Gaussian Interference Channel Capacity to Within One Bit: the General Case

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***Abstract*— The characterization of the capacity region of the two-user Gaussian interference channel has been an open problem for thirty years. The understanding on this problem has been limited. The best known achievable region is due to Han-Kobayashi but its characterization is very complicated. It is also not known how tight the existing outer bounds are. In this work, we extend our results of [1] to general (i.e. possibly asymmetric) channels for the complete capacity region. We show that the existing outer bounds can in fact be arbitrarily loose in some parameter ranges, and by deriving new outer bounds, we show that a simplified Han-Kobayashi type scheme can achieve to within a single bit the capacity for all values of the channel parameters. Using our results, we provide a natural generalization of the point-to-point classical notion of degrees of freedom to interference-limited scenarios.**

I. Introduction

Interference is a central phenomenon in wireless communication when multiple uncoordinated links share a common communication medium. Most state-of-the-art wireless systems deal with interference in one of two ways:

- orthogonalize the communication links in time or frequency, so that they do not interfere with each other at all;
- allow the communication links to share the same degrees of freedom, but treat each other's interference as adding to the noise floor.

It is clear that both approaches can be sub-optimal. The first approach entails an *a priori* loss of degrees of freedom in both links, no matter how weak the potential interference is. The second approach treats interference as pure noise while it actually carries information and has structure that can potentially be exploited in mitigating its effect.

These considerations lead to the natural question of what is the best performance one can achieve without making any *a priori* assumptions on how the common resource is shared. A basic information theoretic model to study this question is the two-user Gaussian interference channel (GIFC), where two point-to-point links with additive white Gaussian noise interfere with each other (Figure 1). The capacity region of this channel is the set of all simultaneously achievable rate pairs (R_1, R_2) in the two interfering links, and characterizes the fundamental tradeoff between the performance achievable in the two links in face of interference. Unfortunately, the problem of characterizing this region has been open for over thirty years. The only case in which the capacity is known is in the *strong* interference case, where each receiver has a better reception of the other user's signal than the intended receiver [2]–[4]. The best known strategy for the other cases is due to Han-Kobayashi [2]. This strategy is a natural one and involves splitting the transmitted information of both users into two parts: private information to be decoded only at own receiver and common information that can be decoded at both receivers. By decoding the common information, part of the interference can be cancelled off, with the remaining private information from the other user treated as noise. The Han-Kobayashi (HK) strategy allows arbitrary splits of each user's transmit power into the private and common information portions as well as time sharing between multiple such splits. Unfortunately, the optimization among such myriads of possibilities is not well-understood, so while it is clear that it will be no worse than the above-mentioned strategies as it includes them as special cases, it is not very clear how much improvement can be obtained and in which parameter regime would one get significant improvement. More importantly, it is also not clear how close to capacity are the achievable rates of the scheme and whether there will be other strategies that can do significantly better.

The main result in this paper is that a very simple HK type scheme with a single private-common power split can in fact achieve within 1 bit/s/Hz of the capacity of the channel for *all* values of the channel parameters. That is, for all rate pairs (R_1, R_2) in the interference channel capacity region, $(R_1 - 1, R_2 - 1)$ is achievable by this simple strategy. This result is particularly relevant in the high SNR regime, where the achievable rates are high and in fact grow unbounded as the noise level goes to zero. Through this result, we are able to characterize the interference channel capacity region to within one bit.

[1]The ordering of the authors in this paper is alphabetical.

[2]This work was done when R. Etkin was a graduate student at Berkeley and H. Wang was visiting.

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In [1] we presented our results for the symmetric capacity of symmetric GIFCs, i.e. the maximum rate that can be simultaneously achieved by both users. In this conference paper, we present our results for the complete capacity region of general asymmetric GIFCs.

The key feature of our simple HK scheme is that the power of the private information of each user should be set such that it is received at the level of the Gaussian noise at the other receiver. In this way, the interference caused by the private information has a small effect on the other link as compared to the impairments already caused by the noise. At the same time, quite a lot of private information can be conveyed in the own link if the direct gain is appreciably larger than the cross gain.

To prove that this scheme is within one bit of optimality, we need good outer bounds on the capacity region of the interference channel. The best known outer bound [5] is based on giving extra side information to one of the receivers so that it can decode all of the information from the other user (the Z-channel and related bounds). It turns out that while this bound is sufficiently tight in some parameter regimes, it can be arbitrarily loose in others. We derive new outer bounds and show that very simple HK type schemes can get within 1 bit/s/Hz of this outer bound for all range of parameters. Our outer bounds are motivated by the bounding techniques of [7] used to establish the capacity region of a class of deterministic interference channels.

The rest of the paper is structured as follows. In Section II we describe the model. The main results are described in Section III. Using our results, we derive in Section IV a notion of generalized degrees of freedom.

II. Model

In this section we describe the model to be used in the rest of this work. We consider a two-user GIFC. In this model there are two transmitter-receiver pairs, where each transmitter wants to communicate with its corresponding receiver (cf. Figure 1). This channel is represented by the equations:

$$y_1 = h_{11}x_1 + h_{21}x_2 + z_1,\; y_2 = h_{12}x_1 + h_{22}x_2 + z_2,$$

where for $i = 1, 2$, $x_i \in \mathbb{C}$ is subject to a power constraint P_i, i.e., $E[|x_i|^2] \le P_i$, and the noise processes $Z_i \sim \mathcal{CN}(0, N_0)$ are i.i.d. over time.

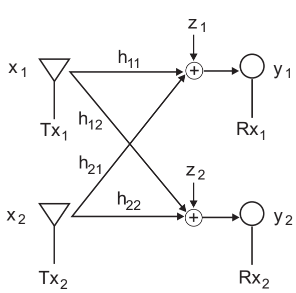

Fig. 1. Two-user Gaussian interference channel.

It is easy to see that the capacity region of the interference channel depends only on four parameters: the signal to noise and interference to noise ratios. For $i = 1, 2$, let $\mathsf{SNR}_i = |h_{ii}|^2 P_i/N_0$ be the signal to noise ratio at receiver i, and $\mathsf{INR}_1 = |h_{21}|^2 P_2/N_0$ ($\mathsf{INR}_2 = |h_{12}|^2 P_1/N_0$) be the interference to noise ratio at receiver 1 (2). As will become apparent from our analysis, this parameterization in terms of SNR and INR is more natural for the interference channel, because it puts in evidence the main factors that determine the channel capacity.

For a given block length n, user i communicates a message $m_i \in \{1, \ldots, 2^{nR_i}\}$ by choosing a codeword from a codebook $\mathcal{C}_{i,n}$, with $|\mathcal{C}_{i,n}| = 2^{nR_i}$. The codewords $\{\mathbf{c}_i(m_i)\}$ of this codebook must satisfy the average power constraint:

$$\frac{1}{n}\sum_{t=1}^{n} |c_i(m_i)[t]|^2 \le P_i$$

Receiver i observes the channel outputs $\{y_i[t] : t = 1, \ldots, n\}$ and uses a decoding function $f_{i,n} : \mathbb{C}^n \to \mathbb{N}$ to get the estimate $\hat{m}_i$ of the transmitted message m_i. The receiver is in error whenever $\hat{m}_i \ne m_i$. The average probability of error for user i is given by $\epsilon_{i,n} = E[P(\hat{m}_i \ne m_i)]$, where the expectation is taken with respect to the random choice of the transmitted messages m_1 and m_2. Note that due to the interference among users, the probability of error of each user may depend on the codeword transmitted by the other user.

A rate pair (R_1, R_2) is achievable if there exists a family of codebook pairs $\{(\mathcal{C}_{1,n}, \mathcal{C}_{2,n})\}_n$ with codewords satisfying the power constraints P_1 and P_2 respectively, and decoding functions $\{(f_{1,n}(\cdot), f_{2,n}(\cdot)\}_n$, such that the average decoding error probabilities $\epsilon_{1,n}, \epsilon_{2,n}$ go to zero as the block length n goes to infinity.

The capacity region $\mathcal{R}$ of the interference channel is the closure of the set of achievable rate pairs.

III. Main Results

In order to derive an inner bound for the GIFC capacity region we will use a simple communication scheme that is a special case of the general type of schemes introduced by Han and Kobayashi in [2]. Let us first describe the HK setup that is usually employed for the GIFC. For a given block length n user i chooses a private message from codebook $\mathcal{C}^u_{i,n}$ and a common message from codebook $\mathcal{C}^w_{i,n}$. These codebooks satisfy the power constraints P_{ui} and P_{wi} with $P_{ui}+P_{wi} = P_i$. The sizes of these codebooks are such that $|\mathcal{C}^u_{i,n}|\cdot|\mathcal{C}^w_{i,n}| = 2^{nR_i}$. After selecting the corresponding codewords user i transmits the signal $\mathbf{x}_i = \mathbf{c}^u_i + \mathbf{c}^w_i$ by adding the private and common codewords. The private codewords must be decoded by the own receiver, while the common codewords must be decoded by both receivers.

The HK scheme allows to generate the codebooks using arbitrary input distributions, and allows to do time sharing between multiple strategies. A characterization of the HK achievable region in terms of single letter expressions is given in [2, Theorem 4.1]. Recently, a simplified yet equivalent HK achievable region was given in [6].

We will consider a simple scheme where the codebooks are generated by using i.i.d. random samples of a Gaussian

$\mathcal{CN}(0,\sigma^2)$ random variable with $\sigma^2 = P_{ui}, P_{wi}$. In addition, we fix the choice of private and common message powers, i.e. we do not time share between multiple strategies with different private/common message power splits. We denote by INR_{pj} the interference to noise ratio at receiver j due to the private message transmitted by user i, i.e.

$$\mathsf{INR}_{pj} = \frac{|h_{ij}|^2 P_{ui}}{N_0}$$

for $i,j = 1,2,\ i \neq j$. This HK scheme is denoted by $\mathrm{HK}(\mathsf{INR}_{\mathrm{p}2}, \mathsf{INR}_{\mathrm{p}1})$, and the corresponding achievable region is denoted by $\mathscr{R}(\mathsf{INR}_{p2}, \mathsf{INR}_{p1})$.

The channel is in weak interference when $\mathsf{SNR}_1 > \mathsf{INR}_2$ and $\mathsf{SNR}_2 > \mathsf{INR}_1$. In this case we choose P_{ui} such that INR_{pj} is as close to 1 as possible. When $\mathsf{INR}_{pj} = 1$ the interference created by the private message has the same power as the Gaussian noise. Note that setting $\mathsf{INR}_{pj} = 1$ is only possible when $\mathsf{INR}_j \geq 1$. When $\mathsf{INR}_j < 1$ we set $\mathsf{INR}_{pj} = \mathsf{INR}_j$. The following theorem states that this scheme achieves rates within 1 bit/s/Hz of capacity.[1]

Theorem 1: The achievable region

$$\mathscr{R}\big(\min(1, \mathsf{INR}_2), \min(1, \mathsf{INR}_1)\big)$$

is within one bit of the capacity region of the GIFC with weak interference. That is, for any rate pair (R_1, R_2) in the interference channel capacity region, $(R_1 - 1, R_2 - 1)$ is in $\mathscr{R}\big(\min(1, \mathsf{INR}_2), \min(1, \mathsf{INR}_1)\big)$. ■

In order to show that the scheme $\mathrm{HK}\big(\min(1, \mathsf{INR}_2), \min(1, \mathsf{INR}_1)\big)$ achieves a region close to capacity, we need to derive good outer bounds to the GIFC capacity region. The HK achievable region of [2], [6] is expressed in terms of bounds for R_1, R_2, $R_1 + R_2$, $2R_1 + R_2$, and $R_1 + 2R_2$. We can obtain an outer bound for the capacity region by computing upper bounds for R_1, R_2, R_1+R_2, $2R_1+R_2$, and R_1+2R_2. The upper bounds for R_1 and R_2 can be obtained from the single user capacity bounds that result from ignoring the effect of interference:

$$R_1 \leq \log(1 + \mathsf{SNR}_1) \quad (1)$$
$$R_2 \leq \log(1 + \mathsf{SNR}_2). \quad (2)$$

Upper bounds for R_1+R_2 can be obtained from the Z-channel bounds that result when a genie provides side information x_1 (x_2) to receiver 2 (1). These bounds are given by:

$$R_1 + R_2 \leq \log(1 + \mathsf{SNR}_2) + \log\left(1 + \frac{\mathsf{SNR}_1}{1 + \mathsf{INR}_1}\right) \quad (3)$$
$$R_1 + R_2 \leq \log(1 + \mathsf{SNR}_1) + \log\left(1 + \frac{\mathsf{SNR}_2}{1 + \mathsf{INR}_2}\right). \quad (4)$$

As was shown in [1] the Z-channel bounds can be arbitrarily loose in some parameter ranges, and therefore, a new sum rate upper bound is required. This is given in the following lemma:

Lemma 1: The sum capacity of the GIFC is upper bounded by:

$$R_1 + R_2 \leq \log\left(1 + \mathsf{INR}_1 + \frac{\mathsf{SNR}_1}{1 + \mathsf{INR}_2}\right) + \log\left(1 + \mathsf{INR}_2 + \frac{\mathsf{SNR}_2}{1 + \mathsf{INR}_1}\right). \quad (5)$$

Proof: Define $s_1 = h_{12}x_1 + z_2$, $s_2 = h_{21}x_2 + z_1$, and consider the genie-aided channel of Figure 2 where a genie provides side information s_1 to receiver 1, and s_2 to receiver 2. In this channel, for any code with block length n we can write:

$$\begin{aligned}
n(R_1 + R_2 - \epsilon_n) &\leq I(\mathbf{x}_1^n; \mathbf{y}_1^n, \mathbf{s}_1^n) + I(\mathbf{x}_2^n; \mathbf{y}_2^n, \mathbf{s}_2^n) \\
&= h(\mathbf{s}_1^n) - h(\mathbf{s}_1^n|\mathbf{x}_1^n) + h(\mathbf{y}_1^n|\mathbf{s}_1^n) - h(\mathbf{y}_1^n|\mathbf{x}_1^n, \mathbf{s}_1^n) \\
&\quad + h(\mathbf{s}_2^n) - h(\mathbf{s}_2^n|\mathbf{x}_2^n) + h(\mathbf{y}_2^n|\mathbf{s}_2^n) - h(\mathbf{y}_2^n|\mathbf{x}_2^n, \mathbf{s}_2^n) \\
&= h(\mathbf{s}_1^n) - h(\mathbf{z}_2^n) + h(\mathbf{y}_1^n|\mathbf{s}_1^n) - h(\mathbf{s}_2^n) \\
&\quad + h(\mathbf{s}_2^n) - h(\mathbf{z}_1^n) + h(\mathbf{y}_2^n|\mathbf{s}_2^n) - h(\mathbf{s}_1^n) \\
&= h(\mathbf{y}_1^n|\mathbf{s}_1^n) + h(\mathbf{y}_2^n|\mathbf{s}_2^n) - h(\mathbf{z}_1^n) - h(\mathbf{z}_2^n) \\
&\leq \sum_{i=1}^{n} \left[h(y_{1i}|s_{1i}) + h(y_{2i}|s_{2i}) - h(z_{1i}) - h(z_{2i})\right] (*)
\end{aligned}$$

where the last inequality follows by the fact that removing conditioning cannot reduce differential entropy, and $\epsilon_n \to 0$ as $n \to \infty$. Using the entropy maximizing property of the circularly symmetric complex Gaussian distribution, and applying Jensen's inequality to a concave and increasing function we obtain:

$$\frac{1}{n}\sum_{i=1}^{n} h(y_{1i}|s_{1i}) \leq \log\left[\pi e\left(N_0 + |h_{21}|^2 P_2 + \frac{|h_{11}|^2 P_1 N_0}{N_0 + |h_{12}|^2 P_1}\right)\right]$$

and

$$\frac{1}{n}\sum_{i=1}^{n} h(y_{2i}|s_{2i}) \leq \log\left[\pi e\left(N_0 + |h_{12}|^2 P_1 + \frac{|h_{22}|^2 P_2 N_0}{N_0 + |h_{21}|^2 P_2}\right)\right].$$

Replacing these inequalities in (*) we obtain the desired upper bound. ■

In order to derive upper bounds for $2R_1 + R_2$ and $R_1 + 2R_2$ we also use a genie to provide side information to the receivers. To derive an upper bound for $2R_1+R_2$ consider the interference channel of Figure 3 where receiver 1 has been split into two virtual receivers Rx_1^a and Rx_1^b. The genie provides side information x_2 to receiver Rx_1^a and side information s_2 to receiver Rx_2.

Receivers Rx_1^a and Rx_1^b need to decode the message transmitted by transmitter 1, and each achieves a rate R_1. Receiver 2 needs to decode the message transmitted by transmitter 2. Using this genie-aided channel we obtain an upper bound for the $2R_1 + R_2$ capacity of the GIFC:

$$2R_1 + R_2 \leq \log(1 + \mathsf{SNR}_1 + \mathsf{INR}_1) + \log\left(\frac{1 + \mathsf{SNR}_1}{1 + \mathsf{INR}_2}\right) + \log\left(1 + \mathsf{INR}_2 + \frac{\mathsf{SNR}_2}{1 + \mathsf{INR}_1}\right). \quad (6)$$

Similarly we can obtain an upper bound for $R_1 + 2R_2$:

$$R_1 + 2R_2 \leq \log(1 + \mathsf{SNR}_2 + \mathsf{INR}_2) + \log\left(\frac{1 + \mathsf{SNR}_2}{1 + \mathsf{INR}_1}\right) + \log\left(1 + \mathsf{INR}_1 + \frac{\mathsf{SNR}_1}{1 + \mathsf{INR}_2}\right). \quad (7)$$

[1] Due to space limitations we only provide sketches of the proofs of our results. The interested reader can refer to [9] (arXiv:cs/0702045) for complete derivations.

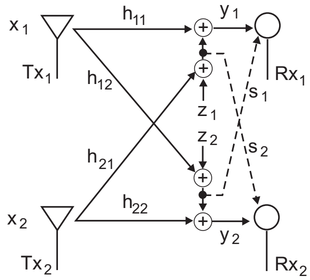

Fig. 2. Genie-aided GIFC for $R_1 + R_2$ upper bound. A genie provides signals s_1 to Rx_1 and s_2 to Rx_2.

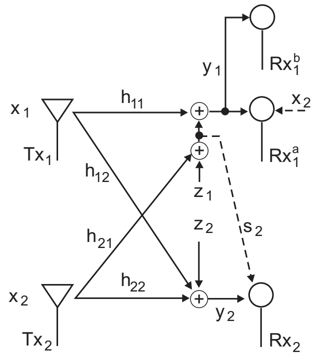

Fig. 3. Genie-aided GIFC for $2R_1 + R_2$ upper bound. Rx_1 is split into two virtual receivers. A genie provides signals x_2 to Rx_1^a and s_2 to Rx_2.

Theorem 1 follows by comparing $\mathscr{R}\big(\min(1, \mathsf{INR}_2), \min(1, \mathsf{INR}_1)\big)$ to the outer bound that results from the intersection of the bounds (1)-(7).

Theorem 1 is applicable only when both links are in weak interference. If one of the links experiences weak interference while the other link experiences strong interference, we need to change the communication scheme. Consider the "mixed interference" channel where $\mathsf{INR}_1 \geq \mathsf{SNR}_2$ and $\mathsf{INR}_2 < \mathsf{SNR}_1$, i.e. receiver 1 observes strong interference and receiver 2 observes weak interference. In any achievable scheme receiver 1 is able to decode the message of its own transmitter. After decoding this message it can subtract it from the received signal to get a cleaner version of the interfering signal. Because of the strong interference condition, receiver 1 observes a stronger version of the message transmitted by transmitter 2 than receiver 2, and can decode this message whenever receiver 2 can do so. Therefore, the message transmitted by transmitter 2 is common information, decodable by both receivers. Therefore, for this mixed interference channel, we choose $\mathsf{INR}_{p2} = 0$ in our simple HK scheme, so that user 2 sends only common information. User 1 still sets INR_{p1} as close to 1 as possible. The following theorem states that this scheme achieves a region within 1 bit/s/Hz of capacity:

Theorem 2: The achievable region

$$\mathscr{R}\big(\min(1, \mathsf{INR}_2), 0\big)$$

is within one bit of the capacity region of the Gaussian interference channel when $\mathsf{INR}_1 \geq \mathsf{SNR}_2$, $\mathsf{INR}_2 < \mathsf{SNR}_1$. ■

An equivalent theorem can be stated for the mixed interference channel where $\mathsf{INR}_2 \geq \mathsf{SNR}_1$ and $\mathsf{INR}_1 < \mathsf{SNR}_2$:

Theorem 3: The achievable region

$$\mathscr{R}\big(0, \min(1, \mathsf{INR}_1)\big)$$

is within one bit of the capacity region of the Gaussian interference channel when $\mathsf{INR}_2 \geq \mathsf{SNR}_1$, $\mathsf{INR}_1 < \mathsf{SNR}_2$. ■
The proof of these theorems requires to derive new bounds. We specify the bounds required to prove Theorem 3. The bounds (1), (2) and (3) apply with no change to this case. The Z-channel bound (4) needs to be changed because the corresponding Z-channel has strong interference. The sum capacity of the strong-interference Z-channel is known [8]:

$$R_1 + R_2 \leq \log(1 + \mathsf{SNR}_2 + \mathsf{INR}_2). \tag{8}$$

The last bound that we require is an upper bound on $2R_1 + R_2$. The bound (6) needs to be changed in the following way: referring to Figure 3 the genie needs to provide additional side information $s_1 = h_{12}x_1 + z_2$ to receiver Rx_1^a. With this change in the genie-aided channel the $2R_1 + R_2$ capacity can be explicitly computed, and the upper bound is given by:

$$\begin{aligned} 2R_1 + R_2 \quad \leq \quad & \log\left(1 + \mathsf{SNR}_1 + \mathsf{INR}_1\right) + \log\left(1 + \frac{\mathsf{SNR}_1}{1 + \mathsf{INR}_2}\right) \\ & + \log\left(1 + \mathsf{INR}_2 + \frac{\mathsf{SNR}_2}{1 + \mathsf{INR}_1}\right). \end{aligned} \tag{9}$$

The upper bounds required to prove Theorem 2 can be derived in a similar way.

Finally, when the channel is in strong interference, i.e. $\mathsf{INR}_2 \geq \mathsf{SNR}_1$ and $\mathsf{INR}_1 \geq \mathsf{SNR}_2$, the capacity region is known from previous results [2], [4].

IV. Generalized Degrees of Freedom

At high SNR, it is well known that the capacity of a point-to-point AWGN link, in bits/s/Hz, is approximately:

$$C_{\text{awgn}} \approx \log \mathsf{SNR}. \tag{10}$$

The approximation is in the sense that for $\mathsf{SNR} > 0$ dB, the approximation error is within 1 bit. Using our results, we can derive analogous approximations of the interference-channel capacity region.

Let $\mathcal{C}(\mathsf{SNR}_1, \mathsf{SNR}_2, \mathsf{INR}_1, \mathsf{INR}_2)$ denote the capacity region of the interference channel with parameters $\mathsf{SNR}_1, \mathsf{SNR}_2, \mathsf{INR}_1, \mathsf{INR}_2$. Let $\tilde{\mathcal{D}}$ be a scaled version of $\mathcal{C}(\mathsf{SNR}_1, \mathsf{SNR}_2, \mathsf{INR}_1, \mathsf{INR}_2)$ given by:

$$\begin{aligned} \tilde{\mathcal{D}}(\mathsf{SNR}_1, \mathsf{SNR}_2, \mathsf{INR}_1, \mathsf{INR}_2) = \big\{ \big(\frac{R_1}{\log \mathsf{SNR}_1}, \frac{R_2}{\log \mathsf{SNR}_2}\big) : \\ (R_1, R_2) \in \mathcal{C}(\mathsf{SNR}_1, \mathsf{SNR}_2, \mathsf{INR}_1, \mathsf{INR}_2) \big\} \end{aligned}$$

and let

$$\alpha_1 = \frac{\log \mathsf{SNR}_2}{\log \mathsf{SNR}_1},\ \alpha_2 = \frac{\log \mathsf{INR}_1}{\log \mathsf{SNR}_1},\ \alpha_3 = \frac{\log \mathsf{INR}_2}{\log \mathsf{SNR}_1}.$$

We define the generalized degrees of freedom region as:

$$\mathcal{D}(\alpha_1, \alpha_2, \alpha_3) = \lim_{\substack{\mathsf{SNR}_1, \mathsf{SNR}_2, \mathsf{INR}_1, \mathsf{INR}_2 \to \infty \\ \alpha_1, \alpha_2, \alpha_3 \text{ fixed}}} \tilde{\mathcal{D}}(\mathsf{SNR}_1, \mathsf{SNR}_2, \mathsf{INR}_1, \mathsf{INR}_2)$$

With this definition, the capacity region can be approximately expressed as the set of rate pairs (R_1, R_2) such that:

$$R_1 = d_1 \log \mathsf{SNR}_1 \ , \ R_2 = d_2 \log \mathsf{SNR}_2$$

for $(d_1, d_2) \in \mathcal{D}$.

The generalized degrees of freedom d_1, d_2 give a sense of how interference affects communication. In the absence of interference, each user can achieve a rate $R_i \approx \log \mathsf{SNR}_i$.

Due to interference, the single user capacity is scaled by a factor d_i.

Using the bounds (1)-(7) for the interference channel with weak interference we can compute $\mathcal{D}$ explicitly. $\mathcal{D}(\alpha_1, \alpha_2, \alpha_3)$ is given by the set of generalized degrees of freedom pairs (d_1, d_2) that satisfy:

$$\begin{aligned} d_1 \leq 1\ ,\ d_2 &\leq 1 \\ d_1 + \alpha_1 d_2 &\leq \alpha_1 + \max\{1-\alpha_2, 0\} \\ d_1 + \alpha_1 d_2 &\leq 1 + \max\{\alpha_1 - \alpha_3, 0\} \\ d_1 + \alpha_1 d_2 &\leq \max(\alpha_2, 1-\alpha_3) + \max(\alpha_3, \alpha_1 - \alpha_2) \\ 2d_1 + \alpha_1 d_2 &\leq \max(1, \alpha_2) + \max(\alpha_3, \alpha_1 - \alpha_2) + 1 - \alpha_3 \\ d_1 + 2\alpha_1 d_2 &\leq \max(\alpha_1, \alpha_3) + \max(\alpha_2, 1-\alpha_3) + \alpha_1 - \alpha_2. \end{aligned} \tag{11}$$

A similar characterization of $\mathcal{D}$ can be made for the interference channel with mixed interference, i.e. one link with strong interference and the other link with weak interference.

In [1] we analyzed how the symmetric generalized degrees of freedom $d_{\text{sym}} = C_{\text{sym}}/\log \mathsf{SNR}$ varies with the interference level $\alpha = \log \mathsf{INR}/\log \mathsf{SNR}$. Having derived the generalized degrees of freedom region $\mathcal{D}$, we can compute it for the symmetric channel and analyze how it varies for the different interference regimes. For a symmetric channel, the generalized degrees of freedom region $\mathcal{D}$ can be obtained from (11) by setting $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \alpha$, $0 < \alpha < 1$.

Figure 4 shows how the symmetric generalized degrees of freedom vary as a function of the interference level α. The performance of the schemes that treat interference as noise, and that orthogonalize the users is also plotted as a reference. We note that there are 5 different regimes of operation depending on whether $0 \leq \alpha < 1/2$, $1/2 \leq \alpha < 2/3$, $2/3 \leq \alpha < 1$ (weak interference regimes); or $1 \leq \alpha < 2$, $\alpha \geq 2$ (strong interference regimes). The symmetric generalized degrees of freedom can be obtained from the generalized degrees of freedom region by maximizing $d = d_1 = d_2$ for $(d_1, d_2) \in \mathcal{D}$.

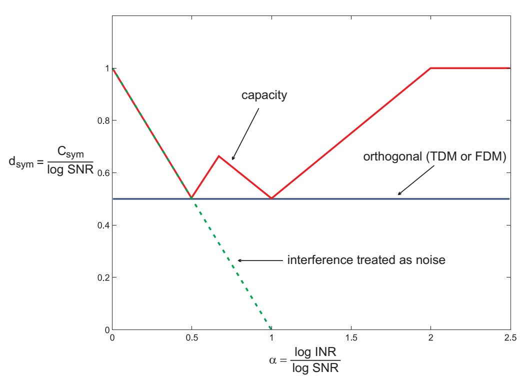

Fig. 4. Symmetric generalized degrees of freedom as a function of the interference level α.

The generalized degrees of freedom region gives a more complete picture of how interference affects communication at the different interference levels. Figures 5 and 6 show how $\mathcal{D}$ varies for the different interference levels α.

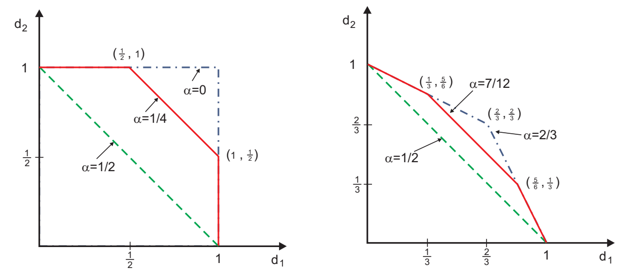

Fig. 5. Generalized degrees of freedom region for a symmetric channel for $0 \leq \alpha \leq 1/2$ (left) and $1/2 \leq \alpha \leq 2/3$ (right).

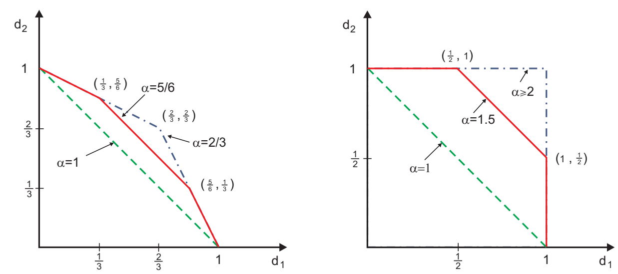

Fig. 6. Generalized degrees of freedom region for a symmetric channel for $2/3 \leq \alpha \leq 1$ (left) and $\alpha \geq 1$ (right).

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