TALK 3

HECKE OPERATORS

(on spaces of modular forms)

Motivation: Recall that cusp forms of weight 12 for $\Gamma(1)$ form a 1-dimensional $\mathbb{C}$ vector space, with basis

$$\Delta = g_2^3 - 27 g_3^2 \qquad \left(\begin{array}{l} g_2 = 60 G_2 \\ g_3 = 140 G_3 \end{array} \right.$$

G_2; G_3 are eisenstein series)

Geometrically; $\Delta(z)$ is the discriminant of elliptic curve

$$\frac{\mathbb{C}}{\mathbb{Z}z + \mathbb{Z}}.$$

Jacobi got the "q expansion" of Δ $(q = e^{2\pi i z}$

$$\Delta(z) = (2\pi)^{12} \cdot q \cdot \prod_{n=1}^{\infty} (1-q^n)^{24}$$

Let $f(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum \tau(n) \cdot q^n$

$n \longmapsto \tau(n)$ is the Ramanujan τ function

Conjectures (Ramanujan):

(a) $|\tau(p)| \leq 2p^{\frac{11}{2}}$

(b) $\begin{cases} \tau(mn) = \tau(m)\tau(n) \text{ if } (m,n) = 1 \\ \tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1}) \\ \quad \text{" } p \text{ is prime } \& \; n \geq 1 \end{cases}$

(b) was proved by Mordell in 1917 in a paper that he introduced Hecke Operators.

To Δ we attach a Dirichlet series

$$L(\Delta, s) = \sum \tau(n)\, n^{-s}$$

<u>Lemma:</u>
It can be shown that $L(\Delta, s)$ has a Euler product expansion

$$L(\Delta, s) = \prod_{p \text{ prime}} \frac{1}{\left(1 - \tau(p)\, p^{-s} + p^{11-2s}\right)}$$

$$\Longleftrightarrow \quad \text{(b) holds.}$$

<u>Proof:</u> For prime p; define

$$L_p(s) \triangleq \sum_{m \geq 0} \tau(p^m)\, p^{-ms}$$

$$= 1 + \tau(p) \cdot p^{-s} + \tau(p^2)(p^{-s})^2 + \cdots$$

If $n \in \mathbb{N}$ is $n = \prod p_i^{a_i}$ then coefficient of (n^{-s}) in $\prod L_p(s)$ is $\prod \tau(p_i^{a_i})$. From (b), this is equal to $\tau(n)$.

$$\therefore \quad \prod L_p(s) = L(\Delta, s).$$

Now consider

$$\left(1 - \tau(p)\, p^{-s} + p^{11-2s}\right) \cdot L_p.$$

Coeff of $(p^{-s})^n$ is $\tau(p^{n+1}) - \tau(p)\tau(p^n) + p^{11}\tau(p^{n-1})$ for $n+1$

& 1 for $n = 0$

Proposition: Let $f(z) = \sum c(n) q^n$ be a modular form of weight $2k$. If f is an eigenfunction for all $T(n)$, then $c(1) \neq 0$, & when we normalize so that $c(1) = 1$, then

$$T(n) f = c(n) \cdot f.$$

Corollary: If $f(z)$ is a normalized eigenform for all $T(n)$, $c(n)$ is real for all n.

Proof: Eigenvalues of a Hermitian operator are real

$$\langle \alpha v, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle = \langle v, \alpha v \rangle$$
$$= \langle v, \lambda v \rangle$$
$$= \bar{\lambda} \langle v, v \rangle$$

Deduce from all of above that if f is a normalized eigenform for all $T(n)$,

$$c(m)\, c(n) = c(m \cdot n) \qquad (m,n) = 1$$
$$c(p)\, c(p^n) = c(p^{n+1}) + p^{2k-1} c(p^{n-1})$$
if p prime; $n \geq 1$.

Just as in case of Δ; this implies that

$$L(f,s) \triangleq \sum c(n) \cdot n^{-s} = \prod_{p \text{ prime}} \frac{1}{(1 - c(p) p^{-s} + p^{2k-1-2s})}$$

Conjectures on $C(p)$ were proved by Deligne

Example: Since space of cusp forms of wt 12 is 1-dimensional, Δ is a simultaneous eigenform for all Hecke Operators. So Ramanujan's for every Λ'' of index mn in Λ, there is a

$$\Lambda \supset \Lambda' \supset \Lambda$$

~~of~~ with Λ' of index n in Λ.

Corollary: Let $\mathcal{H}$ be the $\mathbb{Z}$-subalgebra of End $(\mathcal{D})$ generated by $T(p)$ & $R(p)$ for p prime. Then $\mathcal{H}$ contains $T(n)$ for all n.

Let F be a function $\mathcal{L} \to \mathbb{C}$. Extend F by linearity to $\mathcal{D}$. For any operator T on $\mathcal{D}$, define $T \circ F$ to be the function $\mathcal{L} \to \mathbb{C}$ such that

$$T \circ F(\Lambda) = F(T(\Lambda))$$

ex: $$\left(T(n) \cdot F\right)(\Lambda) = \sum_{[\Lambda : \Lambda'] = n} F[\Lambda']$$

Proposition: Let $F : \mathcal{L} \to \mathbb{C}$ be a homogenous function of weight $2K$. Then $T(n) \cdot F$ is also of weight $2K$. Also if $(m,n) = 1$

$$T(m) \cdot T(n) \cdot F = T(mn) F$$

& if p is prime, $n \geq 1$

$$T(p) \cdot T(p^n) \cdot F = T(p^{n+1}) F + p^{1-2K} T(p^{n-1}) \cdot F$$

But
$$\sum_{0\leq b<d} e^{2\pi i \frac{bm}{d}} = \begin{cases} d & \text{if } d \mid m \\ 0 & \text{else.} \end{cases}$$

Set $m/d = m'$; then
$$T(n)\cdot f(z) = n^{2k-1} \sum_{a,d,m'} d^{-2k+1}\, c(m'd)\, q^{am'}$$
where the sum is over integers a, d, m' s.t. $ad = n$ & $a \geq 1$. The coefficient of q^t in this is
$$\sum_{a \mid \gcd(n,t),\, a\geq 1} a^{2k-1}\cdot c\left(\frac{t}{a}\cdot\frac{n}{a}\right)$$
By substituting m for t in the above, we get the result. Also since $\gamma(m) = 0$ for $m<0$, $T(n)\cdot f$ is holomorphic at ∞.

Corollary:

(a) The coefficients $\gamma(0) = \sigma_{2k-1}(n)\cdot c(0)$
$\gamma(1) = c(n)$

(b) If $n = p$ prime
$$\begin{aligned} \gamma(m) &= c(pm) \quad \text{if } p \nmid m \\ &= c(pm) + p^{2k-1} c\left(\tfrac{m}{p}\right) \quad \text{if } p \mid m \end{aligned}$$

(c) If f is a cusp form; so is $T(n)\cdot f$

Thus the $T(n)$'s act on the vector spaces $M_k(\Gamma(1))$ & $S_k(\Gamma(1))$ & satisfy the identities
$$T(m)\circ T(n) = T(mn) \quad (m,n) = 1$$
$$T(p)\circ T(p^n) = T(p^{n+1}) + p^{2k-1}\cdot T(p^{n-1}) \quad \text{if } p \text{ prime } \& \ n \geq 1$$

Normalised eigenforms:

Proposition: Let $f = \sum c(n) q^n$ be a non-zero modular form of weight $2k$. Let f be a simultaneous eigenform for all $T(n)$ ie

$$T(n)\cdot f = \lambda(n)\cdot f \qquad \lambda(n) \in \mathbb{C}, \; n \geq 1.$$

Then (a) $c(1) \neq 0$

(b) If f is normalised so that $c(1) = 1$;

$$c(n) = \lambda(n).$$

Proof: coeff of q in $T(n) f$ is $c(n)$, also equal to $\lambda(n)\cdot c(1)$. If $c(1)=0$ all $c(n)=0$ & f is constant. Normalise to get $c(n)=\lambda(n)$.

Corollary: 2 normalized eigenforms with same eigenvalues are equal.

Corollary: If $f = \sum c(n) q^n$ is a normalised eigenform for all $T(n)$;

$$c(m)\cdot c(n) = c(mn)$$

$$c(p)\, c(p^n) = c(p^{n+1}) + p^{2k-1} c(p^{n-1}) \qquad p \text{ prim}, \; n \geq 1$$

proof: These relations hold for eigenvalues.

So now;

$$L(f,s) = \sum_{n\geq 1} c(n)\cdot n^{-s} = \prod_p \frac{1}{1 - c(p)\, p^{-s} + p^{2k-1\cdot}}$$

Definition: If f & g modular forms of wt $2k$; & atleast one is a cusp form; the <u>Petersson Inner Product</u> of f & g is

$$\langle f, g\rangle = \iint_D f(z)\cdot\overline{g(z)}\, y^{2k-2}\,dx\,dy$$

Properties:-

(a) linear in first variable, Semi-linear in 2^{nd}

(b) $\langle f, g\rangle = \overline{\langle g, f\rangle}$

(c) $\langle f, f\rangle > 0 \quad \forall f \neq 0$

$\therefore$ If is a positive definite Hermitian form on $S_k(\Gamma)$ & makes $S_k(\Gamma)$ a finite dimensional Hilbert space.

Theorem: For cusp forms f, g of wt $2k$;

$$\langle T(n) f, g\rangle = \langle f, T(n) g\rangle \qquad \forall n.$$

Proof: Suffices to prove it for prime. Exercise.

Eisenstein Series as eigenforms:

$$G_k(z) \triangleq \sum_{(m,n)\neq 0,0} \frac{1}{(mz+n)^k}$$

$$= 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum \sigma_{2k-1}(n)\, q^n;$$

$\therefore \dfrac{G_k(z)}{2\zeta(2k)} = E_k(z)$ is normalized Eisenstein series

$$T(n)\cdot f(z) = \sum_{m\geq 0} \gamma(m)\, q^m$$

Where $\gamma(m) = \sum_{\substack{a \mid m \\ a\geq 0}} a^{2k-1}\, c\left(\frac{mn}{a^2}\right)$

The matrix of $T(n)$ w.r.t a basis for $M_k(\mathbb{Z})$ has integer coefficients, showing that the eigenvalues are algebraic integers.