Tuesday
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~~Traces & Tensors~~

LECTURE NO. 2 ON HODGE THEORY

HODGE THEORY ON HERMITIAN & KAHLER MANIFOLDS

Def: Let $E \to M$ be a cmplx vector bundle. A "hermitian metric" on E is a hermitian inner product on each fiber E_x of E, varying smoothly w/ x, ie s.t. if $\xi = \{\xi_1 \cdots \xi_k\}$ is a frame for E then

trans funcs : $h_{ij}(x) = (\xi_i(x), \xi_j(x))$

are C^∞.

Def: A holomorphic vector bundle w/ a hermitian metric is called a hermitian vector bundle.

Def: A connection D on a cmplx vector bundle $E \to M$ is a map

$$D : A^0(E) \to A^1(E)$$

$A^p(E)$ E valued p-forms

satisfying Leibniz rule

$$D(f \cdot \xi) = df \otimes \xi + f \cdot D(\xi)$$

$\forall$ sections $\xi \in A^0(E)(U)$, $f \in C^\infty(U)$

Let $e = \{e_1 \dots e_n\}$ be a frame for E over U. We can write De_i into

$$De_i = \sum_j \theta_{ij} e_j$$

The matrix $\theta = [\theta_{ij}]$ of 1 forms is called connection matrix of D w/r.t $\{e\}$.

Connection matrix depends on frame. i.e. no "natural" connection. But we can add restriction

(1) Using $T^* = T^{*'} \oplus T^{*''}$, we can write $D = D' + D''$ where

$$D' : A^0(E) \longrightarrow A^{1,0}(E)$$
$$D'' : A^0(E) \longrightarrow A^{0,1}(E)$$

Now we say a D is "compatible" w/ complex structure if $D'' = \bar{\partial}$.

(2) If E is hermitian, D is compatible w/ metric if

$$d(\xi, \eta) = (D\xi, \eta) + (\xi, D\eta)$$

Fact If E is a hermitian v.b on a complex manifold, $\exists!$ unique connection D on E compatible w/ both metric & complex structure. Called Chern connection.

This is known as the "associated metric connection".

Similarly can define operators

$$D: A^p(E) \to A^{p+1}(E)$$

by forcing Leibniz rule

$$D(\psi \wedge \xi) = d\psi \otimes \xi + (-1)^p \psi \wedge D\xi.$$

for $\psi \in A^p(U)$; $\xi \in A^0(E)(U)$.

$\therefore$ Can discuss

$$D^2: A^0(E) \to A^2(E)$$

Notice D^2 is linear over A^0. ie for σ a section of E & f a C^∞ fn.

$$D^2(f\cdot\sigma) = D(df \otimes \sigma + f\cdot D\sigma)$$

$$= -df \wedge D\sigma + df \wedge D\sigma + f\cdot D^2\sigma$$

$$= f\cdot D^2\sigma$$

$\therefore$ $D^2 : A^0(E) \to A^2(E)$ is induced by a bundle map $E \to \Lambda^2 T^* \otimes E$. In other words, D^2 corresp to a global section Θ of bundle

$$\Lambda^2 T^* \otimes \mathrm{Hom}(E, E)$$

$$= \Lambda^2 T^* \otimes (E^* \otimes E)$$

So if $\{e\}$ is a frame for E, then in terms of frame $\{e_i^* \otimes e_j\}$ of $E^* \otimes E$. can represent θ^2 by a matrix of 2-forms ie

$$D^2 e_i = \sum \theta_{ij} \otimes e_j$$

θ_e is called curvature matrix of frame e. It is $k \times k$ matrix of 2 forms.

If $\{e_i' = \sum g_{ij} e_j\}$ is another frame

$$\Theta_{e'} = g \cdot \theta_e \cdot g^{-1}$$

Curvature matrix & connection matrix are related

$$\begin{aligned} D^2 e_i &= D\left(\sum \theta_{ij} \otimes e_j\right) \\ &= \sum \left(d\theta_{ij} - \sum \theta_{ik} \wedge \theta_{kj}\right) \otimes e_j \end{aligned}$$

in matrix notation

$$\Theta_e = d\theta_e - \theta_e \wedge \theta_e \quad \text{[Cartan structure eqn.]}$$

Sanity check: both sides are 2-forms.

Fact — Curvature matrix of metric connection of a hermitian bundle is a hermitian matrix of (1,1)-forms.

Laplacians on complex manifolds:

$$d = \underset{(1,0)}{\partial} + \underset{(0,1)}{\bar\partial}$$

where $\partial^2 = \bar\partial^2 = 0$ & $\partial\bar\partial + \bar\partial\partial = 0$

∂ & $\bar\partial$ have adjoints $\partial^*, \bar\partial^*$ resp & $d^* = \partial^* + \bar\partial^*$

Set $\Delta_\partial = \partial\partial^* + \partial^*\partial$

$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$.

Then $\Delta = \underbrace{(\partial + \bar\partial)}_{d}\underbrace{(\partial^* + \bar\partial^*)}_{d^*} + \underbrace{(\partial^* + \bar\partial^*)}_{d^*}\underbrace{(\partial + \bar\partial)}_{d}$

$= \Delta_\partial + \Delta_{\bar\partial} +$ Cross terms.

Suppose cross terms vanish

$\Delta = \Delta_\partial + \Delta_{\bar\partial}$

If $\phi = \sum \phi_{p,q}$ are harmonic, so are $\phi_{p,q}$.

also if ϕ is real $\overline{\phi_{p,q}} = \phi_{q,p}$

$\therefore$ Obtain Hodge decom on space of harmonic

K forms

$$H^K_\Delta = \bigoplus_{p+q=K} H^{p,q}$$

$$\overline{H^{p,q}_\Delta} = H^{q,p}_\Delta$$

Kahler manifolds:

If $d\omega = 0$, M is Kahler.

Kahler identities :-

Define $L(x) = \omega \wedge x$

$L^* = \Lambda$

$\therefore (Lx, y) = (x, \Lambda y)$

Kahler identity: $[\Lambda, \partial] = i\bar{\partial}^*$

Corollary $[\Lambda, \bar{\partial}] = -i\partial^*$

Now crs terms vanish

(1) $\partial\bar{\partial}^* + \bar{\partial}^*\partial = -\sqrt{-1}\,(\partial\Lambda\partial - \partial^2\Lambda + \Lambda\partial^2 - \partial\Lambda\partial) = 0$

(2) $\Delta_\partial = \Delta_{\bar{\partial}}$; $\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$

Consequences of Kähler:

(1) First Betti No. is even
(rank of $H_1(X,\mathbb{Z})$)

(2) Second Betti No is nonzero.

(3) Holomorphic p forms are closed

(4) Dolbeault theorem

$$H^{p,q}(X) \cong H^q(X, \Omega^p)$$

Where Ω is space of holomorphic p-forms.

Proof: Consider

$$\Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \mathcal{E}^{p,2} \to \cdots$$

$$H^q(\Omega^p) = \frac{\bar\partial \text{ closed } (p,q) \text{ forms}}{\text{exact } (p,q) \text{ forms}}$$

$\cong$ $\Delta_{\bar\partial}$- harmonic (p,q) forms

$\cong$ harmonic p,q forms
(by proportionality)

(5) Hodge Diamond.

$$h^{00} = 1$$

$$h^{10} \qquad\qquad h^{01}$$

$$h^{20} \qquad h^{11} \qquad h^{02}$$

$$h^{n,n-1} \qquad\qquad h^{n-1,n}$$

$$h^{nn} = 1.$$

has vertical & lateral symmetry:

ie $h^{10} = h^{01} = h^{n,n-1} = h^{n-1,n}$

Hodge Decomposition Thm for Kahler

$$H^{p,q}_{BC}(X) \longrightarrow H^{p+q}_{DR}(X, \mathbb{C})$$

is injective

$$H^{l}_{dR}(X, \mathbb{C}) = \bigoplus_{p+q=l} H^{p,q}(X)$$

$$\overline{H^{p,q}(X)} = H^{q,p}(X).$$

Corollary

$$\left.\begin{array}{l} h^{p,q}(X) < \infty \\ h^{0,0}(X) = h^{nn}(X) = 1 \\ h^{p,q} = h^{n-p,n-q}(X) \end{array}\right\} \text{for compact cmplx.}$$

If Kahler;

$$h^{p,q}(X) = h^{q,p}(X)$$

$$b_l(X) = \sum_{p+q=l} h^{p,q}(X)$$

$$b_{odd}(X) = \text{even}$$

$$h^{p,p}(X) \geq 1 \quad 0 \leq p \leq n.$$