ITW2003, Paris, France, March 31 – April 4, 2003

Factor Graphs and Dynamical Electrical Networks[1]

Pascal O. Vontobel[2]
Coordinated Science Lab
University of Illinois
1308 West Main Street
Urbana, IL 61801, USA
vontobel@ifp.uiuc.edu

Hans-Andrea Loeliger
Signal and Information Processing Lab
ETH Zurich
Sternwartstrasse 7
8092 Zurich, Switzerland
loeliger@isi.ee.ethz.ch

Abstract — **Factor graphs are graphical models with origins in coding theory. The sum-product and the max-product algorithms, which operate by message passing on a factor graph, subsume a great variety of algorithms in coding, signal processing, and artificial intelligence. This paper aims at extending the field of possible applications to dynamical electrical networks. Interestingly, the resulting factor graphs have a structure very much akin to a Kalman filter.**

Keywords. Factor graphs, dynamical electrical networks, max-product algorithm, sum-product algorithm, Kalman filter.

I. Introduction

In [7, 8, 9, 6] factor graphs were used to express a one-to-one correspondence between a class of electrical networks and multivariable probability distributions; these factor graphs may also be viewed as variational representations of electrical networks. All these findings were for *static* electrical networks[3] and were essentially based on a result by Dennis [2] who formulated optimization problems whose solution correspond either to the voltages or currents of an electrical network. The obvious question that initiated the present work was if it is possible to associate a factor graph to a *dynamical* electrical network.[4]

This paper answers the above question in the affirmative by trying to show a reasonable way to solve the problem. In the following, we will not use factor graphs as defined in [4], but Forney-style factor graphs (FFGs) which were introduced in [3] (there called normal graphs). For an introduction to FFGs, see e.g. [3, 5, 7, 9].

II. FFGs Corresponding to a Dynamical Electrical Network

For static electrical networks it is fairly obvious that variables in the FFG should represent static voltages and/or static currents. What the variables should represent in the case of dynamical electrical networks is not entirely clear. In the following we try to reveal – as far as the available space allows – what considerations lead to the choices we have made.

Among the possibilities, what the variables of an FFG should be, are the following.

- **(Sampled quantities)** The variables correspond to sampled versions of the continuous voltages and/or currents.
- **(Discrete-time system approximation)** We consider a discrete-time system that approximates the continuous-time system in the sense that it is equivalent to the continuous-time system in the limit $\Delta T \to 0$. In this case, the variables correspond to the voltages and/or currents at each discrete-time step of the discrete-time system.

Concerning the relationship between the electrical network and the FFG we desire the following.

- **(One-to-one relationship)** Each electrical network component should correspond one-to-one to a part of the factor graph.
- **(Direct-form realization)** The parameters of the components of the electrical network should directly appear in the function nodes of the FFG.
- **(No mixing)** Combining the above two requirements we do not allow that parameters of different components are mixed together to get a function node of the FFG.

We think that although an FFG representing sampled quantities would be a desirable possibility, such a solution seems to conflict with the second list above. But it is feasible to get an FFG to a discrete-time system approximation that is equivalent to the continuous-time system in the limit $\Delta T \to 0$. Observe that the explicit formulas for voltages and/or currents over time for the continuous-time system can be quite complicated and can usually involve a mixing of all the parameters of the components of the electrical network: in our solution, the FFG will show the structure of the electrical network and the time axis, whereas the max-product algorithm will lead to the mixing.

Finally, note that with the above constraints it would be relatively easy to give an FFG with voltages and currents whose global function is an indicator function, i.e. whose global function is nonzero when all voltages and currents correspond to a valid solution and zero otherwise (see also the discussion for the static case in [9, 6]). But our aim is an FFG whose global function is a variational representation and where moreover possibly not all variables appear: in the case of static electrical networks, the voltage-mode FFG is a variational representation of only the voltages in an electrical network, whereas the

---

[1] This is an extended version of what appeared in the proceedings of the IEEE Information Theory Workshop 2003 in Paris. It additionally contains a figure and the proofs to the theorems and lemmas.

[2] Supported by NSF Grant CCR 99-84515.

[3] By a *static* electrical networks we mean an electrical network consisting of one-port elements where in all component laws the current through an element is a function only of the present voltage across the element.

[4] By a *dynamical* electrical network we mean an electrical network that contains besides static components also capacitors and inductors.

current-mode FFG is a variational representation of only the currents in an electrical network [9, 6].

III. The Main Idea

Dennis [2] showed that the voltages (or currents) of a linear[5] electrical network solve a constrained quadratic optimization problem, i.e. they minimize a cost function $f(\mathbf{x})$ over $\mathbf{x}$ under some constraints on $\mathbf{x}$. The corresponding FFG then represents $\exp(-f(\mathbf{x}))$ and the constraints on $\mathbf{x}$ [7, 8, 9, 6]: this global function can also be seen as a scaled probability density function. The question is whether we are able to find a suitable optimization problem in the case of dynamical electrical networks.

Note that given a system of equations it is usually not too difficult to find a function $f(\mathbf{x})$ where the zero-gradient points correspond to the solution of the original problem. But such an $f(\mathbf{x})$ can unfortunately not be used for an FFG representing $\exp(-f(\mathbf{x}))$: the zero-gradient points of $f(\mathbf{x})$ are in general not minima, and so do not correspond to the points where $\exp(-f(\mathbf{x}))$ has a maximum.

To get an extremum principle which fulfills the desiderata given in Sec. II, we modify an approach that was used by Collins [1] to formulate an extremum principle whose solution is the solution of the heat equation. His approach essentially boils down to the following.

Let $\mathbf{F}$ and $\mathbf{G}$ be two square real matrices, where $\mathbf{F}$ is symmetric (i.e. $\mathbf{F} = \mathbf{F}^\mathsf{T}$) and positive definite and where $\mathbf{G}$ is skew-symmetric (i.e. $\mathbf{G} = -\mathbf{G}^\mathsf{T}$).[6] Assume that we are interested in the solution $\mathbf{x}$ of

$$(\mathbf{F}+\mathbf{G})\mathbf{x} = \mathbf{x}_0 \quad \Longleftrightarrow \quad \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{x} = \mathbf{x}_0, \tag{1}$$

where $\mathbf{x}_0$ is some given real vector. The paper [1] offers the following approach to get an extremum principle whose solution is equivalent to the solution of (1). We first define the adjoint problem (whose solution for a given $\mathbf{x}'_0$ is $\mathbf{x}'$)

$$(\mathbf{F}+\mathbf{G})^\mathsf{T}\mathbf{x}' = \mathbf{x}'_0 \quad \Longleftrightarrow \quad \mathbf{F}\mathbf{x}' - \mathbf{G}\mathbf{x}' = \mathbf{x}'_0. \tag{2}$$

We now define the transformations

$$\begin{cases} \mathbf{x}^\mathrm{a} \triangleq \mathbf{x} + \mathbf{x}' \\ \mathbf{x}^\mathrm{b} \triangleq \mathbf{x} - \mathbf{x}' \end{cases} \qquad \begin{cases} \mathbf{x} = \frac{1}{2}(\mathbf{x}^\mathrm{a} + \mathbf{x}^\mathrm{b}) \\ \mathbf{x}' = \frac{1}{2}(\mathbf{x}^\mathrm{a} - \mathbf{x}^\mathrm{b}) \end{cases} \tag{3}$$

and similarly for $\mathbf{x}_0^\mathrm{a}$ and $\mathbf{x}_0^\mathrm{b}$. Adding (2) to (1) gives the first equation in the left-hand side of (4), whereas subtracting (2) from (1) gives the second equation in the left-hand side of (4).

$$\begin{cases} \mathbf{F}\mathbf{x}^\mathrm{a} + \mathbf{G}\mathbf{x}^\mathrm{b} = \mathbf{x}_0^\mathrm{a} \\ \mathbf{F}\mathbf{x}^\mathrm{b} + \mathbf{G}\mathbf{x}^\mathrm{a} = \mathbf{x}_0^\mathrm{b} \end{cases} \Longleftrightarrow \begin{pmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{G} & \mathbf{F} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}^\mathrm{a} \\ \mathbf{x}^\mathrm{b} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_0^\mathrm{a} \\ \mathbf{x}_0^\mathrm{b} \end{pmatrix}. \tag{4}$$

In the following lemma we do not formulate an extremum principle for $\mathbf{x}$ and $\mathbf{x}'$ directly, but for $\mathbf{x}^\mathrm{a}$ and $\mathbf{x}^\mathrm{b}$: through the transformation (3) the desired $\mathbf{x}$ (and $\mathbf{x}'$) can then be found.

Lemma 1 *The minimum of*

$$\frac{1}{2}{\mathbf{x}^\mathrm{a}}^\mathsf{T}\mathbf{F}\mathbf{x}^\mathrm{a} + \frac{1}{2}{\mathbf{x}^\mathrm{b}}^\mathsf{T}\mathbf{F}\mathbf{x}^\mathrm{b} - {\mathbf{x}_0^\mathrm{a}}^\mathsf{T}\mathbf{x}^\mathrm{a} \tag{5}$$

[5] This can be generalized to larger classes of electrical networks [2, 7, 9].

[6] If $\mathbf{A}$ is a matrix, $\mathbf{A}^\mathsf{T}$ is its transpose.

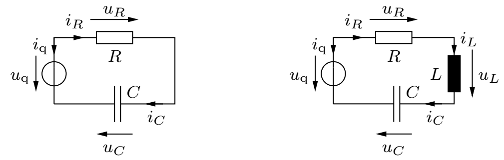

Figure 1: Left: series-RC electrical network. Right: series-RLC electrical network.

over the pair $(\mathbf{x}^\mathrm{a}, \mathbf{x}^\mathrm{b})$ *under the constraint* $\mathbf{F}\mathbf{x}^\mathrm{b} + \mathbf{G}\mathbf{x}^\mathrm{a} = \mathbf{x}_0^\mathrm{b}$. *is attained by the pair* $(\mathbf{x}^\mathrm{a}, \mathbf{x}^\mathrm{b})$ *which solves (4).*

Proof: See App. A □

IV. An Example: Resistor and Capacitor in Series

In order to show how one can obatin an FFG for a dynamical electrical network, we consider a specific example from where the general case can be derived. Therefore, let us consider a series-RC electrical network as shown in Fig. 1(left) with an ideal voltage source u_q in series with a resistor R and a capacitor C. The voltage and currents will be called u_q, u_R, u_C, i_q, i_R, and i_C, respectively. (Note that we choose the arrows of R and C in the same direction, but opposite to the direction of the arrow of the source.) Note that although the derivation involves quite a bit of formulas, the final FFG is relatively simple and can essentially be derived directly from the electrical network once the underlying principle is recognized.

The dynamical electrical network in Fig. 1(left) satisfies the following equations

$$\begin{cases} u_R(t) + u_C(t) - u_\mathrm{q}(t) & = 0 & \text{(KVL)} \\ i_R(t) + i_\mathrm{q}(t) & = 0 & \text{(KCL)} \\ i_C(t) + i_\mathrm{q}(t) & = 0 & \text{(KCL)} \\ u_R(t) - R \cdot i_R(t) & = 0 & \text{(CL)} \\ i_C(t) - C \cdot \frac{\mathrm{d}}{\mathrm{d}t} u_C(t) & = 0 & \text{(CL)} \end{cases} \tag{6}$$

for all times t, where KVL, KCL, and CL stand for Kirchhoff voltage law, Kirchhoff current law, and component law, respectively. The value of the ideal voltage source can vary with time an can be specified arbitrarily. To keep the derivation simple, we do not show the derivation with initial conditions (see also the remark on page 4). A discrete-time version with time step ΔT can be written as

$$\begin{cases} u_R(k) + u_C(k) - u_\mathrm{q}(k) & = 0 & \text{(KVL)} \\ i_R(k) + i_\mathrm{q}(k) & = 0 & \text{(KCL)} \\ i_C(k) + i_\mathrm{q}(k) & = 0 & \text{(KCL)} \\ u_R(k) - R \cdot i_R(k) & = 0 & \text{(CL)} \\ i_C(k) - C \cdot \frac{u_C(k+1) - u_C(k-1)}{2\Delta T} & = 0 & \text{(CL)} \end{cases} \tag{7}$$

for all k, where $u_R(k)$ stands for the voltage across R at step k, etc. In the end, we will be interested in the limit $\Delta T \to 0$ where the discrete-time system is essentially equivalent to the continuous-time system. Inspired by the results in Sec. III we now define the adjoint system (which is essentially the timereversed version of the system in (7)) by

$$
\left\{
\begin{array}{rll}
u'_R(k) + u'_C(k) - u'_q(k) & = 0 & \text{(KVL)} \\
i'_R(k) + i'_q(k) & = 0 & \text{(KCL)} \\
i'_C(k) + i'_q(k) & = 0 & \text{(KCL)} \\
u'_R(k) - R \cdot i'_R(k) & = 0 & \text{(CL)} \\
i'_C(k) + C \cdot \frac{u'_C(k+1) - u'_C(k-1)}{2\Delta T} & = 0 & \text{(CL)}
\end{array}
\right. \tag{8}
$$

We use the transformations

$$
\begin{cases}
u^{\mathrm{a}}_R(k) \triangleq u_R(k) + u'_R(k) \\
u^{\mathrm{b}}_R(k) \triangleq u_R(k) - u'_R(k)
\end{cases}
\qquad
\begin{cases}
u_R(k) = \frac{1}{2}\left(u^{\mathrm{a}}_R(k) + u^{\mathrm{b}}_R(k)\right) \\
u'_R(k) = \frac{1}{2}\left(u^{\mathrm{a}}_R(k) - u^{\mathrm{b}}_R(k)\right)
\end{cases}
$$

for all k, and introduce similar definitions for $i^{\mathrm{a}}_R(k)$, $i^{\mathrm{b}}_R(k)$, $u^{\mathrm{a}}_C(k)$, $u^{\mathrm{b}}_C(k)$, $i^{\mathrm{a}}_C(k)$, $i^{\mathrm{b}}_C(k)$, $u^{\mathrm{a}}_{\mathrm{q}}(k)$, $u^{\mathrm{b}}_{\mathrm{q}}(k)$, $i^{\mathrm{a}}_{\mathrm{q}}(k)$, and $i^{\mathrm{b}}_{\mathrm{q}}(k)$, respectively. Adding (8) to (7) we obtain

$$
\left\{
\begin{array}{rl}
u^{\mathrm{a}}_R(k) + u^{\mathrm{a}}_C(k) - u^{\mathrm{a}}_{\mathrm{q}}(k) & = 0 \\
i^{\mathrm{a}}_R(k) + i^{\mathrm{a}}_{\mathrm{q}}(k) & = 0 \\
i^{\mathrm{a}}_C(k) + i^{\mathrm{a}}_{\mathrm{q}}(k) & = 0 \\
u^{\mathrm{a}}_R(k) - R \cdot i^{\mathrm{a}}_R(k) & = 0 \\
i^{\mathrm{a}}_C(k) - C \cdot \frac{u^{\mathrm{b}}_C(k+1) - u^{\mathrm{b}}_C(k-1)}{2\Delta T} & = 0
\end{array}
\right. \tag{9}
$$

for all k. Subtracting (8) from (7) we obtain

$$
\left\{
\begin{array}{rl}
u^{\mathrm{b}}_R(k) + u^{\mathrm{b}}_C(k) - u^{\mathrm{b}}_{\mathrm{q}}(k) & = 0 \\
i^{\mathrm{b}}_R(k) + i^{\mathrm{b}}_{\mathrm{q}}(k) & = 0 \\
i^{\mathrm{b}}_C(k) + i^{\mathrm{b}}_{\mathrm{q}}(k) & = 0 \\
u^{\mathrm{b}}_R(k) - R \cdot i^{\mathrm{b}}_R(k) & = 0 \\
i^{\mathrm{b}}_C(k) - C \cdot \frac{u^{\mathrm{a}}_C(k+1) - u^{\mathrm{a}}_C(k-1)}{2\Delta T} & = 0
\end{array}
\right. \tag{10}
$$

for all k. We are now in a position to formulate the following theorem.

Theorem 2 *The minimum of*

$$
\sum_k \left[\frac{(u^{\mathrm{a}}_R(k))^2}{2R} + \frac{(i^{\mathrm{b}}_R(k))^2}{2/R} + u^{\mathrm{b}}_{\mathrm{q}}(k) \cdot i^{\mathrm{b}}_{\mathrm{q}}(k) \right] \tag{11}
$$

over all voltages (except voltages of ideal voltage sources) of system "a" *and all currents (except currents of ideal current sources) of system* "b" *under the constraints*

$$
\left\{
\begin{array}{rl}
u^{\mathrm{a}}_R(k) + u^{\mathrm{a}}_C(k) - u^{\mathrm{a}}_{\mathrm{q}}(k) & = 0, \\
i^{\mathrm{b}}_R(k) + i^{\mathrm{b}}_{\mathrm{q}}(k) & = 0, \\
i^{\mathrm{b}}_C(k) + i^{\mathrm{b}}_{\mathrm{q}}(k) & = 0, \\
i^{\mathrm{b}}_C(k) - C \cdot \frac{u^{\mathrm{a}}_C(k+1) - u^{\mathrm{a}}_C(k-1)}{2\Delta T} & = 0,
\end{array}
\right. \tag{12}
$$

for all k, is achieved by the solution that is part of the solution vector to the equations in (9) and (10).

Proof: See App. B □

The minimization problem in Th. 2 can be turned into a factor graph as shown in Fig. 2, where the ideal voltage sources have been set to have the values $u^{\mathrm{a}}_{\mathrm{q}0}(k)$ and $u^{\mathrm{b}}_{\mathrm{q}0}(k)$ (derived from $u_{\mathrm{q}0}(k)$ and $u'_{\mathrm{q}0}(k)$), respectively.[7] By omitting the edge

[7] See e.g. [5, 7, 9] for the conventions used to draw the function nodes of the FFG. For clarity purposes we only show half the FFG: there is a second part with voltages of system "a" at odd times and currents of system "b" at even times.

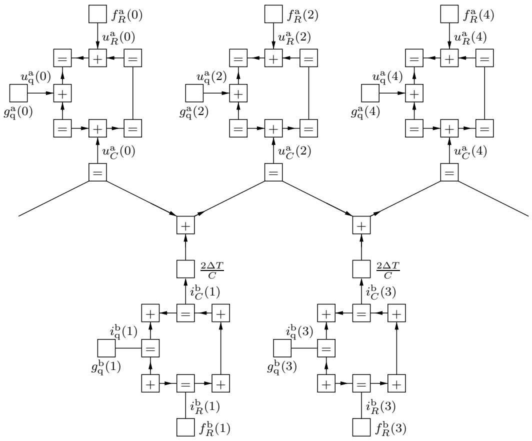

Figure 2: FFG for series-RC electrical network where the abbreviations $f^{\mathrm{a}}_R(k) \triangleq \exp(-u^{\mathrm{a}}_R(k)^2/2R)$, $f^{\mathrm{b}}_R(k) \triangleq \exp(-i^{\mathrm{b}}_R(k)^2/(2/R))$, $g^{\mathrm{a}}_{\mathrm{q}}(k) \triangleq \delta(u^{\mathrm{a}}_{\mathrm{q}}(k) - u^{\mathrm{a}}_{\mathrm{q}0}(k))$, and $g^{\mathrm{b}}_{\mathrm{q}}(k) \triangleq \exp(-u^{\mathrm{b}}_{\mathrm{q}0}(k) i^{\mathrm{b}}_{\mathrm{q}}(k))$ have been used.

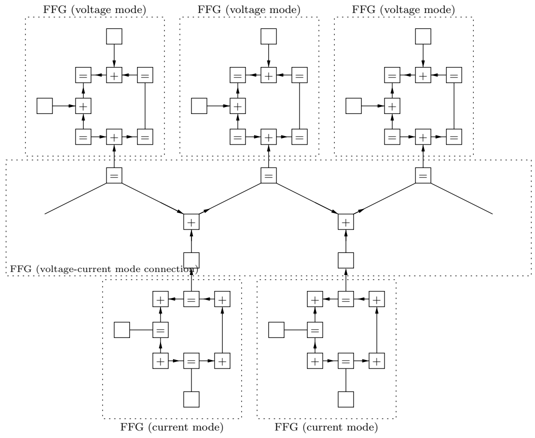

Figure 3: Highlighted structure of FFG for series-RC electrical network.

and function labels, Fig. 3 tries to highlight the structure of the FFG in Fig. 2 by putting edges and function nodes that belong together into boxes (note the topological similarity between the electrical network in Fig. 1 and the voltage/current mode FFGs, respectively [9, 6].) To show another example for the above procedure, we consider the case of the series-RLC network as given in Fig. 1(right): Fig. 4 shows the resulting FFG, whereas Fig. 5 highlights its structure. We have the following comments and observations.

- In the case of static electrical networks one can derive two different FFGs [9, 6]. In the voltage-mode FFG only voltages appear: the voltages that maximize the global function are equivalent to the voltages that appear in the solution of the electrical network. In the current-mode FFG only currents appear: the currents that maximize the global function are equivalent to the currents that appear in the solution of the electrical network. As noted

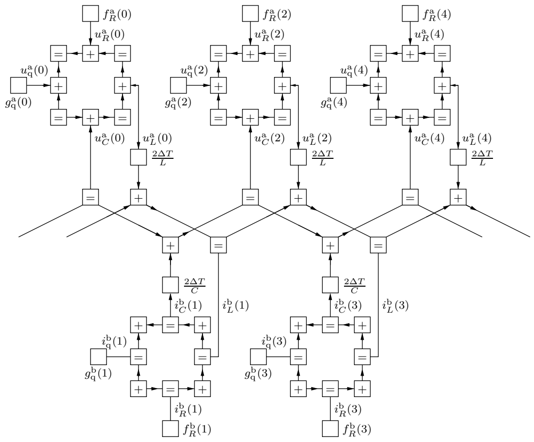

Figure 4: FFG for series-RLC electrical network. (The same abbreviations have been used as in Fig. 2.)

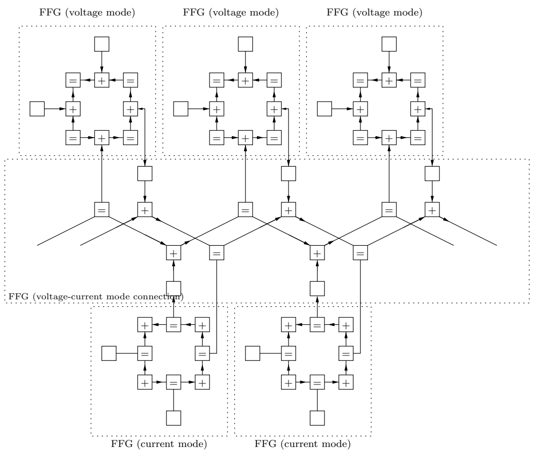

Figure 5: Highlighted structure of FFG for series-RLC electrical network.

in [9], the voltage-mode FFG and the current-mode FFG are related by a sort of Lagrange duality. For dynamical electrical networks we need now both voltage-mode and current mode FFG: from system "a" we take the voltages and from system "b" we take the currents. (In that respect, Lemma 1 is slightly different, as there all variables appear in the optimization problem.)

- To keep the derivation simple we have omitted initial conditions. Usually, $u_C(0)$ is given by an initial value. For the the adjoint system we can impose some terminal condition; besides that, we can choose the value of its sources. There is a very simple and practical choice: set all sources and all terminal conditions in the adjoint system to zero, then all voltages and currents in the adjoint system are zero for all times. The voltages and currents we are interested in are then simply the corresponding sums of the voltages and currents in systems "a" and "b" divided by two, which in this case reduce to $u_R(k) = u_R^{\mathrm{a}}(k) = u_R^{\mathrm{b}}(k)$, etc.

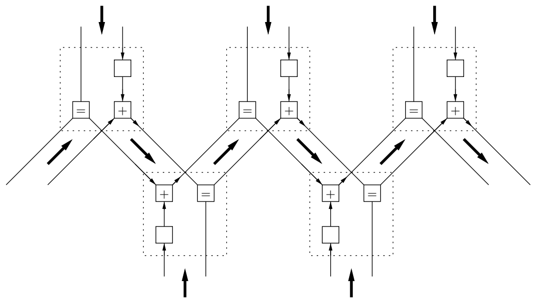

Figure 6: Message passing (in the forward direction) in the voltage-current mode connection FFG for the series-RLC electrical network. (We implicitely join pairs of edges to one edge representing a vector variable in order to obtain an FFG that is a tree.)

- The FFG in Fig. 2 can be modified so that it is a tree. To find the configuration that gives the valid configuration which maximizes the global function, we can then apply the max-product algorithm. In the case of linear electrical networks, the global function is a scaled jointly Gaussian density and the max-product algorithm is equivalent to the sum-product algorithm (see e.g. [5]).
- Interestingly, the voltage-current mode connection FFG in Fig. 3 is akin to the main part of the FFG of a Kalman filter (see e.g. [5]). In [5] it is shown how the max-product (sum-product) update rules can be derived in the jointly Gaussian case for the blocks that appear in Fig. 3. In the case of the series-RLC network, the voltage-current mode connection FFG in Fig. 5 is an intertwined version of the main part of two Kalman filter FFGs, see Fig. 6 (How the update rules for the messages in that case can be derived relatively easily, will be be treated somewhere else.)
- The expression (11) in Th. 2 can be multiplied by any constant without changing the solution. Multiplying (11) by ΔT leads to an FFG whose global function has some additional desirable properties for $\Delta T \ll 1$.

V. Conclusions

We have shown a possible way of associating to a dynamical electrical network an FFG where the voltages and currents over time can be obtained by the help of the max-product (sum-product) algorithm. Whereas the FFG represents a global function where the "unmixed" parameters of the electrical network appear, the max-product (sum-product) algorithm does the correct "mixing" to obtain the solutions over time.

A. Proof to Lemma 1

Setting the gradient of the Lagrangian

$$\mathcal{L} \triangleq \frac{1}{2}{\mathbf{x}^{\mathrm{a}}}^{\mathsf{T}}\mathbf{F}\mathbf{x}^{\mathrm{a}} + \frac{1}{2}{\mathbf{x}^{\mathrm{b}}}^{\mathsf{T}}\mathbf{F}\mathbf{x}^{\mathrm{b}} - {\mathbf{x}_0^{\mathrm{a}}}^{\mathsf{T}}\mathbf{x}^{\mathrm{a}} \tag{13}$$

$$- \boldsymbol{\lambda}^{\mathsf{T}}\left[\mathbf{F}\mathbf{x}^{\mathrm{b}} + \mathbf{G}\mathbf{x}^{\mathrm{a}} - \mathbf{x}_0^{\mathrm{b}}\right] \tag{14}$$

equal to zero we obtain

$$\mathbf{0} \overset{!}{=} \frac{\partial}{\partial \mathbf{x}^{\mathrm{a}}}\mathcal{L} = \mathbf{F}\mathbf{x}^{\mathrm{a}} - \mathbf{x}_0^{\mathrm{a}} - \mathbf{G}^{\mathsf{T}}\boldsymbol{\lambda} = \mathbf{F}\mathbf{x}^{\mathrm{a}} - \mathbf{x}_0^{\mathrm{a}} + \mathbf{G}\boldsymbol{\lambda}, \quad (15)$$

$$\mathbf{0} \overset{!}{=} \frac{\partial}{\partial \mathbf{x}^{\mathrm{b}}}\mathcal{L} = \mathbf{F}\mathbf{x}^{\mathrm{b}} - \mathbf{F}^{\mathsf{T}}\boldsymbol{\lambda} = \mathbf{F}\mathbf{x}^{\mathrm{b}} - \mathbf{F}\boldsymbol{\lambda} = \mathbf{F}(\mathbf{x}^{\mathrm{b}} - \boldsymbol{\lambda}), \quad (16)$$

$$\mathbf{0} \overset{!}{=} \frac{\partial}{\partial \boldsymbol{\lambda}}\mathcal{L} = -(\mathbf{F}\mathbf{x}^{\mathrm{b}} + \mathbf{G}\mathbf{x}^{\mathrm{a}} - \mathbf{x}_0^{\mathrm{b}}), \quad (17)$$

were we used $\mathbf{F}^{\mathsf{T}} = \mathbf{F}$ and $\mathbf{G}^{\mathsf{T}} = -\mathbf{G}$. The matrix $\mathbf{F}$ is invertible because it is positive definite: from (16) follows then $\boldsymbol{\lambda} = \mathbf{x}^{\mathrm{b}}$. Inserting this solution into (15) we obtain $\mathbf{0} = \mathbf{F}\mathbf{x}^{\mathrm{a}} - \mathbf{x}_0^{\mathrm{a}} + \mathbf{G}\mathbf{x}^{\mathrm{b}}$. This last equation together with the equation in (17) are the desired equations in (4).

B. Proof to Theorem 2

Setting the gradient of the Lagrangian

$$\begin{aligned}\mathcal{L} = \sum_k \Bigg[& \frac{(u_R^{\mathrm{a}}(k))^2}{2R} + \frac{(i_R^{\mathrm{b}}(k))^2}{2/R} + u_{\mathrm{q}}^{\mathrm{b}}(k) \cdot i_{\mathrm{q}}^{\mathrm{b}}(k) && (18)\\ & - \lambda_1(k) \cdot (u_R^{\mathrm{a}}(k) + u_C^{\mathrm{a}}(k) - u_{\mathrm{q}}^{\mathrm{a}}(k)) \\ & - \lambda_2(k) \cdot \left(i_R^{\mathrm{b}}(k) + i_{\mathrm{q}}^{\mathrm{b}}(k)\right) - \lambda_3(k) \cdot \left(i_C^{\mathrm{b}}(k) + i_{\mathrm{q}}^{\mathrm{b}}(k)\right) \\ & - \lambda_4(k) \cdot \left(i_C^{\mathrm{b}}(k) - C \cdot \frac{u_C^{\mathrm{a}}(k+1) - u_C^{\mathrm{a}}(k-1)}{2\Delta T}\right) \Bigg] && (19)\end{aligned}$$

equal to zero we obtain

$$\left\{\begin{array}{lll} 0 & \overset{!}{=} \frac{\partial \mathcal{L}}{\partial u_R^{\mathrm{a}}(k)} & = \frac{1}{R} \cdot u_R^{\mathrm{a}}(k) - \lambda_1(k), \\ 0 & \overset{!}{=} \frac{\partial \mathcal{L}}{\partial i_R^{\mathrm{b}}(k)} & = R \cdot i_R^{\mathrm{b}} - \lambda_2(k), \\ 0 & \overset{!}{=} \frac{\partial \mathcal{L}}{\partial u_C^{\mathrm{a}}(k)} & = -\lambda_1(k) - \frac{C}{2\Delta T} \cdot (\lambda_4(k+1) - \lambda_4(k-1)), \\ 0 & \overset{!}{=} \frac{\partial \mathcal{L}}{\partial i_C^{\mathrm{b}}(k)} & = -\lambda_3(k) - \lambda_4(k), \\ 0 & \overset{!}{=} \frac{\partial \mathcal{L}}{\partial i_{\mathrm{q}}^{\mathrm{b}}(k)} & = u_{\mathrm{q}}^{\mathrm{b}}(k) - \lambda_2(k) - \lambda_3(k), \\ 0 & \overset{!}{=} \frac{\partial \mathcal{L}}{\partial \lambda_1(k)} & = -(u_R^{\mathrm{a}}(k) + u_C^{\mathrm{a}}(k) - u_{\mathrm{q}}^{\mathrm{a}}(k)), \\ 0 & \overset{!}{=} \frac{\partial \mathcal{L}}{\partial \lambda_2(k)} & = -(i_R^{\mathrm{b}}(k) + i_{\mathrm{q}}^{\mathrm{b}}(k)), \\ 0 & \overset{!}{=} \frac{\partial \mathcal{L}}{\partial \lambda_3(k)} & = -(i_C^{\mathrm{b}}(k) + i_{\mathrm{q}}^{\mathrm{b}}(k)), \\ 0 & \overset{!}{=} \frac{\partial \mathcal{L}}{\partial \lambda_4(k)} & = -\left(i_C^{\mathrm{b}}(k) - \frac{C}{2\Delta T}(u_C^{\mathrm{a}}(k+1) - u_C^{\mathrm{a}}(k-1))\right). \end{array}\right. \quad (20)$$

Let us define $\lambda_1(k) \triangleq i_R^{\mathrm{a}}(k)$, $\lambda_2(k) \triangleq u_R^{\mathrm{b}}(k)$, $\lambda_3(k) \triangleq u_C^{\mathrm{b}}(k)$, $\lambda_4(k) \triangleq -u_C^{\mathrm{b}}(k)$, $i_C^{\mathrm{a}}(k) \triangleq i_R^{\mathrm{a}}(k)$. Substituting these definitions and rearranging we obtain

$$\left\{\begin{array}{rl} u_R^{\mathrm{a}}(k) + u_C^{\mathrm{a}}(k) - u_{\mathrm{q}}^{\mathrm{a}}(k) & = 0, \\ u_R^{\mathrm{a}}(k) - R \cdot i_R^{\mathrm{a}}(k) & = 0, \\ i_R^{\mathrm{a}}(k) - i_C^{\mathrm{a}}(k) & = 0, \end{array}\right. \quad (21)$$

and

$$\left\{\begin{array}{rl} u_R^{\mathrm{b}}(k) - u_C^{\mathrm{b}}(k) - u_{\mathrm{q}}^{\mathrm{b}}(k) & = 0, \\ u_R^{\mathrm{b}}(k) - R \cdot i_R^{\mathrm{b}} & = 0, \\ i_R^{\mathrm{b}}(k) + i_{\mathrm{q}}^{\mathrm{b}}(k) & = 0, \\ i_C^{\mathrm{b}}(k) + i_{\mathrm{q}}^{\mathrm{b}}(k) & = 0, \end{array}\right. \quad (22)$$

and

$$\left\{\begin{array}{rl} i_C^{\mathrm{a}}(k) - \frac{C}{2\Delta T} \cdot (u_C^{\mathrm{b}}(k+1) - u_C^{\mathrm{b}}(k-1)) & = 0, \\ i_C^{\mathrm{b}}(k) - \frac{C}{2\Delta T}(u_C^{\mathrm{a}}(k+1) - u_C^{\mathrm{a}}(k-1)) & = 0. \end{array}\right. \quad (23)$$

(We omitted the $0 = -\lambda_3(k) - \lambda_4(k)$ equation which turns into $0 = 0$.) So these equations are equivalent to the equations in (9) and (10). Note that the the equations in (21) describe the system "a", the equations in (22) describe the system "b", and finally the equations in (23) give the interconnection between system "a" and "b".

References

[1] W. D. Collins. Dual extremum principles for the heat equation. *Proc. Roy. Soc. Edinburgh Sect. A*, 77(3-4):273–292, 1977.

[2] J. B. Dennis. *Mathematical Programming and Electrical Networks*. The Technology Press of The Massachusetts Institute of Technology, Cambridge, Mass., 1959.

[3] G. D. Forney, Jr. Codes on graphs: normal realizations. *IEEE Trans. on Inform. Theory*, 47(2):520–548, 2001.

[4] F. R. Kschischang, B. J. Frey, and H.-A. Loeliger. Factor graphs and the sum-product algorithm. *IEEE Trans. on Inform. Theory*, IT–47(2):498–519, 2001.

[5] H.-A. Loeliger. Least squares and Kalman filtering on Forney graphs. In R. E. Blahut and R. Koetter, editors, *Codes, Graphs, and Systems*. Kluwer, 2002. Festschrift in honor of G. D. Forney, Jr.

[6] P. O. Vontobel. Factor graphs, electrical networks, and entropy. In *Proc. 40th Allerton Conf. on Communications, Control, and Computing*, Allerton House, Monticello, Illinois, October 2–4 2002. Available online under `http://www.isi.ee.ethz.ch/publications`.

[7] P. O. Vontobel. *Kalman Filters, Factor Graphs, and Electrical Networks*. Post-Diploma Project, ETH Zurich, 2002. Available online under `http://www.isi.ee.ethz.ch/publications`.

[8] P. O. Vontobel, D. Lippuner, and H.-A. Loeliger. Kalman filters, factor graphs, and electrical networks. In *Proc. Fifteenth Intern. Symp. on Math. Theory of Networks and Systems*, Notre Dame, IN, USA, Aug. 12–16 2002. Online proceedings available under `http://www.nd.edu/~mtns/`.

[9] P. O. Vontobel and H.-A. Loeliger. On factor graphs and electrical networks. In D.S. Gilliam and J. Rosenthal, editors, *Mathematical Systems Theory in Biology, Communication, Computation, and Finance, IMA Volumes in Math. & Appl.* Springer Verlag, 2003.