NOTES FOR FIRST TALK (5/28)

Lecture 1: Introduction & low genus examples.

- g, n are nonnegative integers.

Defn: 1.1 An "n-pointed Riemann Surface" $(C; x_1, \ldots, x_n)$ of genus g is a compact Riemann Surface C of genus g together w/ an ordered n-tuple of points $(x_1, \ldots, x_n)$ on C.

Defn: 1.2 2 n-ptd Riemann surfaces $(C; x_1, \ldots x_n)$ & $(C'; x_1', \ldots, x_n')$ are "isomorphic" if there is a biholomorphism $f: C \to C'$ s.t. $f(x_i) = x_i'$ for $1 \leq i \leq n$.

We are interested in studying isomorphism classes of these n-ptd Riemann Surfaces.

Defn: 1.3 The moduli space of n-pointed Riemann Surfaces, denoted $M_{g,n}$, is defined as the space of isomorphism classes of n-ptd CRS of genus g.

ie; $$M_{g,n} \triangleq \left\{ \begin{array}{l} \text{isom. classes of n-ptd} \\ \text{CRS of genus } g \end{array} \right\}.$$

Defn: 1.4 The isom class of $(C; x_1, \ldots, x_n)$ in $M_{g,n}$ is called moduli point of $(C; x_1, \ldots, x_n)$ and denoted by $[C; x_1, \ldots, x_n]$

2 prelimpts: There are atleast 2 notions of genus of a CRS.

(i) Analytic genus $\triangleq \dim H^0(C, \Omega^1_C)$

(ii) Topological " $\triangleq \frac{\text{rank}}{2} H_1(C, \mathbb{Z})$

Both are equivalent by Hodge Theorem that says $H^1(C, \mathbb{C}) \cong \{\text{holomorphic 1 forms}\} \oplus \{\text{Anti holomorphic 1 forms}\}$

Also "complex curve" & "Riemann Surface" mean same thing.

GENUS ZERO.

Fact 1.5: From Riemann Roch, follows that any genus 0 CRS, is biholomorphic to Riemann Sphere $\mathbb{P}^1$. Thus M_0 is a single pt $[\mathbb{P}^1]$.

Defn 1.6 An automorphism of a RS, X, is just a biholomorphism $f: X \to X$. Set of all automorphism forms a group $\mathrm{Aut}(X)$.

Fact 1.7 The automorphism group of $\mathbb{P}^1$ is $PGL_2(\mathbb{C}) \cong PSL_2(\mathbb{C})$ acting as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto \frac{az+b}{cz+d}.$$

Fact 1.8 Aut $\mathbb{P}^1$ acts 3-transitively on $\mathbb{P}^1$. in other words, given 2 ordered 3-tuples (a_1, a_2, a_3) & (b_1, b_2, b_3), there exists

an automorphism f of $\mathbb{P}^1$ s.t $f(a_i) = b_i$ $i=1,2,3$.
Further f is unique.

Fact 1.9 The automorphism group of a n-ptd R.S is finite iff the stability condition below is satisfied

$$2g + n - 2 > 0$$

Prop 1.10 Since $\mathrm{Aut}(\mathbb{P}^1)$ acts 3 transitively on $\mathbb{P}^1$,

every 1-pt RS of genus 0 is isom to $(\mathbb{P}^1; \infty)$
" 2-pt " to $(\mathbb{P}^1; 0, \infty)$
" 3-pt " to $(\mathbb{P}^1; 0, 1, \infty)$

$\therefore$ If $0 \leq n \leq 3$; $M_{g,n}$ is just one pt.

So first interesting case is $n=4$.

If $(X; x_1, x_2, x_3, x_4)$ is a 4 pt. RS of genus 0, then there is a unique biholomorphism
$f: X \longrightarrow \mathbb{P}^1$ s.t
$$f(x_2) = 0$$
$$f(x_3) = 1$$
$$f(x_4) = \infty$$

Value of $f(x_1)$ is forced by these conditions. Since $\{x_i\}$ are distinct & f is a biholomorph
$f(x_1) \in \mathbb{C} - \{0, 1\}$ and is

an invariant of $(X; x_1, x_2, x_3, x_4)$.

Fact 1.11 If g is any bihilom. $g : X \to \mathbb{P}^1$, $f(x_1)$ is the cross ratio $(g(x_1) : g(x_2) : g(x_3) : g(x_4))$

where cross ratio of 4 distinct pts $p_1\ p_2\ p_3\ p_4$ on $\mathbb{P}^1$ is

$$= \frac{(p_1 - p_3)/(p_2 - p_3)}{(p_1 - p_4)/(p_2 - p_4)}$$

Prop 1.12 $M_{0,4}$ can be identified with $\mathbb{C} - \{0,1\}$. The moduli pt. $\{\mathbb{P}^1; x_1 x_2 x_3 x_4\}$ ~~can~~ is identified w/ cross ratio $(x_1 : x_2 : x_3 : x_4) \in \mathbb{C} - \{0,1\}$.

Now generalise to $n \geq 4$.

From previous discussion, every n-pt RS of genus 0 is isom to exactly one of form $\{\mathbb{P}^1; 0, 1, \infty, y_1 \dots y_{n-3}\}$

To say this is n-pt means $0, 1, \infty, y_1 \dots y_{n-3}$ are all distinct. In other words $(y_1, \dots, y_{n-3}) \in (\mathbb{C} - \{0,1\})^{n-3} - \Delta$.

where $\Delta = \bigcup_{j<k} \Delta_{jk}$ is union of diagonals

$$\Delta_{jk} = \{(y_1, \dots, y_{n-3}) : y_j = y_k\}$$

This is an affine algebraic variety since its complement is a divisor in an affine space.

Theorem 1.13 If $n \geq 3$ $M_{0,n}$ is a smooth affine algebraic variety of dimension $n-3$ isomorphic to

$$(\mathbb{C} - \{0,1\})^{n-3} - \Delta .$$

Defn 1.14 UNIVERSAL CURVE

For $n \geq 3$; there is a 'universal' genus 0 curve $M_{0,n} \times \mathbb{P}^1 \longrightarrow M_{0,n}$ equipped with n disjoint sections $\sigma_1 \dots \sigma_n$ s.t

$$\sigma_j([\mathbb{P}^1; x_1 \dots x_n]) = ([\mathbb{P}^1; x_1 \dots x_n], x_j)$$

w/ the universal property that if $f : X \longrightarrow T$ is any family of smooth genus 0 curves over a smooth variety T & if family has n-sections $s_1 \dots s_n$ (disjoint), then $\exists$ a holomorphic map $\phi_f : T \to M_{0,n}$ s.t pullback of universal family is f & pullback of σ_j is s_j.

GENUS 1

Defn: 1.15 A lattice in a fin. dim real vector space V is a finitely generated ($\therefore$ free abelian) subgroup Λ of V with the property that a basis of Λ as abelian group is also basis as a real vector space.

A lattice in $\mathbb{C}$ is a subgroup generated over $\mathbb{R}$ by 2 complex numbers that are indep over $\mathbb{R}$.

Ext 1.16 If Λ is a lattice in V; $\dim_{\mathbb{R}} V = d$; then V/Λ is a compact manifold of real dimension d, diffeomorphic to d-torus $(\mathbb{R}/\mathbb{Z})^d$.

$\therefore$ If Λ is lattice in $\mathbb{C}$, $\mathbb{C}/\Lambda$ is compact Riemann surface of genus 1. The converse is also true

Theorem 1.17: If C is a compact Riemann surface of genus 1, then $\exists$ a lattice Λ in $\mathbb{C}$ and an isomorphism $\mu : C \to \mathbb{C}/\Lambda$. If $x_0 \in C$, then we can choose μ such that $\mu(x_0) = 0$.

Proof: $\because g = 1$; $H_1(C, \mathbb{Z})$ is free of rank 2. Fix a nonzero holomorphic diff ω on C. Every other holomorphic diff is multiple of ω. The period lattice of C is

$$\Lambda = \left\{ \int_c \omega : c \in H_1(C, \mathbb{Z}) \right\}$$

This is a subgroup of $\mathbb{C}$.

Λ is a lattice in $\mathbb{C}$ since if a, b are basis for $H_1(C, \mathbb{Z})$; $\int_a \omega$ & $\int_b \omega$ are linearly independent over $\mathbb{R}$.

Let $E = \mathbb{C}/\Lambda$. We now want to construct a bihol. map from C to E.

Define mapping $\nu(x) = \int_\gamma \omega$.

Where x_0 is a prechosen base pt on C & γ is a smooth path from x_0 to x.

This map ν is a biholomorphism.

Corollary 1.18 If C is CRS of genus 1, $\mathrm{aut}(C)$ acts transitively on C. Consequently, the natural map $\mathcal{M}_{1,1} \to \mathcal{M}_1$

$$[C; x] \mapsto [C] \text{ is a bijection.}$$

Proof: Since every CRS of genus 1 is of form $\mathbb{C}/\Lambda$ we have

$$\mathbb{C}/\Lambda \to \mathrm{Aut}(\mathbb{C}/\Lambda)$$

$$a + \Lambda \mapsto z + a + \Lambda.$$

$\therefore$ If C is a genus 1 curve; $[C; x_0] = [C; y_0]$ for any $x_0, y_0 \in C$.

Moduli Space of ECs is $\mathcal{M}_{1,1}$.

Prop 1.19: If Λ_1 & Λ_2 are lattices in $\mathbb{C}$; $\mathbb{C}/\Lambda_1 \cong \mathbb{C}/\Lambda_2$ iff $\exists\ \lambda \in \mathbb{C}^*$ s.t $\Lambda_1 = \lambda \Lambda_2$

Construction of $\mathcal{M}_1$:

Complx structure on a R.S $\mathbb{C}$ gives it a canonical orientation (@ each pt, +ve is going counter clockwise in a local chart). If C is compact, this allows us to define the intersection No. of 2 transversally intersecting curves on $\mathbb{C}$. It depends only on the homology classes of the 2 curves & gives an intersection pairing

$$\langle\ ,\ \rangle : H_1(C,\mathbb{Z}) \otimes H_1(C,\mathbb{Z}) \to \mathbb{Z}$$

Let α, β be a basis for $H_1(C,\mathbb{Z})$ s.t $\langle \alpha, \beta \rangle = +1$. Then it is called a positive basis or a frame. 2 framed genus 1 surfaces (C, α, β) & (C', α', β') are isomorphic if $\exists$ biholomorphic f s.t

$$f: C \to C'$$
$$\alpha \mapsto \alpha'$$
$$\beta \mapsto \beta'$$

Defn: 1.20: Let $\mathcal{X}_1 = \{$ isom classes of framed R.S of genus 1 $\}$

Defn 1.21 If $(\alpha\ \beta)$ & $(\alpha'\ \beta')$ are 2 frames for C, then $\exists$ a unique element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{Z})$ s.t.

$$\begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

This defines a $SL_2(\mathbb{Z})$ action on $[C, \alpha, \beta]$.

Prop: 1.22 $M_1 \simeq SL_2(\mathbb{Z}) \backslash X_1$

At present X_1 is just a set. But we will show it is a Riemann surface itself. We know from Theorem 1.17 that every element of X_1 is of form $[\mathbb{C}/\Lambda : \alpha, \beta]$. But from Alg Top. there is a natural isomorphism

$$\Lambda \simeq H_1(C, \mathbb{Z}).$$ Whereby a basis of $H_1(C, \mathbb{Z})$ corresponds to a basis of Λ.

Fact: 1.23 If α, β is a basis for $H_1(\mathbb{C}, \mathbb{Z})$ & $\omega_1\ \omega_2$ basis for Λ, α, β is +ve iff ω_2/ω_1 has +ve imaginary part.

$$\therefore \quad X_1 = \{[\mathbb{C}/\Lambda : \omega_1, \omega_2] : \operatorname{Im}(\omega_2/\omega_1) > 0\} \Big/ \mathbb{C}^*$$

Where $\mathbb{C}^*$ action is defined by

$$\lambda \cdot [\mathbb{C}/\Lambda : \omega_1, \omega_2] = [C/\lambda\Lambda : \lambda\omega_1, \lambda\omega_2]$$

Since basis $\omega_1\ \omega_2$ determines the lattice

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$$

Can dispense w/ lattice & define

(2) $$X_1 = \{(\omega_1, \omega_2) : \omega_1\ \omega_2 \in \mathbb{C} \ \& \ \mathrm{Im}(\omega_2/\omega_1) > 0\} / \mathbb{C}^*$$

where $\mathbb{C}^*$ acts on (ω_1, ω_2) by scalar multiplication.

—

Denote upper half plane $\{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ by $\mathbb{H}$. Each $\tau \in \mathbb{H}$ determines an element

$$[\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) : 1, \tau] \text{ in } X_1.$$

defining a function $\psi : \mathbb{H} \to X_1$.

But because of (2); this is a bijection.

Fact 1.24. $PSL_2(\mathbb{C})$ acts on $\mathbb{P}^1$ by fractional linear transform

(i) $T \in PSL_2(\mathbb{C})$ satisfies $T(\mathbb{H}) \subset \mathbb{H}$ iff $T \in PSL_2(\mathbb{R})$.

$\therefore PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by fract linear trans.

(ii) $\psi : \mathbb{H} \to X_1$ is $SL_2(\mathbb{Z})$ equivariant ie if $T \in SL_2(\mathbb{Z})$;

$$T(\psi(\tau)) = \psi(T\tau)$$

Theorem 1.25 $\quad M_1 \cong \mathcal{M}_{1,1} \cong SL_2(\mathbb{Z}) \backslash \mathbb{H}.$

Theorem 1.26. More concretely; Let C be a genus 1 Riemann Surface. Then $[C]$ goes to the $SL_2(\mathbb{Z})$ orbit of $\cdot \in M_1$

$$\left(\int_\beta \omega \Big/ \int_\alpha \omega \right) \in \mathbb{H}.$$

where ω is any nonzero element of $H^0(C, \Omega^1_C)$ & (α, β) is a +ve basis of $H_1(C, \mathbb{Z})$.

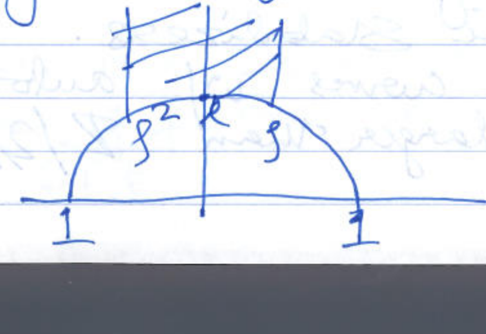
Understanding $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ (to motivate concept of orbifold)

Riemann Surface structure on $\mathbb{H}$ descends to R.S.S on $SL_2(\mathbb{Z}) \backslash \mathbb{H}$.

Fact 1.27 Action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ is generated by

$$z \to z+1$$
$$z \to -1/z$$

$\therefore$ The fundamental domain for this action is



M_1 Can be obtained from this domain by identifying sides & arcs to give "pinched straw"!
This action is not free. There are 2 fixed points i & $\frac{1+\sqrt{3}i}{2}$ $(=\sqrt[3]{-1})$.

Remark 1·28 If action were free, $SL_2(\mathbb{Z}) \setminus \mathbb{H}$ would be a manifold w/ fundamental group $SL_2(\mathbb{Z})$. Instead M_1 is homeomorphic to $\mathbb{R}^2$ (disc).

Discussion 1·29 Conceivable that M_1 is biholomorphic to a disc. But this is not the case since M_1 can be compactified by adding one point ∞. $\therefore$ M_1 is biholomorphic to $\mathbb{C}$!

Remark 1·30 Compactification is $\overline{M_1}$.

Remark 1·31 Shows M_1 is a variety since all CRS are canonically complex curves.

Automorphisms.

Automorphisms of an elliptic curve are intimately linked to set of elements of $SL_2(\mathbb{Z})$ that stabilize pts v in $\mathbb{H}$ corresp-to it.

2 pts lying in same orbit of $SL_2(\mathbb{Z})$ have isomorphic stabilizers. $\therefore$ to look for elliptic curves w/ automorphism group larger than $\mathbb{Z}/2\mathbb{Z}$, one has to look at

points in fundamental domain w/ stabilizer larger than $\mathbb{Z}/2\mathbb{Z}$.

Fact 1.31 Stabilizer of $\tau \in \mathbb{H}$ in $SL_2(\mathbb{Z})$ is

$\mathbb{Z}/2\mathbb{Z}$ if $\tau \notin$ orbit of i & ρ

$\mathbb{Z}/4\mathbb{Z}$ if $\tau \in$ orbit of i

$\mathbb{Z}/6\mathbb{Z}$ if $\tau \in$ orbit of ρ.

Theorem 1.32 If $(C; x_0)$ is an elliptic curve,

$$\mathrm{Aut}(C; x_0) = \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if } (C, x_0) \cong \mathbb{C}/\mathbb{Z}[i] \\ \mathbb{Z}/6\mathbb{Z} & \cong \mathbb{C}/\mathbb{Z}[\rho] \\ \mathbb{Z}/2\mathbb{Z} & \text{else} \end{cases}$$

Ex: Fermat cubic $x^3+y^3+z^3=0$ has $\mathbb{Z}/6\mathbb{Z}$ $\Rightarrow$ it is isom to $\mathbb{Z}/\mathbb{Z}[\rho]$

Families of Genus 1 & Elliptic Curves:

Defn 1.33 Consider X is a cmplx analytic manifold $f: X \to T$ holomorphic s.t each fiber is genus 1 curve. Call this a family of genus 1 curves

If there is a section $s: T \to X$ of f, then fiber of f over $t \in T$ is elliptic curve w/ identity $s(t)$.

Construction 1.34 Each family gives rise to a map called period mapping:

$$\phi_f : T \to \mathcal{M}_1$$

$$t \mapsto [X_t] \text{ of fiber of } f \text{ at } t.$$

Theorem 1.35 (1) ϕ_f is holomorphic.

(2) $\phi_f : T \to \mathcal{M}_1$ is locally liftable to a map to $\mathbb{H}$

(3) $\phi_f : T \to \mathcal{M}_1$ is globally lifted to a map $\tilde{T} \xrightarrow{\tilde{\Phi}_f} \mathbb{H}$ s.t.

$$\begin{tikzcd} \tilde{T} \arrow[r, "\tilde{\phi}_f"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{H} \arrow[d] \\ T \arrow[r, "\phi_f"] & \mathcal{M}_1 \end{tikzcd}$$

(4) there is a homomorphism $\phi_{f*} : \pi_1(T, t_0) \to SL_2(\mathbb{Z})$

s.t. $\tilde{\phi}_f(\gamma . x) = \phi_{f*}(\gamma) . \tilde{\phi}_f(x)$

$\forall x \in \tilde{T};\ \gamma \in \pi_1(T, t_0).$

Corollary 1.36 Not every holomorphic map is a period mapping of a family $\therefore$ not every holomorphic map is locally liftable

Fact 1.37 identity $M_1 \longrightarrow M_1$ is NOT locally liftable

Corollary 1.38 There does not exist a universal elliptic curve (or even genus 1 curve)

ORBIFOLDS.

Def 1.39 Orbifold is locally the quotient of a manifold by a group acting properly discontinuously s.t all isotropy groups are finite

ex: M_1 is quotient of $\mathbb{H}$ by $SL_2(\mathbb{Z})$

Basically manifold is locally $\mathbb{R}^n$
orbifold is " $G\backslash\mathbb{R}^n$

Every manifold is orbifold but not conversely.

To each point of an orbifold is associated a finite isotropy group. Pts w/ trivial isotropy are called non singular. Others called singular.

efn: An exhaustive decomposition of X_Q into strata is called a cell decomposition.

Defn 1.40 The Euler Characteristic of an orbifold is

$$\chi(Q) = \sum_{c_i} (-1)^{\dim c_i} \frac{1}{|G_{c_i}|}$$

where c_i is i^{th} cell in a cell decomps of X_Q

Example 1.41 $(0,1)$ is a 1 dim ~~stri~~ manifold. Its χ ~~ht~~ is 1 in both manifold & orbifold sense.

Now let $\mathbb{Z}/2\mathbb{Z}$ act on $(0,1)$ by reflection. Quotient is $(0, \frac{1}{2}]$. This is an orbifold w/ istoropy grps $\mathbb{Z}/2\mathbb{Z}$ for $\{\frac{1}{2}\}$ & other pts nonsingular.

$$\chi(\text{orbifold}) = \frac{1}{2}$$

$\therefore$ Satisfies covering condition for orbifolds.

$$\chi(\text{manifold}) = 2\,\chi(\text{orbifold})$$