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On the Degrees-of-Freedom of the K-User Gaussian Interference Channel

Raul Etkin
Hewlett-Packard Laboratories
Palo Alto, CA 94304
Email: raul.etkin@hp.com

Erik Ordentlich
Hewlett-Packard Laboratories
Palo Alto, CA 94304
Email: erik.ordentlich@hp.com

***Abstract*—The degrees-of-freedom of a K-user Gaussian interference channel (GIFC) has been defined to be the multiple of $(1/2)\log_2 P$ at which the maximum sum of achievable rates grows with increasing P. In this paper, we establish that the degrees-of-freedom of three or more user, real, scalar GIFCs, viewed as a function of the channel coefficients, is discontinuous at points where all of the coefficients are non-zero rational numbers. More specifically, for all $K > 2$, we find a class of K-user GIFCs that is dense in the GIFC parameter space for which $K/2$ degrees-of-freedom are exactly achievable, and we show that the degrees-of-freedom for any GIFC with non-zero rational coefficients is strictly smaller than $K/2$. These results are proved using new connections with number theory and additive combinatorics.**

I. Introduction

The time-invariant, real, scalar K-user Gaussian interference channel (GIFC), as introduced in [1], involves K transmitter-receiver pairs in which each transmitter attempts to communicate a uniformly distributed, finite-valued message to its corresponding receiver by sending a signal comprised of n real numbers. Each receiver observes a component-wise linear combination of possibly *all* of the transmitted signals plus additive memoryless Gaussian noise, and seeks to decode, with probability close to one, the message of its corresponding transmitter, in spite of the interfering signals and noise. The time averages of the squares of the transmitted signal values are required to not exceed certain power constraints. A K-tuple of rates $(R_1, \ldots, R_K)$ is said to be achievable for a GIFC if the transmitters can increase the sizes of their message sets as 2^{nR_i} with the signal length n, and signal in such a way that the power constraints are met and the receivers are able to correctly decode their corresponding messages with probability converging to 1, as n grows to infinity. The set of all achievable K-tuples of rates is known as the capacity region of the GIFC. Determining it, as a function of the channel coefficients (specifying the linear combinations mentioned above), power constraints, and noise variances, has been an open problem in information theory for over 30 years.

A complete solution for even the two-user case, which has received the most attention to date, is still out of reach. The best known coding scheme for two users is that presented in [2]. In some ranges of channel coefficients, such as for strong interference, the capacity region is completely known for two users [1]. Still for other ranges, the maximum achievable sum-of-rates is known [3], [4], [5], [6]. For the general two-user case, [7] determines the capacity region to within a $1/2$ bit margin (1 bit for the complex case) using a carefully chosen version of the scheme of [2], and a new genie-aided outerbound.

The case of $K > 2$ users has, until very recently, received less attention. Much of the recent effort on $K > 2$, beginning with [8] and continuing in e.g., [9], [10] has focused on characterizing the growth of the capacity region in the limit of increasing signal-to-noise ratio (SNR) corresponding, for example, to fixing the noise variances and channel coefficients and letting the power constraints tend to infinity. Specific attention has been directed at the growth of the maximum sum of achievable rates. If there were no interference, the maximum achievable rate corresponding to each transmitter-receiver pair would grow like $(1/2)\log_2 P$ in the limit of increasing power, which follows from the well known formula of $(1/2)\log_2(1 + P/N)$ for the capacity of a single user additive Gaussian noise channel with power constraint P and noise variance N. Thus, the maximum sum of achievable rates would grow as $(K/2)\log_2 P$ if there were no interference. This motivates the expectation that, in the general case, the maximum sum of achievable rates would grow as $(d/2)\log_2 P$ for some constant $d \leq K$, depending on the channel coefficients, where d has been dubbed the degrees-of-freedom of the underlying GIFC. Although determining d for a given GIFC is, in principle, simpler than determining the capacity region, it has turned out to be a difficult problem in its own right, for $K > 2$.[1]

A positive development in the study of the degrees-of-freedom of GIFCs with more than two users has been the discovery of a new coding technique known as *interference alignment*, which involves carefully choosing the transmitted signals so that the interfering signals "align" benignly at each receiver [9]. Interference alignment has been shown, under some conditions which we summarize below, to achieve nearly $d = K/2$ degrees-of-freedom, which is half of the degrees-of-freedom in the case of no interference at all. Interference alignment is not possible to implement for two users and its discovery thus had to wait until the focus shifted to more users.

[1]The degrees-of-freedom is known to be 1 for all two-user GIFCs, unless there is no interference [8].

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Another new phenomenon in network information theory that has recently emerged as the number of users studied was increased, is the technique of *indirect decoding*, which is crucial for achieving the capacity region of certain three-user broadcast channels [12]. Again, this technique is not relevant in the two-user case, and could not have been discovered in the study thereof.

In this paper, we find a new information theoretic phenomenon concerning interference channels that is not manifest in the two-user case. In particular, we find that the degrees-of-freedom (and therefore the capacity region at high signal-to-noise ratio) of real, scalar GIFCs with $K > 2$ users is very sensitive to whether the channel coefficients determining the linear combinations of signals at each receiver are rational or irrational numbers. Next, we formally explain our results and their significance in the context of the growing literature on $K > 2$ user GIFCs.

II. Main results

We shall use a matrix H to denote the direct and cross gains of a time invariant, real, scalar K-user (GIFC) [1] with the (i, j)-th entry $h_{i,j}$ specifying the channel gain from transmitter i to receiver j. Thus, the signal observed by receiver $j \in \{1, \ldots, K\}$ at time index $t = 1, 2, \ldots$ is given by $y_{j,t} = z_{j,t} + \sum_{i=1}^{K} x_{i,t} h_{i,j}$ where $x_{i,t}$ is the real valued signal of transmitter $i \in \{1, \ldots, K\}$ at time t and $z_{j,t}$ is additive Gaussian noise with variance σ_j^2, independent across time and users. Fixing a block length n, the transmitted signals $\{x_{i,t}\}$ are required to satisfy the average power constraints $\sum_{t=1}^{n} x_{i,t}^2 \leq nP_i$ for some collection of powers $P_1, \ldots, P_K$. For $H \in \mathbb{R}^{K \times K}$, and $\boldsymbol{\sigma}, \mathbf{P} \in \mathbb{R}_+^K$, we let $\mathcal{C}(H, \boldsymbol{\sigma}, \mathbf{P})$ denote the capacity region of a GIFC with gain matrix H, receiver noise variances given by the corresponding components of $\boldsymbol{\sigma}$, and average (per codeword) power constraints given by the components of $\mathbf{P}$, defined formally below.

Definition 1: The capacity region $\mathcal{C}(H, \boldsymbol{\sigma}, \mathbf{P})$ of the GIFC is defined as the set of rate-tuples $R_1, \ldots, R_K$ for which there exists a sequence of block length n message sets and power-constrained coding schemes satisfying $\lim_{n\to\infty} \max_{1\leq i\leq K} Pr(W_i \neq g_i(\mathbf{y}^n)) = 0$, where W_i and $g_i(\cdot)$ are the transmitted message and decoding function of user i, and where the probability of error is taken with respect to the distribution induced by the random messages, the coding scheme, and the channel, as specified above.

Following [8], we define the degrees-of-freedom of H as

$$DoF(H) = \limsup_{P\to\infty} \frac{\max_{\mathbf{R}\in\mathcal{C}(H,\mathbf{1},P\mathbf{1})} \mathbf{1}^t\mathbf{R}}{(1/2)\log_2 P}, \tag{1}$$

where $\mathbf{1}$ denotes the vector of all ones. The degrees-of-freedom of a GIFC characterizes the behavior of the maximum achievable sum rate as the SNR tends to infinity, with the gain matrix fixed.

A fully connected GIFC is one for which $h_{i,j} \neq 0$ for all i and j. It was shown in [8] that for fully connected H, $DoF(H) \leq K/2$. If H is not fully connected, the degrees-of-freedom can be as high as K, such as when H is the identity matrix where all cross gains are zero. Little was known about tightness of the $K/2$ bound for $K > 2$ until it was shown in [9] that for *vector* GIFCs and an appropriate generalization of $DoF(\cdot)$ to include a normalization by the input/output vector dimension, the degrees-of-freedom of "almost all" fully connected vector GIFCs approaches $K/2$ when the vector dimension tends to infinity.[2] In addition, an example of a fully connected two-dimensional vector GIFC achieving exactly $K/2$ degrees-of-freedom was also given in [9]. The key tool introduced in [9] to establish these results is the technique of interference alignment, which involves the transmitters signaling over linear subspaces that, after component-wise scaling by the cross gains, *align* into interfering subspaces which are linearly independent with the directly received subspaces, allowing for many interference free dimensions over which to communicate. For real, scalar GIFCs, it was shown in [10], using a different type of interference alignment, that the degrees-of-freedom of certain fully connected GIFCs also approaches $K/2$ when the cross gains tend to zero. Yet a different type of interference alignment is used in [11] to find new achievable rates for a non-fully connected GIFC in which interference occurs only at one receiver. To our knowledge, the problem of determining or computing the degrees-of-freedom of general GIFCs is still open.

As in [10], in this paper we consider only fully connected scalar, real GIFCs and establish the following results on the degrees-of-freedom.

Theorem 1: If all diagonal components of a fully connected H are irrational algebraic numbers and all off-diagonal components are rational numbers then $DoF(H) = K/2$.

Theorem 2: For $K > 2$, if all elements of a fully connected H are rational numbers then $DoF(H) < K/2$.

The following corollary is then immediate from Theorems 1 and 2, and the well known fact that irrational algebraic numbers are dense in the real numbers.

Corollary 1: For $K > 2$, the function $DoF(H)$ is discontinuous at all fully connected H with rational components.

Theorem 1 demonstrates the existence of fully connected, real K-user GIFCs with exactly $K/2$ degrees-of-freedom. In contrast to the result of [10], Theorem 1 is non-asymptotic (in H). The underlying achievability scheme is based on an interference alignment phenomenon that differs from the ones used in [9] and [10], and relies on number theoretic lower bounds on the approximability of irrational algebraic numbers by rationals.

Theorem 2 reveals a surprising limitation on $DoF(H)$ when the components are non-zero rational numbers (up to arbitrary pre-post multiplication by diagonal matrices - see Lemma 1 in Section IV). In this case, $DoF(H)$ is strictly bounded away from $K/2$. Previously known techniques for finding outer-bounds to the capacity regions of GIFCs, such as cooperative encoding and decoding [13], [14], genie aided decoding [7], [15], [16], and multiple access bounds [1], [17] are not sensitive to the rationality of the channel parameters

[2] The $K/2$ bound of [8] extends to the fully connected vector case as well.

and hence do not suffice to establish Theorem 2. Instead, our proof of this theorem is based on a new connection between GIFCs with rational H and results from additive combinatorics [19], a branch of combinatorics that is concerned with the cardinalities of sum sets, or sets obtained by adding (assuming an underlying group structure) any element of a set A to any element of a set B.

III. Real, scalar GIFCs with exactly $K/2$ degrees-of-freedom

In this section, we provide a sketch of the proof of Theorem 1, demonstrating the existence of fully connected, real, scalar K-user GIFCs with exactly $K/2$ degrees-of-freedom. The complete proof can be found in [20].

First, we prove a simple lemma (see Lemma 1 in the next section) showing that $DoF(H) = DoF(D_t H D_r)$ for any diagonal matrices D_t and D_r with positive diagonal components. This, in turn, implies that we can transform any H satisfying the assumptions of Theorem 1 to one with irrational, algebraic numbers along the diagonal and integer values in off-diagonal components, while preserving the degrees-of-freedom. We then focus on coding schemes for the new H in which each transmitter is restricted to signaling over the scalar lattice $\{zP^{1/4+\epsilon} : z \in \mathbb{Z}\}$ intersected with the interval $[-P^{1/2}, P^{1/2}]$. The idea is that the integer valued cross gains guarantee that the interfering signal values at each receiver will also be confined to this scalar lattice (though may fall outside of the $P^{1/2}$ interval), while the irrational direct gains place the directly transmitted signal values on a scaled lattice that "stands out" from the interfering lattice. Specifically, this scaled lattice has the property that offsetting the interfering lattice (equal to the original lattice) by each point in the scaled lattice results in disjoint sets. A non-empty intersection would imply that the direct gain could be written as the ratio of two integers, which would contradict its irrationality. An even stronger property holds for *algebraic* irrational direct gains: the distance between any pair of points obtained by adding a point from the scaled lattice to a point from the interfering lattice actually grows with P. This is shown to follow from a major result in number theory stating that for any irrational algebraic number α and any $\gamma > 0$, a rational p/q approximation will have an error of at least $\delta/q^{2+\gamma}$ for some δ depending only on α and γ [18].[3] The next step in the proof is to deal with the noise by coupling this inter-point distance growth with Fano's inequality to show that the mutual information induced between each transmitter-receiver pair by independent, uniform distributions on the original power-constrained lattices, taking interference into account, grows like $(1/4 - \epsilon)\log_2 P$, for arbitrarily small ϵ. This, in turn, implies the existence of a sequence of block codes (with symbols from the original lattice) with sum rate approaching $(K/4)\log_2 P$, and which are correctly decodeable, with high probability, by treating interference as noise.

[3] For irrational algebraic numbers of degree two (solutions to quadratic equations with integer coefficients), such as $\sqrt{2}$, the approximation bound holds with $\gamma = 0$ and is known as Liouville's Theorem (established in 1844). The validity of the bound for general algebraic numbers was a longstanding open problem in number theory and was finally established in 1955 by K. F. Roth, for which he was awarded the Fields Medal.

IV. Degrees-of-freedom for rational H

In this section, we give a sketch of the proof of Theorem 2, which establishes that the degrees-of-freedom of any fully connected, real, scalar GIFCs is bounded strictly below $K/2$, for $K > 2$. The proof uses several lemmas, which are stated at the end of this section. The complete proof of the theorem (and lemmas) can be found in [20].

Most of the work in the proof is to establish the theorem for $K = 3$ users. The theorem for $K > 3$ will then follow from an extension of the averaging argument of [8], used therein to obtain the $K/2$ degrees-of-freedom upper bound from a bound of 1 on the degrees-of-freedom for $K = 2$. In this case, the $K = 3$ bound is averaged over all three-tuples of users (transmitters and corresponding receivers), as opposed to pairs of users in [8].

Given a $K = 3$ user GIFC with fully connected, rational H, using Lemma 1 (invariance property) and eliminating cross links, we can upper bound $DoF(H)$ by $DoF(\tilde{H})$ where

$$\tilde{H} = [\tilde{h}_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & p & 0 \\ 1 & q & 1 \end{bmatrix},$$

where p and q are integers (see Figure 1).

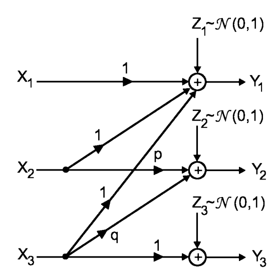

Fig. 1. A three-user Gaussian IFC with channel matrix $\tilde{H}$.

The main step in the proof of the overall theorem is establishing that $DoF(\tilde{H}) < 3/2$, which is formally carried out in Lemma 2 below, and proceeds as follows. First, it is shown (Lemma 3) that a deterministic channel obtained by eliminating all noise sources and restricting the power-constrained codewords to have integer valued symbols results in at most a power-constraint-independent loss in the achievable sum rate. Therefore, the degrees-of-freedom (according to the obvious generalization) of this deterministic interference channel (IFC) is no smaller than $DoF(\tilde{H})$. Next, it is shown using a Fano's inequality based argument that if the degrees-of-freedom of the deterministic IFC is at least $3/2$ there would exist finite sets of n-dimensional integer valued vectors $\mathcal{X}_2$ and $\mathcal{X}_3$ such that the corresponding independent random variables $\mathbf{x}_2^n$ and $\mathbf{x}_3^n$, uniformly distributed on these sets, induce discrete entropies satisfying $H(\mathbf{x}_2^n) \approx n(1/4)\log_2 P$, $H(\mathbf{x}_3^n) \approx n(1/4)\log_2 P$, $H(\mathbf{x}_2^n + \mathbf{x}_3^n) \approx n((1/4)+\epsilon)\log_2 P$, and $H(p \cdot \mathbf{x}_2^n + q \cdot \mathbf{x}_3^n) \approx n((1/2) - \epsilon)\log_2 P$, for the integers p and q defining the channel and ϵ arbitrarily small. These entropy relations suggest that the cardinality of the support of $p \cdot \mathbf{x}_2^n + q \cdot \mathbf{x}_3^n$ is much larger than that of $\mathbf{x}_2^n + \mathbf{x}_3^n$. However, tools from additive combinatorics (through Lemma 4) can be used to show that this is impossible for integer valued p and q, leading to a contradiction, and thereby implying that the deterministic channel must have degrees-of-freedom strictly smaller than $3/2$. Unfortunately, the link between the entropy and the cardinality of the support of a sum of independent, uniformly distributed random variables is sufficiently weak that a somewhat more involved argument (incorporating Lemma 5 and Theorem 3) is ultimately required to reach the above conclusions. The overall intuition behind the proof, however, is as outlined.

Lemma 1 (Invariance property): For any matrix H and diagonal matrices D_t and D_r with positive diagonal components $DoF(D_t H D_r) = DoF(H)$.[4]

Lemma 2: Let $p, q \in \mathbb{Z}$, $p, q \neq 0$, and $\tilde{H} = [\tilde{h}_{ij}]$ as defined above, with the corresponding GIFC depicted in Figure 1. Then $DoF(\tilde{H}) \leq \frac{3}{2} - \epsilon(p,q)$, with $\epsilon(p,q) > 0$. In particular, this holds for

$$\epsilon(p,q) = \frac{1}{12d(p,q)+2},$$

where $d(p,q)$ is as in Lemma 4 below.

Lemma 3: Given a gain matrix H and power constraints $\mathbf{P} = (P_1, \ldots, P_K)$, let $\mathcal{C}_D(H, \mathbf{P})$ denote the capacity region of the deterministic IFC defined by

$$\bar{y}_i(t) = \sum_{j=1}^{K} h_{ji} \bar{x}_j(t) \ , \ i = 1, \ldots, K$$

where the inputs are constrained to be integers (i.e. $\bar{x}_i(t) \in \mathbb{Z}$, $i = 1, \ldots, K$, $t = 1, 2 \ldots$) and satisfy an average power constraint $\frac{1}{n}\sum_{t=1}^{n} \bar{x}_i(t)^2 \leq P_i$ for all i. Then, $\mathbf{R} \in \mathcal{C}(H, \mathbf{1}, \mathbf{P}) \Rightarrow (\mathbf{R} - \mathbf{\Delta}) \in \mathcal{C}_D(H, \mathbf{P})$ with $\mathbf{\Delta} = (\delta_1, \ldots, \delta_K)$, $\delta_i = \frac{1}{2}\log_2(1 + 2\sum_{j=1}^{K} h_{ji}^2)$, $i = 1, \ldots, K$, where $\mathcal{C}(H, \mathbf{1}, \mathbf{P})$ is the capacity region of the corresponding GIFC (see Definition 1).

Lemma 4: Let $p, q \in \mathbb{Z}$, $T \in \mathbb{R}$ with $T \geq 1$, and $A, B \subseteq G$ for some abelian group G. Let $A + B = \{a + b : a \in A, b \in B\}$ and $p \cdot A + q \cdot B = \{pa + qb : a \in A, b \in B\}$. If $|A + B| \leq T|A|^{1/2}|B|^{1/2}$ then $|p \cdot A + q \cdot B| \leq T^{d(p,q)}|A|^{1/2}|B|^{1/2}$ for $d(p,q) = 2\max\{|p|, |q|\} + 5$, where $|S|$ denotes the cardinality of the set S.

Lemma 5: Given an abelian group G and $A, B \subseteq G$, let X and Y be independent, uniform random variables with support sets A and B, with $|A| \geq |B|$, such that

$$H(X+Y) \leq (1+\epsilon)\log_2|A|$$

for some $\epsilon > 0$. Then, for any given $c > 1$ there exists a set $F \subseteq A \times B$ such that

$$|F| \geq |A||B|\frac{c-1}{c}$$

and

$$|A \overset{F}{+} B| \leq \left(|A|^{1/2+c\epsilon}|B|^{-1/2}\right)|A|^{1/2}|B|^{1/2},$$

where $A \overset{F}{+} B = \{a + b : (a,b) \in F\}$ (the partial sum set of A and B with respect to F).

Theorem 3 (Balog-Szemerédi-Gowers theorem): Let $A, B \subseteq G$ for some abelian group G and let $F \subseteq A \times B$ be such that $|F| \geq |A||B|/K$ and $|A \overset{F}{+} B| \leq K'|A|^{1/2}|B|^{1/2}$ for some $K \geq 1$ and $K' > 0$. Then there exists $A' \subseteq A, B' \subseteq B$ such that

$$|A'| \geq \frac{|A|}{4\sqrt{2}K} \ , \ |B'| \geq \frac{|B|}{4K} \ ,$$
$$|A' + B'| \leq 2^{12}K^5(K')^3|A|^{1/2}|B|^{1/2}.$$

Theorem 3 is proved in Chapter 6 of [19].

V. A 3-USER RATIONAL GIFC EXAMPLE

In this section, we will derive lower and upper bounds on the degrees-of-freedom of the 3-user GIFC with channel matrix

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(see Figure 1). This channel is a special case of the one considered in Lemma 2, with $p = 2$ and $q = 1$. From this lemma, we obtain:

$$DoF(H) \leq \frac{3}{2} - \frac{1}{12d(2,1)+2} \tag{2}$$

where $d(p,q) = 2\max\{|p|, |q|\} + 5$ was obtained in Lemma 4. For the special case of $q = 1$ the result of Lemma 4 can be easily strengthened to get $d(p,1) = 2|p| + 3$. Evaluating (2) with $d(2,1) = 7$ we obtain $DoF(H) \leq 1.4884$. It should be possible to improve this bound in a number of ways, such as by improving on $d(2,1) = 7$, and possibly by improving on the power 3 in the term $(K')^3$ appearing in Theorem 3. Another possibility might be to forgo this theorem for a different approach, such as one based on Exercise 2.5.4 of [19].

To get a lower bound on $DoF(H)$, we will describe a communication scheme that aims to achieve good interference alignment at receiver 1 by aligning the interfering signals of transmitters 2 and 3, while achieving good separation at receiver 2 between the signal of transmitter 2 and the interference of transmitter 3.

As was done in [10], we design the communication scheme for a deterministic interference channel and later show how to extend the scheme to the Gaussian channel. We derive the deterministic channel from the Gaussian IFC by removing the Gaussian noise and constraining the inputs to be integers. Let $A_1 = \{0,1\}$, $A_2 = \{0,2,4\}$, $A_3 = \{0,2\}$, and $Q = 8$. We communicate information independently over L levels, without coding over time. The signal of user i at time t is given by

$$x_i(t) = \sum_{\ell=1}^{L} m_{i,\ell}(t) Q^{\ell-1}, \tag{3}$$

[4]The matrices D_t and D_r need only have non-zero diagonal components for the result to hold. We assume positivity for simplicity, as this is all we shall require in this paper.

where $m_{i,\ell}(t) \in A_i$ is the message of user i in level ℓ at time t.

Since $A_1 + A_2 + A_3 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, the signal at receiver 1 can be written as

$$y_1(t) = \sum_{i=1}^{3} x_i(t) = \sum_{\ell=1}^{L} w_{1,\ell}(t) Q^{\ell-1}$$

with $w_{1,\ell}(t) = m_{1,\ell}(t) + m_{2,\ell}(t) + m_{3,\ell}(t)$. Therefore, by computing the Q-ary decomposition of $y_1(t)$ we can recover the sums $m_{1,\ell}(t) + m_{2,\ell}(t) + m_{3,\ell}(t)$ at each of the L levels. In addition, since $A_2 + A_3 = \{0, 2, 4, 6\}$ we have $m_{1,\ell}(t) = 1(w_{1,\ell}(t) \in \{1, 3, 5, 7\})$, so we can directly determine $m_{1,\ell}(t)$ from $w_{1,\ell}(t)$.

Similarly, at receiver 2 we compute

$$\frac{y_2(t)}{2} = x_2(t) + \frac{1}{2} x_3(t) = \sum_{\ell=1}^{L} w_{2,\ell}(t) Q^{\ell-1}$$

with $w_{2,\ell}(t) = m_{2,\ell}(t) + (1/2) m_{3,\ell}(t) \in \{0, 1, 2, 3, 4, 5\}$, from which we can compute $m_{2,\ell}(t) = w_{2,\ell}(t) - [w_{2,\ell}(t) \mod 2]$.

Finally, receiver 3 can directly recover $m_{3,\ell}(t)$ at all levels from the received signal $y_3(t) = x_3(t)$.

To compute the achievable degrees-of-freedom of this scheme we note that since $|x_i(t)| < Q^L$ the transmission power at each transmitter is smaller than Q^{2L}. On the other hand the rate of users 1 and 3 is $L \log_2 2$ while the rate of user 2 is $L \log_2 3$. Therefore, we obtain for the deterministic channel

$$DoF \geq \frac{2L \log_2 2 + L \log_2 3}{\frac{2L}{2} \log_2 8} = \frac{2 + \log_2 3}{3} \approx 1.19499.$$

We now informally argue that the same degrees-of-freedom can be achieved in the Gaussian channel. We essentially use the same multi-level coding scheme, but we now encode the signals of each level over long blocks of time. The lower levels may be severely affected by noise, but as the level ℓ increases, the influence of the noise becomes smaller, ultimately being insignificant. As a result, the amount of redundancy that needs to be added to the signal of level ℓ to ensure low probability of decoding error goes to 0 as ℓ grows to infinity. It follows that for ℓ large enough, the achievable rates in the Gaussian channel at level ℓ approach the achievable rates in the deterministic channel, and since the rates of the lower levels do not affect the degrees-of-freedom, we conclude that $DoF(H) \geq \frac{2+\log_2 3}{3}$ (see [10] for a similar argument).

In summary, using the lower and upper bounds that we derived we have,

$$1.19499 \leq DoF(H) \leq 1.4884.$$

Remark 1: The achievable scheme that we described is simple to analyze because there are no "carry overs" across the different levels, and the signals and interference are "orthogonal" in the sense that there is no need to code over time to ensure reliable decoding in the deterministic channel. In choosing the sets A_1, A_2 and A_3 we tried to obtain small $|A_2 + A_3|$ to align the interference at receiver 1 and simultaneously obtain large $|2A_2 + A_3|$ to achieve good signal-interference separation at receiver 2. With these design guidelines, one could optimize the sets A_1, A_2 and A_3 (with possibly larger Q) in order to improve the achievable degrees-of-freedom. In addition, Han-Kobayashi-type schemes [2] at each level where part of the interference is decoded and subtracted may result in better performance than purely orthogonal schemes.

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