Shape Optimization for a Rectangularly Constrained Small Loop Antenna

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Abstract

The theoretical optimal shape of a small loop antenna within a rectangular area restriction is derived. Approaches with cut-off corners and rounded corners are proposed and optimal parameters (such as cut-off radius) are calculated. In a second part, by means of calculus of variation, a proof is given that the rounded-corner approach is indeed the optimal shape in terms of antenna efficiency if a simple model is assumed; the result is valid for any convex shape constraint.

1 Introduction

Small loop antennas, despite their inefficiency, have found wide application in personal communication devices where space is tight, e.g. pagers. They are mostly built as rectangular metal strip loops or as printed circuits. The problem of finding an optimal shape within a certain space equally applies to either form. A loop antenna is considered small, if the loop diameter is small enough with respect to the wavelength for the current to be assumed constant around its circumference. In this paper we assume such a simple model for the small loop antenna with a uniform current around the loop. Furthermore, the antenna loop strip is treated as an infinitesimally thin wire.

The efficiency of an antenna is usually given [1] by

$$\eta = \frac{R_{\text{rad}}}{R_{\text{rad}} + R_{\text{loss}}}, \qquad (1)$$

with R_{rad} being the radiation resistance and R_{loss} the loss resistance, respectively.

The radiation resistance is proportional to the squared aperture of the loop antenna

$$R_{\text{rad}} = 31171\,\Omega \cdot \left(\frac{A}{\lambda^2}\right)^2 = k_1 \cdot A^2, \qquad (2)$$

where A denotes the aperture or area of the loop antenna and λ is the wavelength of the frequency under consideration. The loss resistance is the Ohmic resistance of the metal and therefore proportional to the circumference c of the loop

$$R_{\text{loss}} = k_2 \cdot c. \qquad (3)$$

Thus, (1) can be written as (with $k = k_2/k_1$)

$$\eta = \frac{A^2}{A^2 + k \cdot c} = \left(1 + k \cdot \frac{c}{A^2}\right)^{-1}. \qquad (4)$$

k_1, k_2, and k are constants. In the following, different loop shapes will be investigated with the constraint that the antenna must not use more space than a rectangular reference antenna with the longer side denoted by l and the shorter side denoted by h (see Figs. 1 and 2).

2 Comparison of different shapes

2.1 Rectangular shape

The efficiency of the rectangular shape antenna with circumference $c = 2(h + l)$ can be calculated in a straightforward manner using (4)

$$\eta_{\text{rect}} = \left(1 + k \cdot \frac{2(h+l)}{(hl)^2}\right)^{-1}. \qquad (5)$$

2.2 Rectangular shape with cut-off corners

Intuitively it is clear, that the ratio of area contribution to circumference is particularly small at the corners of the antenna. Cutting the corners (for an illustration see Fig. 1) seems to help up to a certain length a_{opt} which can be found in the following way. The efficiency of the loop antenna with cut-off corners is

$$\eta_{\text{cutoff}} = \left(1 + k \cdot \frac{2(h+l) - a(8 - 4\sqrt{2})}{(hl - 2a^2)^2}\right)^{-1}. \qquad (6)$$

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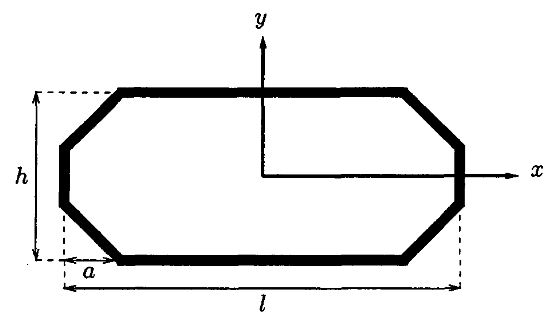

Figure 1: Drawing of rectangular loop antenna shape with cut-off corners.

The optimal length to cut off at each of the four corners, which can be found by differentiating (6) with respect to a and setting it equal to zero, is

$$a_{\text{opt}} = \frac{(h+l) - \sqrt{h^2 + l^2 + (6\sqrt{2} - 7)hl}}{3(2-\sqrt{2})}. \qquad (7)$$

2.3 Rectangular shape with rounded corners

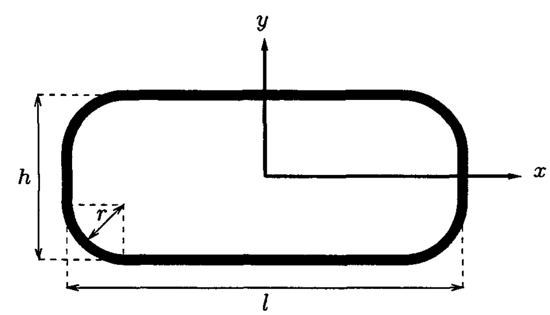

Figure 2: Drawing of rectangular loop antenna shape with rounded corners.

Another means to improve the efficiency is rounding off the corners (see Fig. 2) of the rectangular shape with a resulting efficiency of

$$\eta_{\text{round}} = \left(1 + k \cdot \frac{2(h+l) - 2r(4-\pi)}{(hl - r^2(4-\pi))^2}\right)^{-1}. \qquad (8)$$

There is an optimal radius r of the quarter circles, which can be found by differentiating (8) with respect to r and setting it equal to zero. The optimal radius r_{opt} is thus

$$r_{\text{opt}} = \frac{2(h+l) - \sqrt{4h^2 + 4l^2 + (3\pi - 4)hl}}{3(4-\pi)} \qquad (9)$$

As is shown in Section 3 this is the overall optimal solution.

3 Calculus of variations

The problem can be generalized. We introduce a Cartesian coordinate system and assume that the center of the rectangle is in the origin and the longer side is parallel to the x-direction and the shorter side is parallel to the y-direction (as drawn in Fig. 2). The loop is described by a parametric curve, i.e. $x = x(t)$ and $y = y(t)$, which turns anticlockwise. So, the area A and the circumference c are functions of $x(.)$ and $y(.)$ and become

$$A = \oint -\dot{x}(t)y(t)\mathrm{d}t, \qquad (10)$$

$$c = C(x(.), y(.)) = \oint \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\mathrm{d}t. \qquad (11)$$

In (10) and (11) the limits of the integration have to be taken such that the integral is calculated around the whole loop in the mathematical positive sense, which is indicated by the $\circ$ in $\oint$. The maximization can be reformulated as

$$\begin{aligned}
\eta_{\text{opt}} &= \max_{x(.),y(.)} \left(1 + k \cdot \frac{c}{A^2}\right)^{-1} \\
&= \left(1 + k \cdot \min_{x(.),y(.)} \frac{c}{A^2}\right)^{-1} \\
&= \left(1 + k \cdot \min_{c} \min_{\substack{x(.),y(.) \\ C(x(.),y(.))=c}} \frac{c}{A^2}\right)^{-1} \\
&= \left(1 + k \cdot \min_{c} c \cdot \left(\max_{\substack{x(.),y(.) \\ C(x(.),y(.))=c}} A\right)^{-2}\right)^{-1}.
\end{aligned} \qquad (12)$$

The problem has now been separated into two parts. In a first step, the area A under the constraint of constant circumference c has to be maximized. In the second step, we minimize over all possible circumferences. The first problem is attacked with the help of calculus of variations [2]. The function to be maximized is

$$\oint f(x(t), \dot{x}(t), y(t), \dot{y}(t), t)\mathrm{d}t, \qquad (13)$$

with the Lagrangian function $f(x, \dot{x}, y, \dot{y}, t)$ (subsequently, for the sake of simplicity we will omit parameter t in $x(t)$, $\dot{x}(t)$ etc.)

$$f = f(x, \dot{x}, y, \dot{y}, t) = -\dot{x} \cdot g(x, y) + \lambda \cdot \sqrt{\dot{x}^2 + \dot{y}^2}, \tag{14}$$

with

$$g = g(x, y) = \begin{cases} y & (-\frac{l}{2} < x < \frac{l}{2}, -\frac{h}{2} < y < \frac{h}{2}) \\ +\frac{h}{2} & (-\frac{l}{2} < x < \frac{l}{2}, y > \frac{h}{2}) \\ -\frac{h}{2} & (-\frac{l}{2} < x < \frac{l}{2}, y < -\frac{h}{2}) \\ 0 & \text{(elsewhere)}, \end{cases} \tag{15}$$

and $\lambda \in \mathbb{R}$ denoting a Lagrange multiplier. In the following, we will need the partial derivative of g with respect to y

$$\begin{aligned} g_y = g_y(x, y) &= \frac{\partial}{\partial y} g(x, y) \\ &= \begin{cases} 1 & (-\frac{l}{2} < x < \frac{l}{2}, -\frac{h}{2} < y < \frac{h}{2}) \\ 0 & \text{(elsewhere)}. \end{cases} \end{aligned} \tag{16}$$

The function $g(x, y)$ was chosen in such a way as to find the maximal area which is inside the specified rectangle. In other words, any part of the shape that is outside this rectangle does only contribute to the circumference but not to the area. In order to maximize (13) we have to find the solutions of the two Euler-Lagrange equations

$$f_x - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{x}} \stackrel{!}{=} 0 \tag{17}$$

$$f_y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{y}} \stackrel{!}{=} 0. \tag{18}$$

(f_x denotes the partial derivative of the function f with respect to x and similarly for the other parameters $\dot{x}$, y and $\dot{y}$.) We introduce the curvature κ of a line [2]

$$\kappa = \kappa(\dot{x}, \ddot{x}, \dot{y}, \ddot{y}) = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}. \tag{19}$$

The partial derivatives in (17) and (18) are given by

$$f_x = -\dot{x} g_x, \tag{20}$$

$$f_{\dot{x}} = -g + \lambda \cdot \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \tag{21}$$

$$f_y = -\dot{x} g_y, \tag{22}$$

$$f_{\dot{y}} = \lambda \cdot \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}. \tag{23}$$

Total derivation of (21) and (23) with respect to t and using (19) leads to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{x}} &= -g_x \dot{x} - g_y \dot{y} + \lambda \cdot \frac{\ddot{x}\sqrt{\dot{x}^2 + \dot{y}^2} - \dot{x}\frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}}{\dot{x}^2 + \dot{y}^2} \\ &= -g_x \dot{x} - g_y \dot{y} - \lambda \dot{y} \cdot \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \\ &= -g_x \dot{x} - g_y \dot{y} - \lambda \dot{y} \cdot \kappa, \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{y}} &= \lambda \cdot \frac{\ddot{y}\sqrt{\dot{x}^2 + \dot{y}^2} - \dot{y}\frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}}{\dot{x}^2 + \dot{y}^2} \\ &= \lambda \dot{x} \cdot \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \\ &= \lambda \dot{x} \cdot \kappa. \end{aligned} \tag{25}$$

With this, (17) and (18) can be reformulated

$$-\dot{x} g_x - (-g_x \dot{x} - g_y \dot{y} - \lambda \dot{y} \cdot \kappa) = g_y \dot{y} + \lambda \dot{y} \cdot \kappa \stackrel{!}{=} 0, \tag{26}$$

$$-\dot{x} g_y - \lambda \dot{x} \cdot \kappa \stackrel{!}{=} 0. \tag{27}$$

Finally, reordering both (26) and (27) leads to

$$\dot{y} \cdot \left(\kappa + \frac{g_y}{\lambda}\right) = 0, \tag{28}$$

$$\dot{x} \cdot \left(\kappa + \frac{g_y}{\lambda}\right) = 0. \tag{29}$$

We must separate two cases:

Case 1: $\kappa \neq -g_y/\lambda$. The solution of (28) and (29) is

$$\dot{x} = 0, \qquad \dot{y} = 0, \tag{30}$$

in other words, a stationary point.

Case 2: $\kappa = -g_y/\lambda$. The solution is a curve consisting of arcs whose curvature equals $-g_y/\lambda$. Straight lines are thereby just arcs with curvature zero. Note, that the different arcs must be connected in a smooth way. To check this, we reparameterize $x(t)$ and $y(t)$ to $x(s)$ and $y(s)$ where $s = s(t) = \int_0^t \sqrt{\dot{x}^2(\tau) + \dot{y}^2(\tau)} \mathrm{d}\tau$ is the parameter corresponding to the actual curve length. Thus, we get [2]

$$|\kappa| = \sqrt{x''^2 + y''^2}, \tag{31}$$

where

$$x'' = x''(s) = \frac{\mathrm{d}^2}{\mathrm{d}s^2} x(s), \qquad y'' = y''(s) = \frac{\mathrm{d}^2}{\mathrm{d}s^2} y(s). \tag{32}$$

As κ is a piecewise continuous and bounded function of s, by Eq. (31) x'' and y'' are piecewise continuous and bounded, too. Thus, the continuity of x' and y' follows immediately, which in turn means that the arcs are connected in a smooth way, indeed.

Inspection of (16) reveals the existence of two different regions. In the first region ($-l/2 < x < l/2$, $-h/2 < y < h/2$), the curvature is $-1/\lambda$ and in the second region (everywhere else), the curvature is 0. So in the second region we have straight lines. Summarizing this, the shape of our area is as described in Section 2.3 (see Fig. 2).

Until now, we have solved the inner maximization of (12). The next step is the minimization over all possible circumferences of the rectangular shape with rounded corners. But instead of varying the circumference c, we can vary the radius r of the quarter circles. This is exactly what we have done in Section 2.3, and thus the overall optimal solution is the one given in (8) and (9).

4 A practical example

Suppose an antenna with the dimensions $l = 4\,\text{cm}$ and $h = 1\,\text{cm}$, as typically used in UHF pagers. For small, inefficient loop antennas the exact value of k is not important for the evaluation of the efficiency improvement, because the term with k in Eqs. (8) and (5), respectively, is much larger than l^3, so that k cancels out in the efficiency ratio. With (9) we get $r_{\text{opt}} = 2.05\,\text{mm}$. Fig. 3, where $\frac{\eta_{\text{round}}}{\eta_{\text{rect}}}$ is shown in

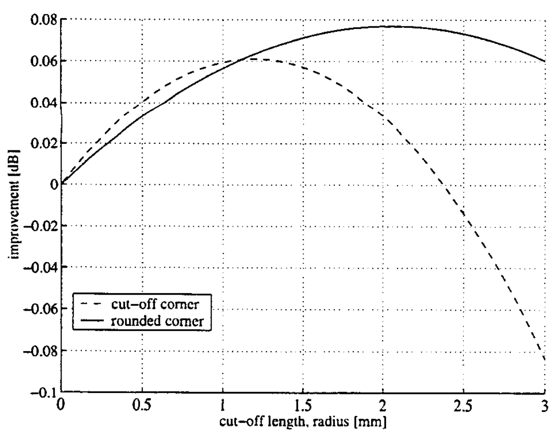

Figure 3: Antenna efficiency improvement for rounded and cut-off corners.

dB, reveals that the potential improvement is within a fraction of a dB. However, this marginal improvement comes without a prize. On the contrary, bending tools for the metal strip are easier to manufacture for a finite radius.

The improvement does not depend on the absolute values of l and h but rather on the ratio of the two dimensions. As might be expected, it is maximal for $l = h$, where an improvement of $0.12\,\text{dB}$ is achieved.

5 Conclusions

It has been shown that the efficiency-optimal shape of a small loop antenna as used in personal communication devices is of a rectangular shape with rounded corners, given an area restriction of a certain rectangle. The optimal radius of the quarter circles has been derived. The solution can easily be generalized to any convex shape in the way that corners are rounded with circle segments. We conjecture that this shape is optimal for single-turn loops as well as for multi-turn loops if the same simple model for the efficiency of a small loop antenna is applied.

Acknowledgment

The first author would like to thank Said Moridi from Philips Research Laboratories and Simon Saunders from University of Surrey for helpful discussions.

References

[1] C.A. Balanis, *Antenna Theory,* John Wiley & Sons, 2nd edition, 1997.

[2] I.N. Bronshtein and K.A. Semendyayev, *Handbook of Mathematics,* Springer, 3rd edition, 1979.