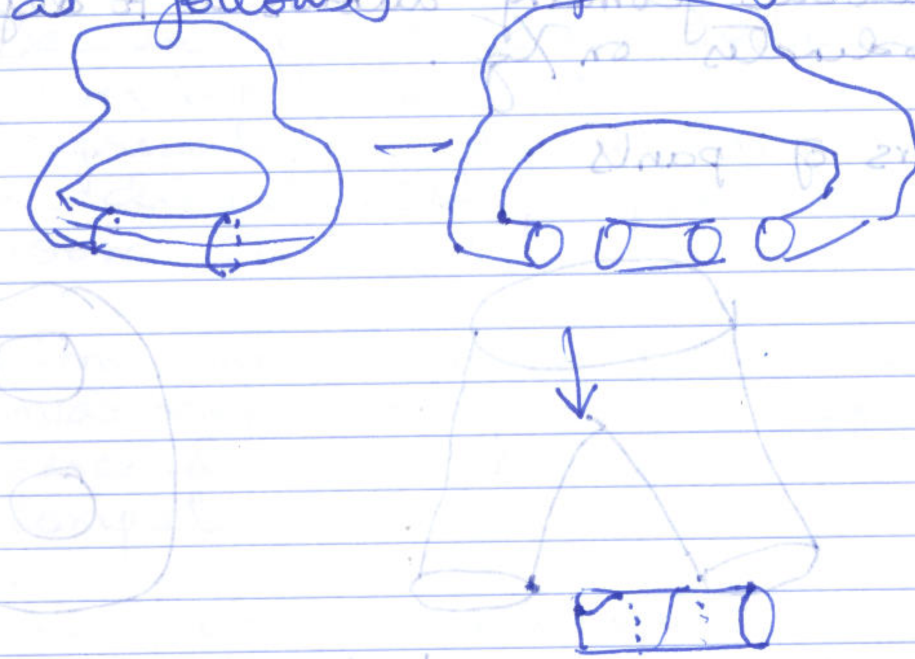
Bers Pants decomposition

Theorem: Riemann surface of genus g ($g \geq 2$) always contains a loop system of $3g-3$ disjoint simple closed geodesics. Regardless of which loop system you choose, cutting R along geodesics always decomposes R into $2g-2$ pants.

$\therefore X_g$

Dehn Twists :

We can "glue" pairs of pants together as follows



Theorem: Any point in X_g can be specified by $3g-3$ lengths & $3g-3$ twists
$\therefore$ X_g is set theoretically $\mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$

Complex structure, metric on X_g making it a manifold. [very hard]

Moduli Space :

M_g is a complex analytic variety whose singularities are finite quotient.
M_g can be regarded as orbifold w/ universal cover X_g & orbifold

fundamental group is Γ_g. More precisely if C is a compact genus g R.S. w/ no non trivial auts,

$$\pi_1^{orb}(\mathcal{M}_g, [C]) \cong \pi_0 \operatorname{Diff}^+ C.$$

connected components

More is known

$$\Phi : \Gamma_S \xrightarrow{\text{homom.}} \operatorname{Out} \pi_1(S)$$

Dehn Nielsen: Φ is injective & $\operatorname{Im}\Phi$ is a subgrp of index 2.