Degrees of Freedom in Underspread MIMO Fading Channels

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Recent information theoretic results suggest that in richly scattered wireless environments, systems with multiple transmit and multiple receive antennas (MIMO systems) can have very large capacities. In particular, Foschini and Gans [1] considered a channel with n_t transmit and n_r receive antennas, with Rayleigh flat faded channel gains i.i.d. across antenna pairs, and showed that at high SNR, the capacity of this channel grows like $\min(n_t, n_r) \log \mathsf{SNR}$ for large SNR. This yields a $\min\{n_t, n_r\}$-fold increase in capacity over a channel with a single transmit and a single receive antenna. The parameter $\min\{n_t, n_r\}$ can be interpreted as the number of *degrees of freedom* (d.o.f.) of the channel: the dimension of the space over which communication can take place.

The above result assumes that the receiver can perfectly track the fading gains of the channel (so-called perfect channel state information CSI at the receiver). In high mobility applications, this may not be a reasonable assumption. Moreover, in the high SNR regime where the amount of noise is small, it is conceivable that the impact of channel uncertainty on performance is more pronounced. This leads to the question: what is the high SNR capacity of time-varying fading channels without the prior assumption of CSI? Towards this end, Lapidoth and Moser [2], building on earlier work by Taricco and Elia [3], recently showed a contrasting result: that at high SNR, the first order term of the capacity is $\log \log \mathsf{SNR}$, regardless of what the number of transmit and receive antennas is. Thus, not only does capacity grow much slower than in the case with perfect CSI, but this result also suggests that without CSI, the performance gain from having multiple antennas, if any, will only appear as a second-order effect. The increase in the number of the degrees of freedom has minimal impact.

Does this then suggest that the perfect CSI results are very fragile? Even though [2] explicitly incorporates the channel variation (and the associated uncertainty) in its model, this result is still asymptotic in the SNR and thus one has to be careful in interpreting its regime of validity. In particular, since the channel variation process is fixed while the SNR is taken to infinity, it is conceivable that the $\log \log \mathsf{SNR}$ growth only occurs when the noise level is much smaller than the amount of channel variation from one sample to the next. However, typical wireless channels are *underspread*, which means that this variation is small. Thus, one has to look at the effect of the SNR and the amount of channel variation *simultaneously* to get a more complete picture.

In this work, we would like to put forth such a picture. Suppose the fading process for each channel gain is Gauss-Markov and define SPR to be the inverse of the one-step MMSE prediction error. For underspread channels, SPR is large. We propose that the capacity $C(\mathsf{SNR}, \mathsf{SPR})$ of a high SNR underspread MIMO fading channel (without CSI) can be described in three regimes:

Regime 1: $\mathsf{SNR}, \mathsf{SPR} \gg 1, \mathsf{SNR} < \mathsf{SPR}$

$$C(\mathsf{SNR}, \mathsf{SPR}) \sim \min\{n_t, n_r\} \log \mathsf{SNR}.$$

Regime 2: $\mathsf{SNR}, \mathsf{SPR} \gg 1, \mathsf{SPR} \leq \mathsf{SNR} < \exp\left(\mathsf{SPR}^{\min\{n_t, n_r\}}\right)$

$$C(\mathsf{SNR}, \mathsf{SPR}) \sim \min\{n_t, n_r\} \log \mathsf{SPR}.$$

Regime 3: $\mathsf{SNR}, \mathsf{SPR} \gg 1, \mathsf{SNR} \geq \exp\left(\mathsf{SPR}^{\min\{n_t, n_r\}}\right)$

$$C(\mathsf{SNR}, \mathsf{SPR}) \sim \log \log \mathsf{SNR}.$$

In the first regime, the SNR is smaller than the SPR. The system is noise-limited and its capacity behaves as though there is perfect CSI at the receiver. In the second regime, the SNR is now larger than the SPR and the system is now limited by channel uncertainty. However, when the SNR gets much larger, the Lapidoth-Moser regime kicks in and the system is again noise-limited, albeit with a much smaller growth rate. The important point is that in both regime 1 and 2, the capacity is proportional to the degrees of freedom in the channel.

To get a feeling of the values of SNR that separate the three regimes consider an $n_r = n_t = 4$ system. For urban environments with mobile speeds in the order of $5 - 50$ km/h, with carrier frequencies ranging from 800 Mhz to 5 Ghz the threshold between regimes 1 and 2 can range from 17.4 to 40 dB, while the threshold that separates regimes 2 and 3 can range from $4.1 \cdot 10^7$ to $4.3 \cdot 10^{16}$ dB. For indoor environments these thresholds are even larger. We conclude that typical wireless scenarios fall in regimes 1 and 2 but very rarely in 3.

In [4] we propose a communication scheme based on interleaving, decision-oriented channel estimation and weighted minimum Euclidean distance decoding that achieves the full number of degrees of freedom of the channel in the first two regimes.

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References

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