# Spectrum Sharing for Unlicensed Bands

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***Abstract*— We study a spectrum sharing problem in an unlicensed band where multiple systems coexist and interfere with each other. We first analyze a cooperative setting where all the systems collaborate to achieve a common goal. Under the assumptions that the systems communicate with Gaussian signals and treat interference as noise, we study the structure of the optimal power allocations. We show that any Pareto efficient vector of rates can be achieved with piece-wise constant power allocations. Moreover, if a strong interference condition among all the systems is satisfied, we show that frequency division multiplexing is optimal.**

**We then consider a non-cooperative situation, where the systems act in a selfish and rational way, and investigate how the lack of cooperation can affect performance. Using game theory, we first analyze the possible outcomes of a one shot game, and observe that in many cases an inefficient solution results. We show that by extending the game definition to that of a repeated game, the possibility of building reputations and applying punishments allows to enlarge the set of achievable rates. We present examples that show that in many cases, the performance loss due to lack of cooperation is small. We also provide a converse theorem that proves that our results are tight and quantify the best achievable performance in a non-cooperative scenario.**

## I. Introduction

We consider a spectrum sharing scenario in an unlicensed band. Multiple systems coexist in the same area and communicate using the same portion of spectrum. For concreteness one can consider present-day 802.11 networks, bluetooth systems, walkie-talkies, etc. operating in the same unlicensed band, e.g. ISM, UNII, etc., and interfering with each other. With the future advent of sophisticated frequency-agile radios which can operate over a wider and wider bandwidth, the potential for sharing will only increase. We are interested in designing spectrum sharing rules and protocols that allow the systems to share the bandwidth in a way that is fair, efficient and compatible with the incentives of the individual systems. We assume that spectrum is a scarce resource, so that efficiency is a concern.

A spectrum sharing rule is any method followed by each system which results in their sharing the spectrum. Clearly, the rule in which all systems attempt to maximize their rates while treating the other systems as interference can result in unfair and/or inefficient co-existence operating points.

Many of the fairness issues in spectrum sharing arise due to asymmetries between the systems. Figure 1 shows three different examples where two systems operate in asymmetric situations. In scenario (a) both systems have similar power capabilities (e.g. two 802.11 systems) but due to the locations of the transmitters and receivers, one system receives large interference while the other does not. Scenarios (b) and (c) describe situations where a high power system (e.g. 802.11 system) shares spectrum with a low power system (e.g. bluetooth system). In (b) all the gains are comparable, so intuitively the weak system is in disadvantage. In (c) due to asymmetry in the gains both systems can interfere with each other and one can imagine that a more fair situation may result.

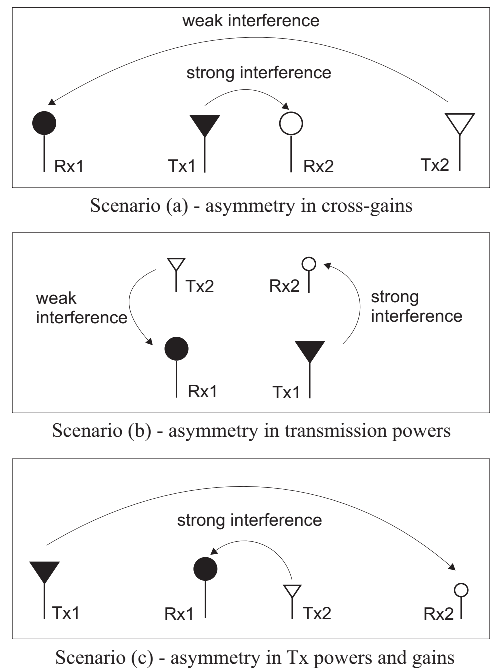


Fig. 1. Three examples of asymmetric situations between two systems sharing the same band. The sizes of the antennas represent power capabilities, and smaller distances indicate higher gains.

We address the question of whether these systems can coexist and share spectrum in a fair and efficient manner. A central aspect of the problem is the issue of cooperation: the systems have different objectives and there may not be any incentive to achieve a common goal. We investigate how the lack of cooperation may affect the overall performance of the systems.

Another important element of the problem is the amount of knowledge that each system has about the others. For example, a system may or may not know how many other systems are sharing the same band, the corresponding channel gains, transmission powers, communication rates, and codes. In this paper, we will assume that the number of systems, channel gains, and maximum transmission powers are common knowledge. However we don't require the systems to know the rates or codebooks used by the other systems.

Current regulations provide a spectral mask that limits the total power and power spectral density that each transmitter can use. This tries to limit the amount of interference generated by each transmitter onto other systems' receivers. This approach, however, has a number of drawbacks. First, it may severely constrain the data rate when there are no other systems around. In addition, it does not provide any performance guarantee nor does it avoid unfair situations. Consider for example scenario (b) in Fig. 1 where the powerful 802.11 system interferes the weak bluetooth system. Is this an unavoidable consequence of the asymmetry between the 802.11 and bluetooth systems, or is just an indication of inefficient spectrum sharing rules ?

Systems may need to spread their signals over the available bandwidth in order to satisfy the spectral masks imposed by regulation while achieving high data rates. Even though spreading may work properly in cellular systems with tight power control rules, this may not be a good spectrum sharing technique in the scenarios that we are considering. Under reasonable assumptions on the choice of the utility functions we investigate the structure of the optimal power allocations, and find that in many practical cases forcing the systems to spread their signals is suboptimal. In particular, we show that when the interference between systems is high, frequency division multiplexing (FDM) is the optimal spectrum sharing technique.

We investigate how the lack of cooperation may affect performance using non-cooperative game theory. We start by formulating a one shot game, and show by extending (for the case of flat fading with white background noise) a result of [3] that in low interference situations, the spread spectrum equilibrium is the only possible outcome of the game. And in many cases, the rates that result from the spread spectrum equilibrium are poor for one or more systems. This is a negative result from the point of view of designing a standard, as it would be desirable to have multiple equilibrium points to choose from. But if we repeat the stage-game indefinitely, the possibility of building reputations and applying punishments leads to a much richer set of equilibria. We show that in this repeated game, any vector of rates in the achievable region that is component-wise larger than the spread-spectrum rates can be supported. Therefore, if the cooperative optimal rate vector is component-wise greater than the spread spectrum rate vector, there is no performance loss due to lack of cooperation.

In summary, if we incorporate punishment in the spectrum sharing rule, then it is possible to find operating points which are fair, efficient and incentive compatible. In theory, the spectrum sharing rule can just mandate all systems to operate at such a point, and since it is incentive compatible, there is no reason why any system would leave it on its own.

This rest of this work is organized as follows. In Section II we present the model to be used in the following sections. Section III considers the structure of the optimal power allocations assuming cooperation between systems. In Section IV we analyze the non-cooperative situations. Finally in Section V we present some conclusions and open problems.

## II. Channel Model

We model a situation in which $M$ systems, each formed by a single transmitter-receiver pair, coexist in the same area. Consider an $M$ user Gaussian interference channel in discrete time defined by:

$$y_i[n] = \sum_{j=1}^{M} h_{j,i} x_j[n] + z_i[n];\ \text{i=1,...,M} \quad (1)$$

where $x_i, y_i, z_i \in \mathbb{C}$ and the noise processes are i.i.d. over time with $z_i \sim \mathcal{CN}(0, N_0)$[1]. By assuming that the channel from each transmitter to each receiver has a single tap we are restricting attention to the case of flat fading. Note also that the i.i.d. assumption on the noise process results in white background noise.

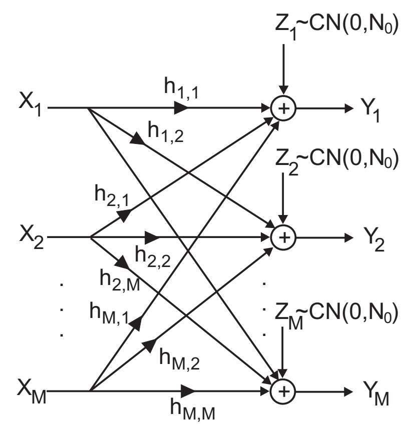


Fig. 2. MxM Gaussian interference channel.

We assume that the underlying continuous time channel is band-limited to a bandwidth $W$, and that the sampling is done at the Nyquist rate resulting in $WT$ complex degrees of freedom in the time interval $[0, T]$. The input of user $i$ has an average power constraint $P_i$.

Unfortunately, except for few special cases, the capacity region of the interference channel is unknown (see for example [1] for a summary of the known results about the interference channel). The best inner bound known to date is given by Han and Kobayashi in [2], whose computation is in general prohibitively complex. Instead of using the best known bounds, in the rest of this work we will assume that each system treats the received interference as noise. This leads to a more tractable inner bound of the capacity region. For many applications it may be reasonable to assume that each system does not know

[1] $\mathcal{CN}(0, \sigma^2)$ is a circularly symmetric complex Gaussian random variable with zero mean and variance $\sigma^2$.

the codebooks used by the other systems, and as a result the interference must be treated as noise. Even if the codebooks are known, practical limitations such as decoder complexity, uncertainty in the estimation of $\{h_{j,i}\}$, delay constraints, etc., may preclude the use of interference cancellation techniques. Therefore the assumption of treating interference as noise may be realistic in these scenarios as well.

Finally, we will assume that the systems use random Gaussian codebooks, which means that the transmitted signals look like white Gaussian processes. Under these assumptions, using the capacity expression for the single user Gaussian channel, we can determine the maximum rate that system $i$ can achieve for specific power allocations[2]:

$$R_i = \int_0^W \log\left(1 + \frac{c_{i,i} p_i(f)}{N_0 + \sum_{j \neq i} c_{j,i} p_j(f)}\right) df \qquad (2)$$

where $p_i(f)$ is the power spectral density of the input signal of system $i$, and where for convenience we defined $c_{i,j} = |h_{i,j}|^2$. Note that due to the power constraints, $p_i(f)$ must satisfy:

$$\int_0^W p_i(f) df \leq P_i \qquad (3)$$

Having introduced the channel model, we can describe the three scenarios introduced in Figure 1 by giving their parameters. Without loss of generality we can assume in all cases that $c_{1,1} = c_{2,2} = 1$, $N_0 = 1$ and $W = 1$. In scenario (a) we can assume $P_1 = P_2 = 10$, $c_{1,2} = 10$ and $c_{2,1} = 0.5$. For scenario (b) we set $P_1 = 10$, $P_2 = 1$, and $c_{1,2} = c_{2,1} = 1.1$. Finally in (c) we set $P_1 = 10$, $P_2 = 1$, $c_{1,2} = 0.5$ and $c_{2,1} = 10$.

The spectrum sharing problem that we consider is to determine power allocations $\{p_i(f)\}$ for the $M$ systems, that maximizes a given global utility function while satisfying the power constraints. In the next section we study the structure of the optimal power allocations for any reasonable choice of global utility.

## III. Optimal Spectrum Allocations

Imagine that in the three scenarios described in Figure 1 we want to maximize some global utility function $U(R_1, R_2)$. We are interested in determining the maximum value of $U$ and the corresponding power spectral allocations that achieve it. In this section we will show how to solve this problem efficiently.

Let $\mathcal{R}$ be the achievable rate region:

$$\mathcal{R} = \Bigg\{ \mathbf{R} : R_i = \int_0^W \log\left(1 + \frac{c_{i,i} p_i(f)}{N_0 + \sum_{j \neq i} c_{j,i} p_j(f)}\right) df \wedge \int_0^W p_i(f) df \leq P_i \wedge p_i(f) \geq 0 \text{ for } i = 1, \ldots, M \Bigg\} \qquad (4)$$

where $\mathbf{R} = (R_1, R_2, \ldots, R_M)$ and let $\mathcal{R}^*$ be the set of Pareto optimal points of $\mathcal{R}$:

$$\mathcal{R}^* = \Big\{ (R_1, \ldots, R_M) \in \mathcal{R} : R_i \geq \tilde{R}_i \; \forall (R_1, \ldots, R_{i-1}, \tilde{R}_i, R_{i+1}, \ldots, R_M) \in \mathcal{R}, \text{ for } i = 1, \ldots, M \Big\} \qquad (5)$$

[2]In all cases we use $\log(\cdot)$ for a base 2 logarithm.

In words, a rate allocation is Pareto optimal (or efficient) if it is not possible to increase the rate of any system without decreasing the rate of some other system.

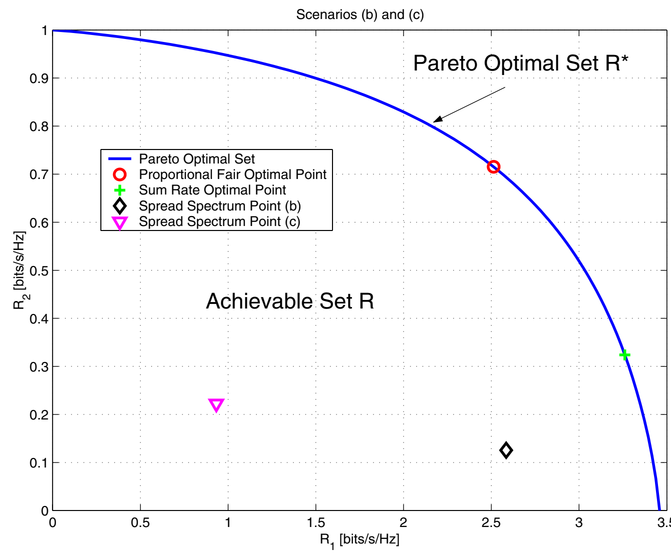


Fig. 3. Achievable set $\mathcal{R}$ and Pareto efficient set $\mathcal{R}^*$ for scenarios (b) and (c). Indicated in the figure are the spread spectrum rates, and optimal sum rate, and proportional fair points.

Figure 3 shows the achievable set and Pareto optimal set for scenarios (b) and (c). The reason why for this specific choice of parameters in both scenarios we obtain the same sets will be explained later in this section.

The choice of the utility function will strongly influence the fairness in the resulting allocations. For example we may consider $U_{sum}(R_1, R_2) = R_1 + R_2$ if we are interested in maximizing the total sum rate. While in scenario (a) this choice of utility results in an optimal operating point where $R_1 = R_2$, in scenarios (b) and (c) the resulting optimal allocations are very unfair for system 2 ($R_2 \ll R_1$) (see Figure 3). A more fair allocation results from choosing the proportional fair metric $U_{PF}(R_1, R_2) = \log(R_1) + \log(R_2)$ proposed in [9]. By applying the $\log(\cdot)$ function to each rate, we give higher priority to the system in disadvantage. We can see in Figure 3 how in scenarios (b) and (c) the use of the proportional fair metric results in a more fair allocation. Note that in scenario (a) the use of $U_{PF}$ results in the same rates as when $U_{sum}$ is used.

For any utility function that is component-wise monotonically increasing in $(R_1, \ldots, R_M)$, the optimal rate allocation must occur in a point of $\mathcal{R}^*$. So it is of interest to obtain a simple characterization for $\mathcal{R}$ and $\mathcal{R}^*$. For example, if we knew $\mathcal{R}^*$ the problem of maximizing $U_{sum}$ reduces to finding the point in $\mathcal{R}^*$ that is tangent to the line of slope $(-1)$.

At first glance, computing $\mathcal{R}$ requires to search over all possible power allocations $p_i(f)$ that satisfy the power constraint. Since $p_i(f)$ are functions with arbitrarily many degrees of freedom, the computation of $\mathcal{R}$ seems to be an infinite dimensional problem. However the following theorem shows that we can restrict attention to piecewise constant power allocations, and as a result, the problem of computing $\mathcal{R}$ has finite dimension.

*Theorem 1:* Any point in the achievable rate region $\mathcal{R}$ defined in (4) can be obtained with $M$ power allocations that are piecewise constant in the intervals $[0,w_1),[w_1,w_2),\ldots,[w_{2M-1},W]$, where $w_i \leq w_{i+1}, i = 1,\ldots,2M-2$, for some choice of $\{w_i\}_{i=1}^{2M-1}$.

Note that once we fix the choice of intervals to obtain a point in $\mathcal{R}$, the $M$ power allocations are constant in the same intervals.

For the special case of channels satisfying a pairwise high interference condition (which is satisfied with the choice of parameters in scenarios (a), (b) and (c)), it turns out that the optimal power allocations are orthogonal, and hence the characterization of $\mathcal{R}^*$ is further simplified.

*Theorem 2:* Let $(R_1,\ldots,R_M)$ be a Pareto efficient rate vector achieved with power allocations $\{p_i(f)\}_{i=1,\ldots,M}$ which are piece-wise constant in the intervals $[0,w_1),[w_1,w_2),\ldots,[w_{2M-1},W]$. If $\frac{c_{i,j}c_{j,i}}{c_{i,i}c_{j,j}} > 1$ then the power allocations $p_i(f)$ and $p_j(f)$ are orthogonal, i.e. $p_i(f)p_j(f) = 0$ for $f \in [0,W]$.

The condition $\frac{c_{i,j}c_{j,i}}{c_{i,i}c_{j,j}} > 1$ means that for systems $i$ and $j$, the product of the channel cross gains $c_{i,j}c_{j,i}$ is greater than the product of the channel direct gains $c_{i,i}c_{j,j}$. Note that the condition can be satisfied even if one of the cross gains is small, by having the other cross gain large enough. Also, note that the condition is independent of the power constraints $\{P_i, P_j\}$ and noise variance $N_0$. In our three examples, we chose $c_{1,1} = c_{2,2} = 1$, so to check whether the condition of Theorem 2 is satisfied, we only need to check that $c_{1,2}c_{2,1} > 1$. In scenario (a) we have $c_{1,2}c_{2,1} = 10\cdot 0.5 = 5$, in scenario (b) we have $c_{1,2}c_{2,1} = 1.1\cdot 1.1 = 1.21$, and in scenario (c) we have $c_{1,2}c_{2,1} = 0.5\cdot 10 = 5$, so in the three cases the condition is met.

In particular, if $\frac{c_{i,j}c_{j,i}}{c_{i,i}c_{j,j}} > 1$ for any $i \neq j$, $j = 1,\ldots,M$, we can achieve any Pareto efficient rate vector with frequency division multiplexing (FDM). In this case, the the maximization of any weighted sum of rates reduces to the optimization problem:

$$\begin{aligned} &\text{Maximize} && \sum_{i=1}^{M} \alpha_i W_i \log\left(1 + \frac{c_{i,i}P_i}{W_i N_0}\right) \\ &\text{subject to:} && \sum_{i=1}^{M} W_i = W \wedge W_i \geq 0, i = 1,\ldots,M \end{aligned} \tag{6}$$

which, as can be easily verified, is a concave optimization problem and can be efficiently solved. It can also be verified that the maximization of $U_{PF} = \sum_{i=1}^{M} \log(R_i)$ is a concave problem as well. These results allow to easily compute the rates in Figure 3.

Note that since the Pareto efficient rates are obtained with orthogonal allocations when $c_{1,2}c_{2,1} > 1$ (for direct gains equal to 1), the actual values of the cross gains $c_{1,2}$ and $c_{2,1}$ have no influence on the achievable region. This explains why scenarios (b) and (c) result in the same achievable region and optimal rates.

When the conditions of Theorem 2 are not satisfied, we can use techniques such as Lagrangian methods to solve the problem of maximizing $U(R_1,\ldots,R_M)$ with tractable complexity.

Throughout this section we have implicitly assumed that the $M$ systems cooperate to maximize a global utility function by choosing appropriate power allocations. This assumption may be realistic when the different systems are jointly designed with a common goal, are complying with some standard or regulation, or are in fact transmitter-receiver pairs of a single global system.

However, in a spectrum sharing scenario where regulations may be lax and systems may be competing with one another to gain access to the shared medium, assuming selfish behavior may be more realistic. In the next section we analyze how the lack of cooperation among systems may affect the set of achievable rates.

## IV. Non-cooperative Scenarios

We will consider the same model introduced in Section II under the assumption that the different systems behave selfishly and rationally. We associate to each system $i$ a utility function $U_i(R_i)$, which we assume concave and increasing in $R_i$. The systems are selfish in the sense that they only try to maximize their own utility. The rationality assumption means that each system will never choose a strictly dominated strategy[3].

We analyze the set of achievable rates in this non-cooperative scenario using non-cooperative game theory. We first consider a static game of complete and perfect information, usually referred to in the literature as the Gaussian Interference Game [4].

The game has $M$ players, the $M$ systems. The strategy space $\mathcal{S}_i$ of system $i$ is the set of power allocations $p_i(f)$, $f \in [0,W]$ that are continuous almost everywhere and that satisfy the power constraint (3). A strategy $s_i$ for user $i$ is the choice of power allocation $p_i(f)$. For a given strategy profile $(s_1,\ldots,s_M)$ the rate of user $i$ is given by (2). For concreteness will consider $U_i(R_i) = R_i$, but we note that all the results of this section apply to any choice of concave and increasing utilities. The players play simultaneously, and know the utility functions of all the other players ($N_0, \{c_{i,j}\}_{i,j}, \{P_i\}_{i=1}^M, W$ are common knowledge). A strategy profile $\{s_i^*\}_{i=1}^M$ is a Nash Equilibrium (N.E.) of the game if

$$R_i(s_1^*,\ldots,s_M^*) \geq R_i(s_1^*,\ldots,s_{i-1}^*,s_i,s_{i+1}^*,\ldots,s_M^*) \quad \text{for all } s_i \in \mathcal{S}_i,\ i = 1,\ldots,M \tag{7}$$

A direct consequence of the flat-fading and white noise assumption is the following fact:

[3]A strategy $s_i$ for player $i$ is strictly dominated by strategy $s_i'$ if $U_i(s_1,\ldots,s_{i-1},s_i',s_{i+1},\ldots,s_M) < U_i(s_1,\ldots,s_{i-1},s_i,s_{i+1},\ldots,s_M))$ for each $(s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_M)$ that can be constructed from the other players' strategy spaces.

*Fact 1:* The frequency-flat allocations $p_i(f) = P_i/W, f \in [0, W]$ for $i = 1, \ldots, M$ form a Nash Equilibrium of the Gaussian Interference Game.

This means that the best possible strategy for a given system is to spread its available power over the total bandwidth whenever all the interfering systems are spreading their signals. Fact 1 can be understood by noting that the best response of a system to a strategy profile of the other systems is to waterfill[4] the available power over the noise+interference seen. When all the other systems use flat allocation, the waterfilling power allocation is flat, and it follows that flat allocations are best responses to each other.

If the players randomize their actions, the (mixed) strategy of each player is the choice of probability distribution used for the randomization. The utility that each user gets is the expected utility, averaged over the random choices of actions of all the players. Taking into account these changes in the definition of the strategies and the utilities, the concept of a mixed strategy Nash equilibrium can be defined exactly as before.

When studying the set of N.E., one needs to consider both pure and mixed strategies. However, in the case of the Gaussian Interference Game it turns out that we need only consider pure strategies.

*Theorem 3:* The Gaussian Interference Game can only have pure strategy Nash equilibria. That is, every mixed strategy N.E. of the game must consist of atomic distributions with a single atom, and therefore is a pure strategy N.E.

If the channel gains across systems are sufficiently small the full-spread N.E. is the only N.E. of the Gaussian game. The following theorem gives a sufficient condition for the uniqueness of the spread spectrum N.E.

*Theorem 4:* If $\sum_{\substack{j=1 \\ j \neq i}}^{M} \frac{c_{j,i}}{c_{i,i}} < 1$ for $i = 1, \ldots, M$ then the spread spectrum N.E. is the only N.E. of the Gaussian Interference Game.

Theorem 4 does not give us any information about the uniqueness of the Nash equilibrium when the condition $\sum_{\substack{j=1 \\ j \neq i}}^{M} \frac{c_{j,i}}{c_{i,i}} < 1$ is not met.

In many cases, the vector of rates that results from the spread spectrum N.E. is not Pareto efficient (i.e. is not in $\mathcal{R}^*$) so there may be a significant performance loss if the $M$ systems operate in this point due to lack of cooperation. And in many cases this inefficient outcome is the only possible outcome of the game. Consider for example a two system scenario (call it (d)) with $c_{1,1} = c_{2,2} = 1$, $c_{2,1} = c_{1,2} = 1/4$, $W = 1$, $N_0 = 1$ and $P_1 = P_2 = P$. Note that in this case the condition of Theorem 4 is satisfied. If both users spread their signals, they obtain rates

$$R_1^{SS} = R_2^{SS} = \log\left(1 + \frac{P}{1 + P/4}\right) \text{[bits/s/Hz]} \qquad (8)$$

[4]The waterfilling power allocation consists of distributing the available power as if pouring water in a container whose bottom is given by the noise+interference level. This power allocation maximizes the rate of the system for a given Gaussian noise+interference power spectral density [5].

which tends to $\log(5)$[bits/s/Hz] as $P \to \infty$. However, if the systems orthogonalize their power allocations using half of the bandwidth each, the resulting rates are:

$$R_1 = R_2 = \frac{1}{2}\log(1 + 2P) \text{[bits/s/Hz]} \qquad (9)$$

which tends to $\infty$ as $P \to \infty$. The regime in which $P \gg N_0$ corresponds to the high signal to noise ratio (SNR) regime. In this regime, when the systems orthogonalize their power allocations they can communicate with an interference free channel, and achieve large data rates. If on the contrary both systems spread their signals, the signal to interference plus noise ratio becomes limited by interference, resulting in a reduced communication rate. This example shows that the inefficiency resulting from choosing the spread spectrum equilibrium can be arbitrarily large.

Scenario (d) shows that there are situations in which the only possible outcome of the game is very inefficient, and as a result, there is a large performance degradation due to lack of cooperation. This negative result can be attributed to the static nature of the game that we defined.

Many wireless systems operate and co-exist with the same set of competing systems over a long period of time (days, months, years). In this context, it may be more reasonable to model the scenario as a repeated (or dynamic) game where systems play multiple rounds, remembering the past experience in the choice of the power allocation in the next round. We will consider an infinite horizon repeated game, where the Gaussian Interference Game is repeated forever. The utility of each player is defined by

$$U_i = (1 - \delta) \sum_{t=0}^{\infty} \delta^t R_i(t) \qquad (10)$$

where $R_i(t)$ is the utility of user $i$ in the stage game at time $t$, and $\delta \in (0, 1)$ is a discount factor that accounts for the delay sensitivity of the systems. At the end of each stage, all the players can observe the outcome of the stage-game and can use the complete history of play to decide on the future action. A strategy in the repeated game is a complete plan of action, that defines what the player will do in every possible contingency in which he may need to act.

One property of this repeated game is that sequences of strategy profiles that form a N.E. in the stage game, form a N.E. in the dynamic game[5]. Furthermore, the dynamic game allows for a much richer set of N.E. This is an advantage from the point of view of policy making or standardization. The systems can agree through a standardization process to operate in any N.E. of the dynamic game. Having many equilibrium points to choose from gives more flexibility in obtaining a fair and efficient resource allocation. A natural question that arises is what set of rate vectors can be supported as a N.E. of the repeated game. The following theorem, a general version of which is due to Friedman [6], [7], gives a sufficient condition

[5]For the reader familiar with game theory, these equilibria are in fact subgame perfect Nash equilibria.

for the rate vector $(R_1, \ldots, R_M)$ to be achievable as the resulting utilities in a N.E. of the repeated game.

*Theorem 5:* Let $R_i^{SS}$ be the rate of system $i$ when all the systems spread their power over the bandwidth $W$, i.e. the rate obtained in the spread spectrum N.E. There exists a sub-game perfect N.E.[6] of the dynamic Gaussian Interference Game with utilities $(U_1, \ldots, U_M) = (R_1, \ldots, R_M)$ whenever $(R_1, \ldots, R_M) \in \mathcal{R}$ and $R_i > R_i^{SS}$ for $i = 1, \ldots, M$ for a discount factor $\delta$ sufficiently close to 1.

Theorem C of [8] states that any utility vector that Pareto dominates the payoffs of a Nash equilibrium of the stage game can be supported by a sub-game perfect N.E. of the repeated game for a discount factor $\delta$ sufficiently close to 1. This Folk theorem is due to Friedman [6], [7], although he considered only Nash equilibria instead of perfect equilibria in his work. In the Gaussian Interference Game the spread-spectrum allocations form a N.E. (see Fact 1), and we can use $(R_1^{SS}, \ldots, R_M^{SS})$ as the payoff vector of the N.E. of the stage game in the Theorem above.

Let $\{p_i(f)\}_{i=1}^M$ be the power allocations that result in the rate vector $(R_1, \ldots, R_M)$ (which always exist since $(R_1, \ldots, R_M) \in \mathcal{R}$). The strategy that each system follows to obtain the rate vector $(R_1, \ldots, R_M)$ in Theorem 5 is the following trigger strategy:

- at $t = 1$: use power allocation $p_i(f)$.
- at $t = t_0$: if at time $t = t_0 - 1$ every user $j \in \{1, \ldots, M\}$ used the power allocation $p_j(f)$ then use $p_i(f)$. Otherwise use the power allocation $P_i/W$ for $f \in [0, W]$

The idea behind this strategy, is to ”cooperate” by using the required power allocation as long as all the other systems cooperated in the previous stages. As soon as at least one system deviates from the ”good” behavior, a punishment is triggered where all the other systems spread their powers forever. Since the rates obtained by the systems once the punishment is triggered are lower than those obtained with cooperation, it is in the system’s own interest to cooperate. Friedman’s analysis shows that if $\delta$ is not too small, the above set of strategies forms a sub-game perfect N.E.. The sub-game perfection property of the N.E. guarantees that each system will indeed apply the punishment once the punishing situation arises. This property makes the threats believable.

Applying these ideas to scenario (d), we can define a trigger strategy where system 1 uses the first half of the bandwidth, and system 2 uses the second half, as long as in all the previous stages both systems complied with this frequency allocation. If at some stage any of the systems stops complying, a punishment is triggered where the systems spread their powers forever. For large enough $P$ this pair of strategies forms a N.E. where each system obtains a utility $1/2 \log(1 + 2P)$. This shows how the punishment strategies within the dynamic game formulation allow us to overcome the inefficiency that we observed in the static game.

[6]The sub-game perfect N.E. is a refined and stronger version of the N.E. concept defined before. It guarantees that the N.E. does not arise due to unbelievable threats.

Theorem 5 gives us a sufficient condition for a rate vector $(R_1, \ldots, R_M)$ to be achievable through a N.E. But if the condition of the theorem is not met we may still have hope to find some other N.E. to support the desired vector of rates. A natural question to ask is if there are other N.E. that result in utilities $(R_1, \ldots, R_M)$ with some $R_i < R_i^{SS}$. The following theorem answers this question negatively and provides a converse to Theorem 5.

*Theorem 6:* The rate $R_i^{SS}$ is the reservation utility of player $i$ in the Gaussian Interference Game. That is, player $i$ can obtain a utility at least as large as $R_i^{SS}$ by using the power allocation $p_i(f) = P_i/W$, $f \in [0, W]$ regardless of the power allocations used by the other players. Therefore, the rate $R_i$ obtained by user $i$ in any N.E. of the Gaussian Interference Game must satisfy $R_i \geq R_i^{SS}$. The same statement holds for the repeated Gaussian Interference Game.

The proof of Theorem 6 shows that for a white Gaussian input, the worst possible Gaussian interference of given power is white. Since the power of each system is bounded to $P_j$, the total interference power seen by system $i$ is bounded to $\sum_{j \neq i} c_{j,i} P_j$. If system $i$ uses a white input, the worst case interference is obtained when all the other systems spread their powers, and it follows that a rate at least as large as $R_i^{SS}$ is always achieved. Therefore, there is no incentive for player $i$ to play any strategy that results in a utility smaller than $R_i^{SS}$.

An immediate consequence of Theorems 5 and 6 is that if the desired operating point $(R_1, \ldots, R_M)$ (i.e. the maximizer of a desired global utility) is component-wise greater than the spreading rate vector $(R_1^{SS}, \ldots, R_M^{SS})$ there is no performance loss due to lack of cooperation. However, when this condition is not satisfied, the best that one can do is to find the point in $\mathcal{R}^*$ that is component-wise greater than $(R_1^{SS}, \ldots, R_M^{SS})$ and that maximizes the desired utility.

Referring to Figure 4 we see that in scenario (b) the optimal sum rate point lies within the achievable region in the non-cooperative setting. However, the optimal proportional fair point lies outside of this set and cannot be supported without cooperation. The best that one can do in the non-cooperative setting is to operate in the point indicated in the figure. In scenario (c) both the optimal sum rate and optimal proportional fair rates are achievable in the non-cooperative setting. Note that while in the cooperative case the specific values of the cross gains had no influence on the achievable region (as long as the strong interference condition is satisfied) this is not true in the non-cooperative setting. This is because large cross gains enable the systems to apply punishments, and hence achieve a good N.E. through believable threats. In scenario (c) the large value of $c_{2,1}$ allows system 2 to punish system 1 whenever it departs from the proportional fair allocation.

To further illustrate the concepts introduced in this and the previous section, consider a two user scenario and assume that we use the proportional fair utility $U_{PF}$ to measure the global performance. Without loss of generality we assume that $c_{1,1} = c_{2,2} = 1$, $W = 1$ and $N_0 = 1$. Also we take $P_1 = P_2 = P$ and analyze the results in terms of the $\mathsf{SNR} = P/N_0$. At a

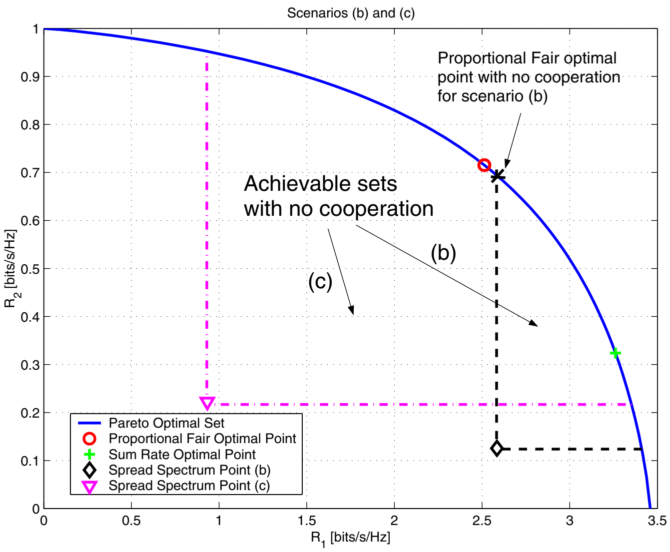


Fig. 4. Achievable rates with no cooperation for scenarios (b) and (c).

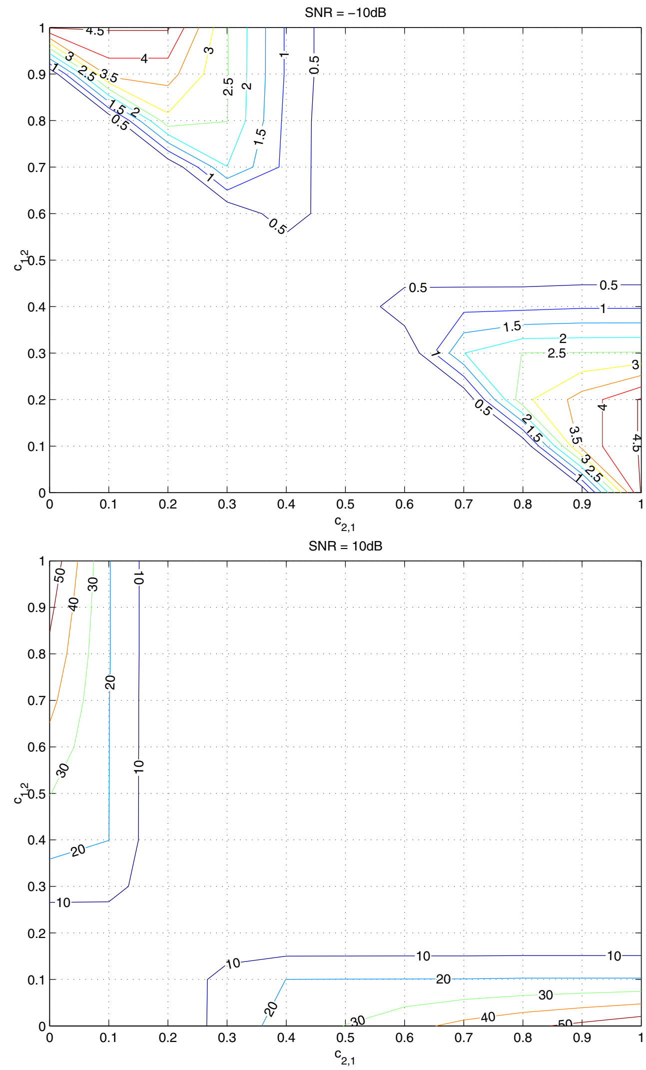


Fig. 5. Contour plots of $(R_i^* - \tilde{R}_i)/R_i^*(\%)$ (percentage of rate loss with respect to the cooperative solution) as a function of the cross gains $c_{1,2}$ and $c_{2,1}$, for SNR $= -10, 10$dB.

given SNR we can control the asymmetry between the two systems by varying the cross gains $c_{1,2}$ and $c_{2,1}$.

For a fixed set of parameters, using the results of Section III we optimize the power allocations to maximize the proportional fair metric, obtaining $R_1^*$ and $R_2^*$ as the resulting rates. In the non-cooperative scenario, $R_1^*$ and $R_2^*$ can only be supported by a N.E. if $R_1^* \geq R_1^{SS}$ and $R_2^* \geq R_2^{SS}$. If these inequalities are not satisfied, we obtain the best possible solution for the non-cooperative case by maximizing $\log(R_1) + \log(R_2)$ subject to the constraint $R_i \geq R_i^{SS}$, $i = 1, 2$, being $\tilde{R}_1$ and $\tilde{R}_2$ the corresponding optimal rates. If $R_i^* = \tilde{R}_i$ for $i = 1, 2$ we conclude that there is no loss due to lack of cooperation. If $R_i^* > \tilde{R}_i$ for $i = 1$ or $i = 2$ we measure the loss due to lack of cooperation using $\max_{i \in \{1,2\}} 100(R_i^* - \tilde{R}_i)/R_i^*$, i.e. the percentage loss in rate for one of the systems. Note that the other system will have a rate larger than the one obtained with cooperation.

In Figure 5 we see that for low SNR, the region of rate pairs $(c_{1,2}, c_{2,1})$ for which there is a loss due to lack of cooperation is large, but this loss is quite small (about 5% in the worst case seen in the figure). As the SNR increases, this region becomes progressively smaller, but the corresponding performance loss is more significant. For SNR $=$ 10dB this performance loss can be as large as 50% of the proportional fair rates, but this only occurs in very asymmetric situations.

At low SNR performance is limited by noise not interference, so whether the systems cooperate or not does not have much influence in performance. At larger SNRs, interference becomes the dominant performance limiter. In very asymmetric situations the spread spectrum point $(R_1^{SS}, R_2^{SS})$ is such that $R_1^{SS}$ or $R_2^{SS}$ is large. In either of these cases, one system obtains a large enough rate with the spread spectrum allocations, and a threat of the other system to apply the spreading punishment is not effective to modify its behavior. Using Theorem 6 we see that if $R_1^{SS}$ or $R_2^{SS}$ is large, the set of rates achievable in the non-cooperative situation is quite limited, and often it does not include the optimal cooperative point. Fortunately, the level of asymmetry required to reach this unfair situation increases with SNR. The same qualitative behavior should be observed with other performance metrics and larger number of users.

The plots of Figure 5 illustrate the performance loss due to lack of cooperation when there is asymmetry in the cross gains between the systems. One can do a similar analysis for the case when the source of asymmetry is the transmission power instead of the cross gains.

## V. Conclusions

For a situation where all the systems cooperate to achieve a common goal, we showed that any Pareto efficient rate vector can be achieved with piece-wise constant power allocations.

This allows to significantly reduce the complexity of the underlying optimization problem. If in addition a strong interference condition among all the systems is satisfied, frequency division multiplexing allows to achieve any Pareto efficient allocation.

Our game theoretic analysis showed that the use of punishment strategies can significantly enlarge the set of rates achievable in a non-cooperative situation. The converse theorem that we presented shows that the rates that can be obtained with our punishment strategies are essentially the best that one can hope for in a non-cooperative setting. Therefore our results are tight and quantify the best achievable performance with lack of cooperation. The two system example that we presented shows that in most situations the performance loss due to lack of cooperation is small, and vanishes with increasing SNR.

We note that we assumed games of perfect and complete information. In practice, some communication may be needed between the systems to exchange parameters, such as channel gains, transmission powers, etc. This parameter exchange may be explicit through a specific communication protocol, or implicit through a learning process. In a non-cooperative environment systems may misrepresent information to increase their utilities. This brings the issue of trust, and opens a whole new set of interesting problems to investigate.

The solution to the optimization problem studied in Section III requires a centralized processor with access to all the systems' parameters. In practice, the systems should be able to optimize their power allocations in a distributed way with only local information about their neighbors. Our future research involves designing distributed algorithms that approximate the solutions obtained from the centralized optimization. The resource allocations resulting from these algorithms may not be optimal, but in many cases can be achieved as a N.E. in a dynamic game. Therefore, in a non-cooperative setting, the distributed resource allocation algorithms should determine the desired N.E. point and the corresponding punishment strategies that achieve it.

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