In Memoriam Ralf Koetter

(10/10/1963 - 02/02/2009)

A Graph-Dynamics Interpretation of the Sum-Product Algorithm

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Overview of Talk

- Introductory example

- Review of some basics
  (factor graphs / SPA / fixed points of the SPA / graph covers)

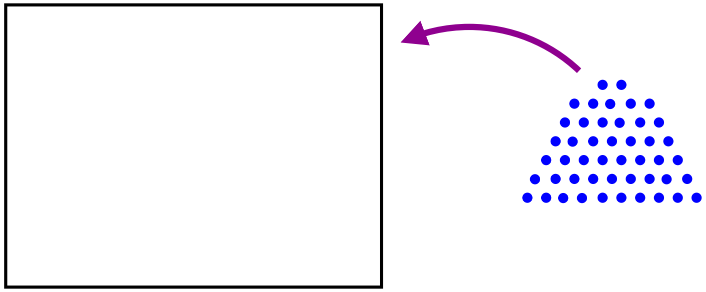
- Re-interpretation of fixed points of the SPA
  in terms of graph covers and valid configurations therein

- Re-interpretation of the transient part of the SPA
  in terms of a graph-dynamical system

Introductory Example

Particles in a Box

Experiment: let us place M particles in a uniformly and independently distributed manner on a very fine lattice bounded by a box.



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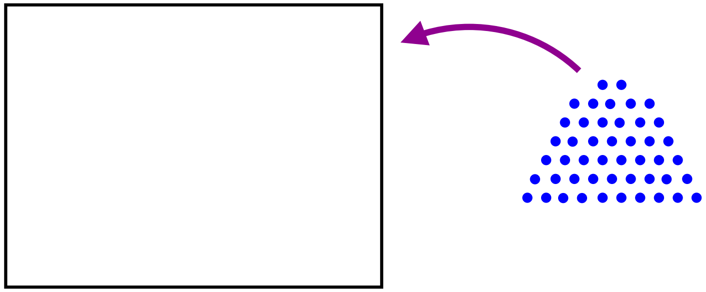

This experiment has many possible outcomes. Here are two of them:

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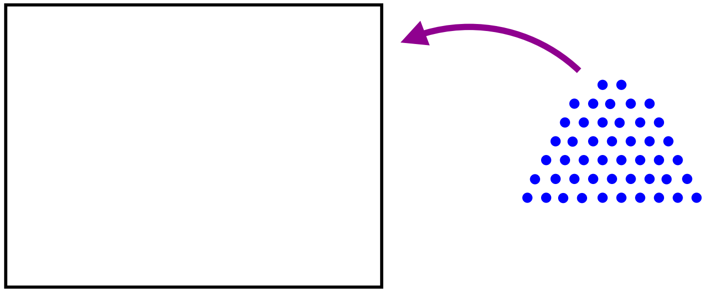

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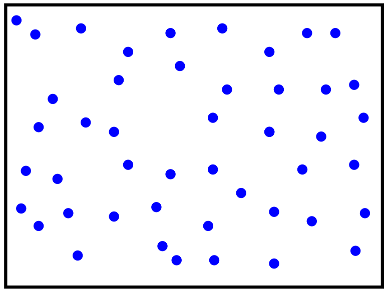

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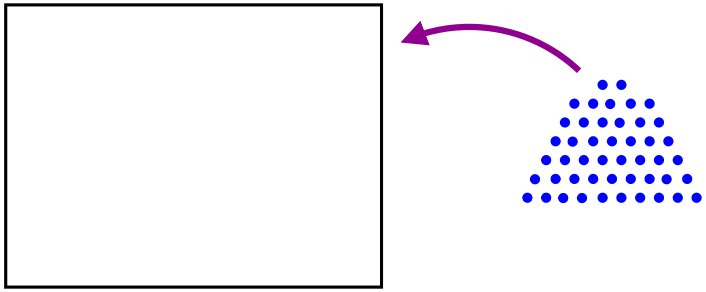

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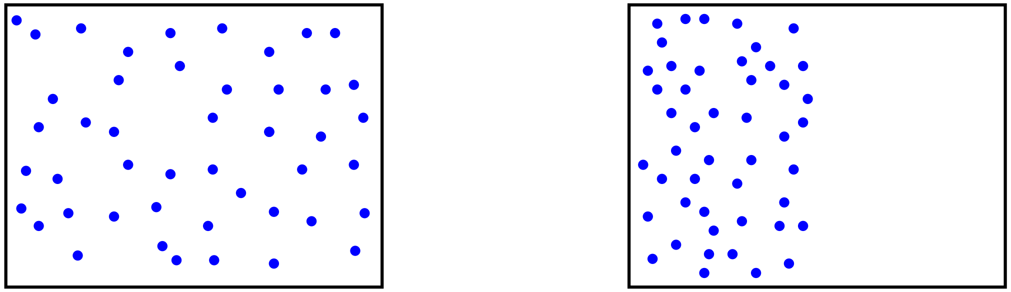

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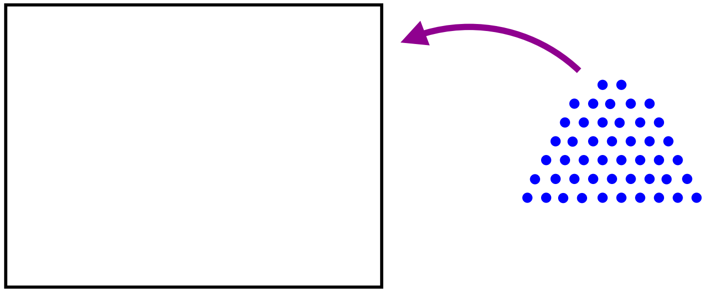

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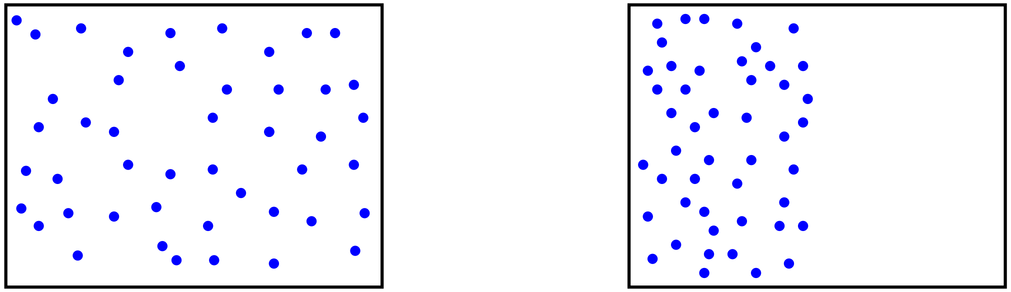

Which one of the above two outcomes is more likely to happen?

Particles in a Box

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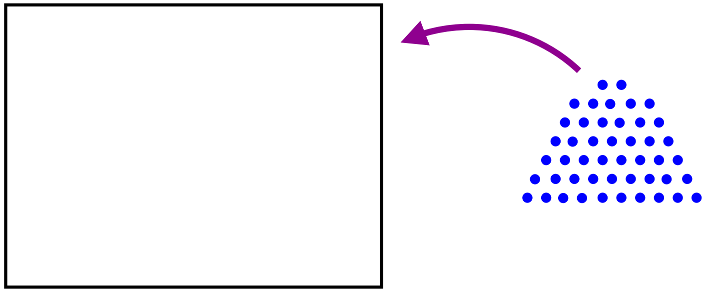

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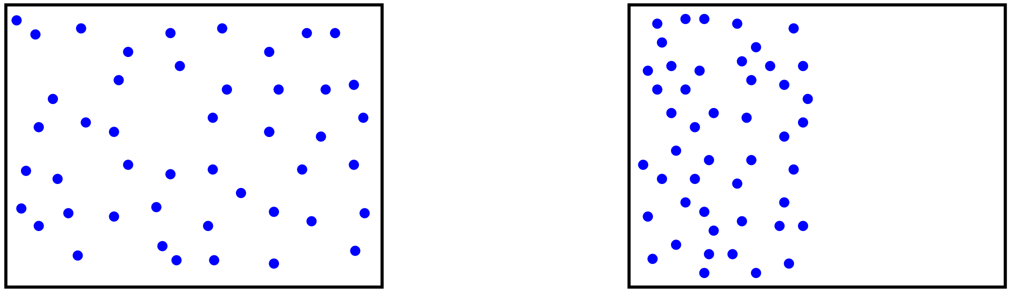
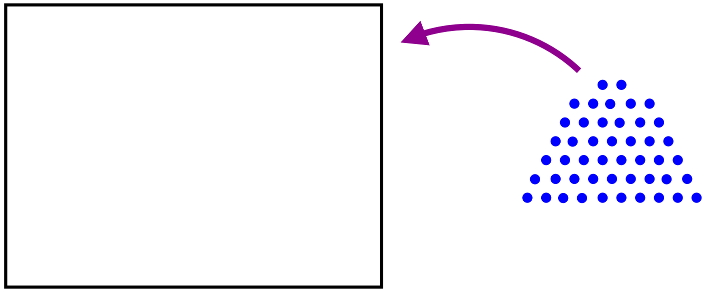


Which one of the above two outcomes is more likely to happen?

Both scenarios are equally likely!

Particles in a Box

Experiment: let us place M particles in a uniformly and independently distributed manner on a very fine lattice bounded by a box.



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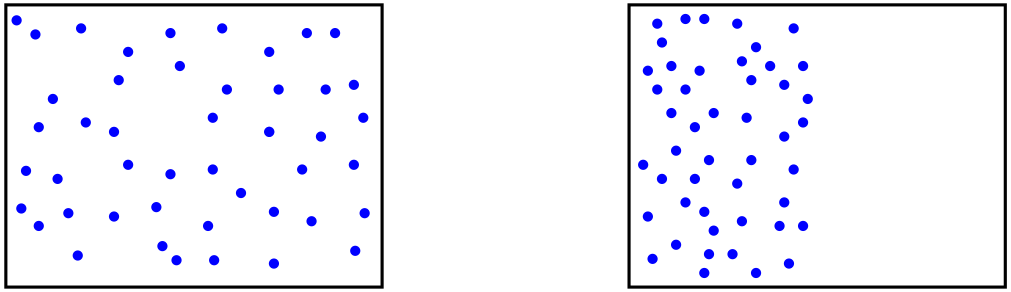
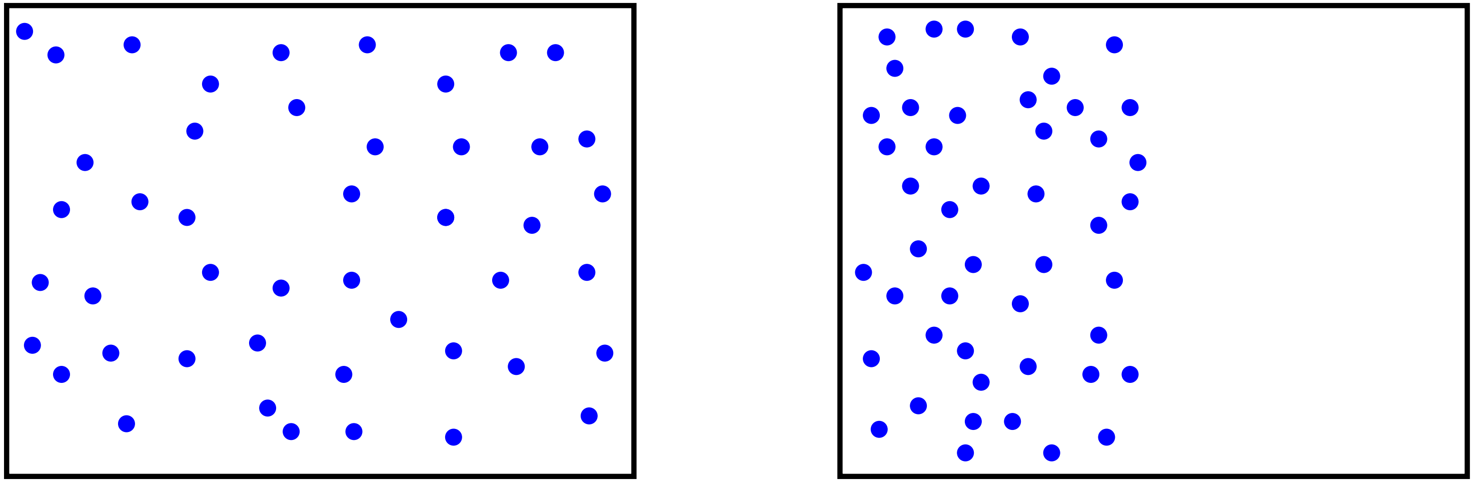


Better question: when doing this experiment again, is the outcome more likely "to look nearly like" outcome 1 or like outcome 2?

Particles in a Box

The results of the previous experiment:



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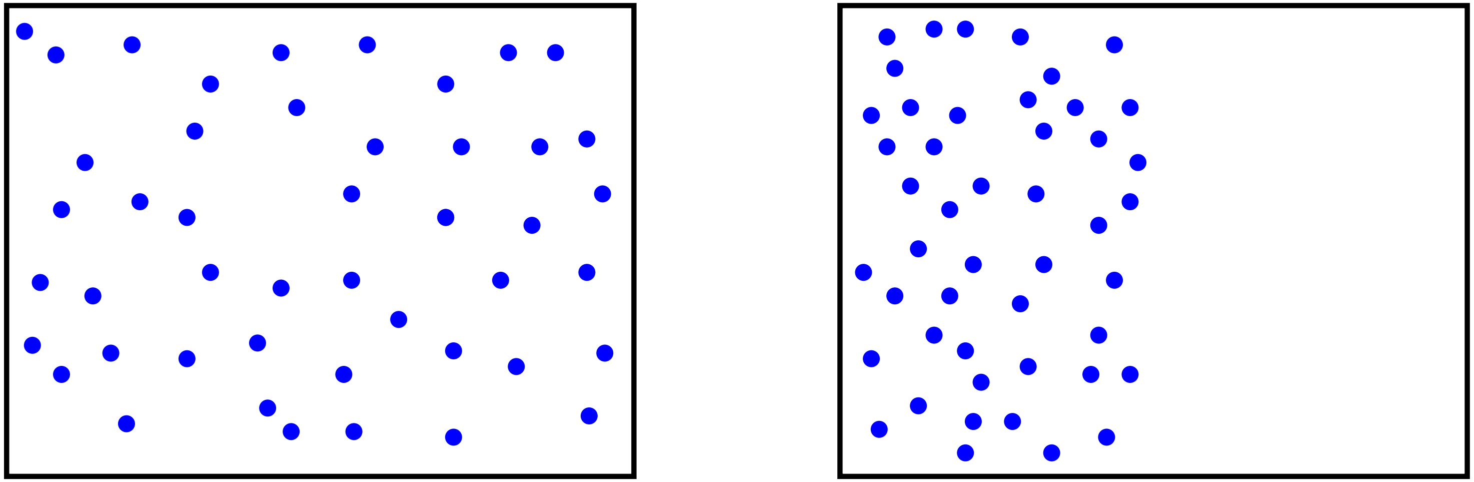

$\text{microstate} =$ Coordinates of all M particles

Particles in a Box

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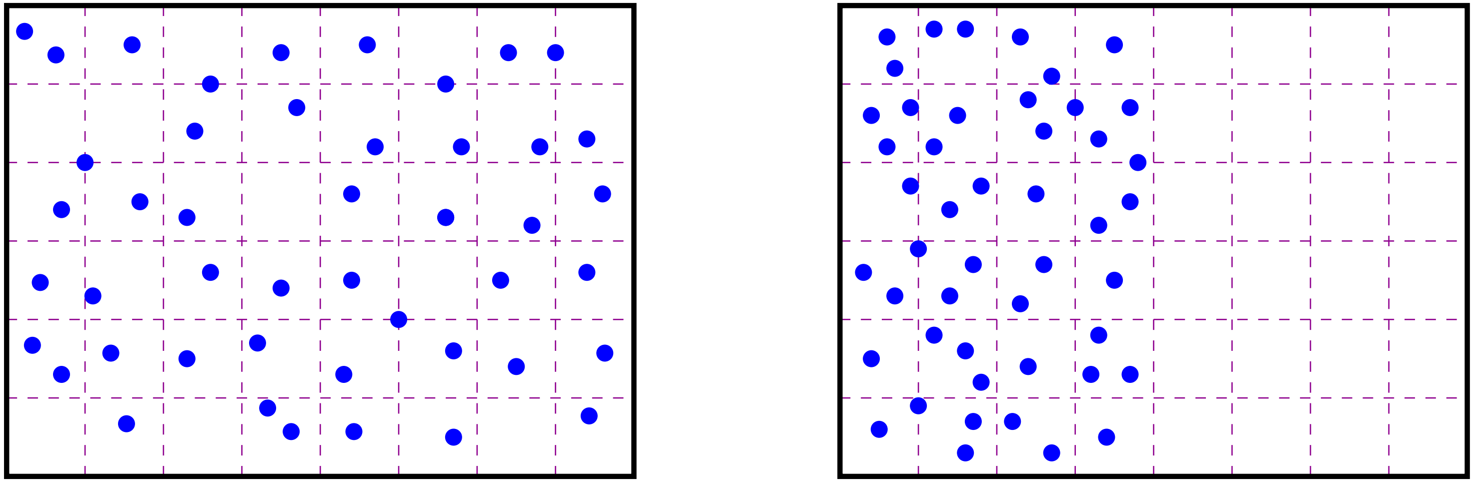


$\color{blue}{\mathrm{microstate}}$ = Coordinates of all M particles

Particles in a Box

The results of the previous experiment:

$$\frac{1}{47}\begin{array}{|cccccccc|}\hline 2 & 1 & 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 1 & 0 & 1 \\ \hline\end{array} \qquad \frac{1}{47}\begin{array}{|cccccccc|}\hline 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 3 & 3 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 3 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ \hline\end{array}$$

$\mathrm{microstate} =$ Coordinates of all M particles

Particles in a Box

The results of the previous experiment:

$\frac{1}{47}$

2	1	1	1	1	1	2	0
1	0	1	1	1	1	1	1
1	1	1	0	1	1	1	1
1	1	1	1	2	0	1	1
2	1	1	1	1	1	1	1
0	1	0	2	1	1	0	1

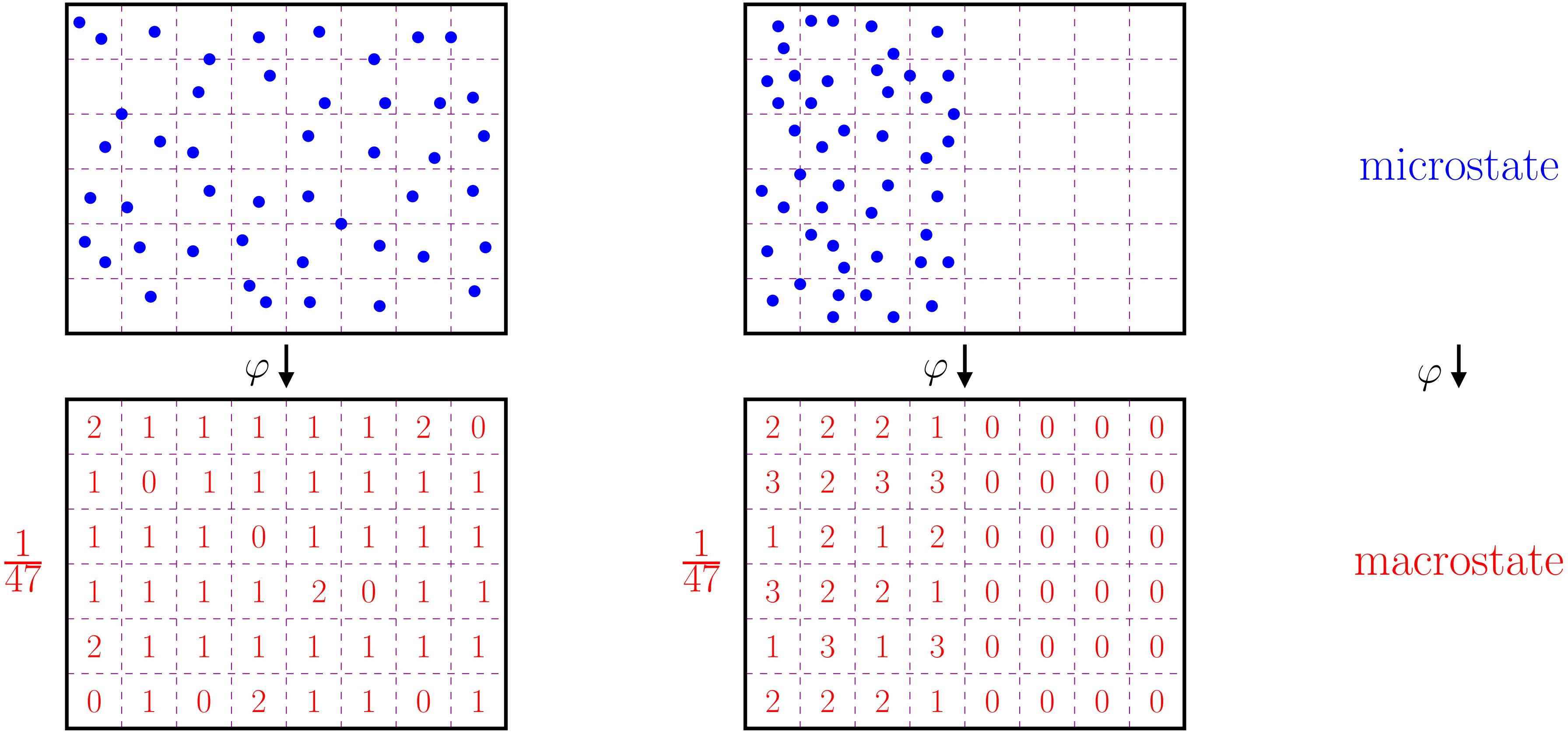
$\frac{1}{47}$

2	2	2	1	0	0	0	0
3	2	3	3	0	0	0	0
1	2	1	2	0	0	0	0
3	2	2	1	0	0	0	0
1	3	1	3	0	0	0	0
2	2	2	1	0	0	0	0

$\text{microstate} =$ Coordinates of all M particles

$\text{macrostate} =$ "Summary" of a microstate

Particles in a Box



microstate

$\frac{1}{47}$

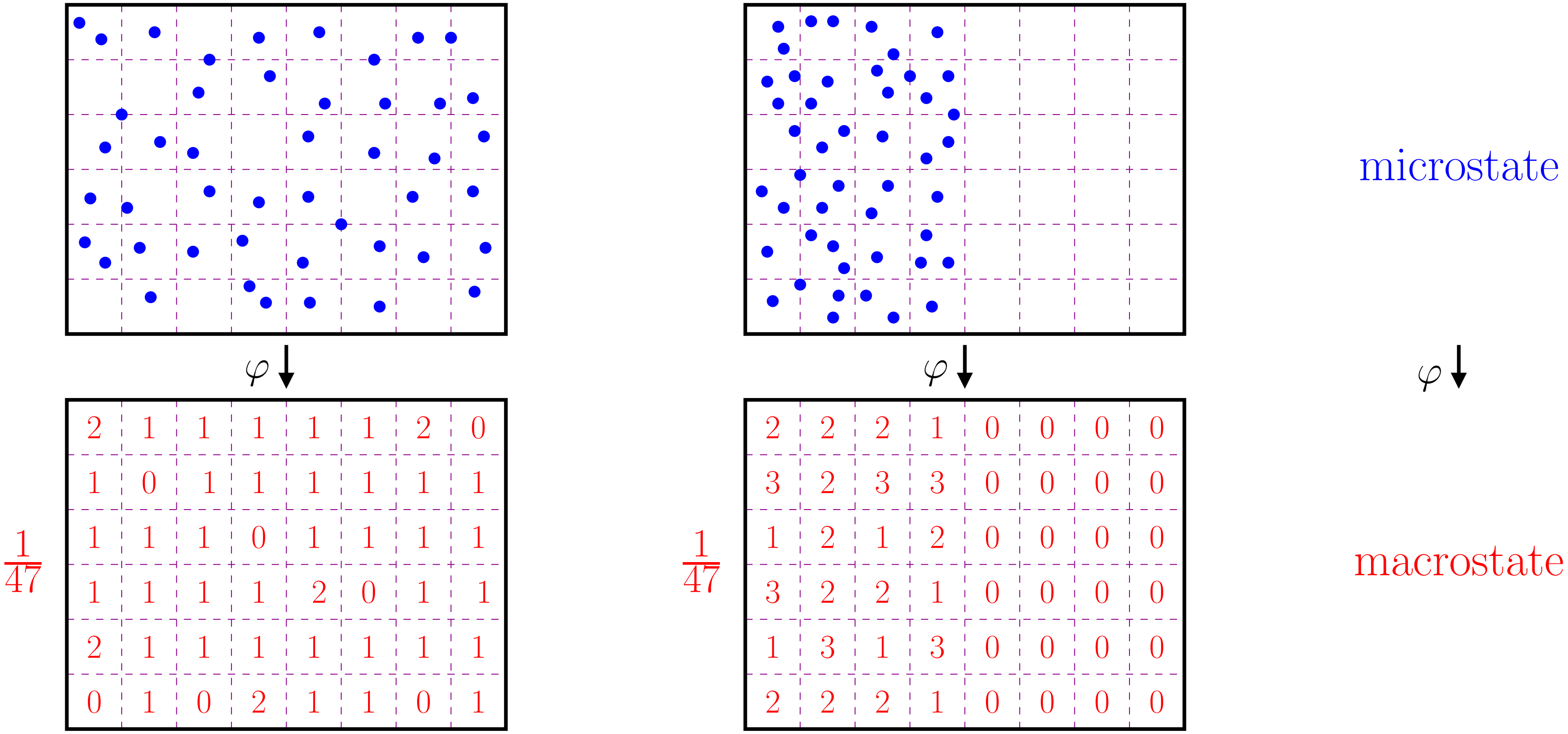
2	1	1	1	1	1	2	0
1	0	1	1	1	1	1	1
1	1	1	0	1	1	1	1
1	1	1	1	2	0	1	1
2	1	1	1	1	1	1	1
0	1	0	2	1	1	0	1

$\frac{1}{47}$

2	2	2	1	0	0	0	0
3	2	3	3	0	0	0	0
1	2	1	2	0	0	0	0
3	2	2	1	0	0	0	0
1	3	1	3	0	0	0	0
2	2	2	1	0	0	0	0

macrostate

Particles in a Box



$\frac{1}{47}$

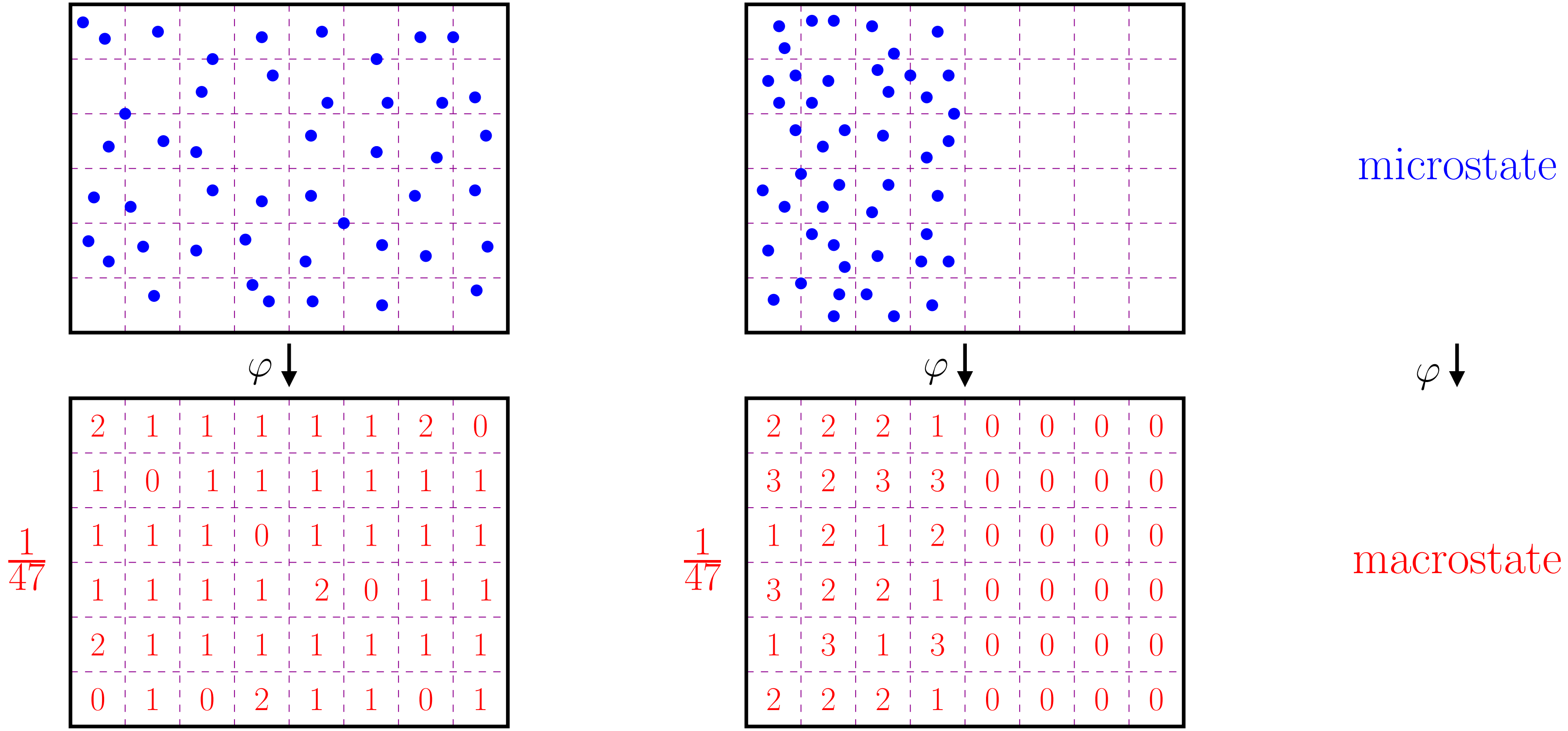
2	1	1	1	1	1	2	0
1	0	1	1	1	1	1	1
1	1	1	0	1	1	1	1
1	1	1	1	2	0	1	1
2	1	1	1	1	1	1	1
0	1	0	2	1	1	0	1

$\frac{1}{47}$

2	2	2	1	0	0	0	0
3	2	3	3	0	0	0	0
1	2	1	2	0	0	0	0
3	2	2	1	0	0	0	0
1	3	1	3	0	0	0	0
2	2	2	1	0	0	0	0

Note: φ is usually a many-to-one mapping.

Particles in a Box



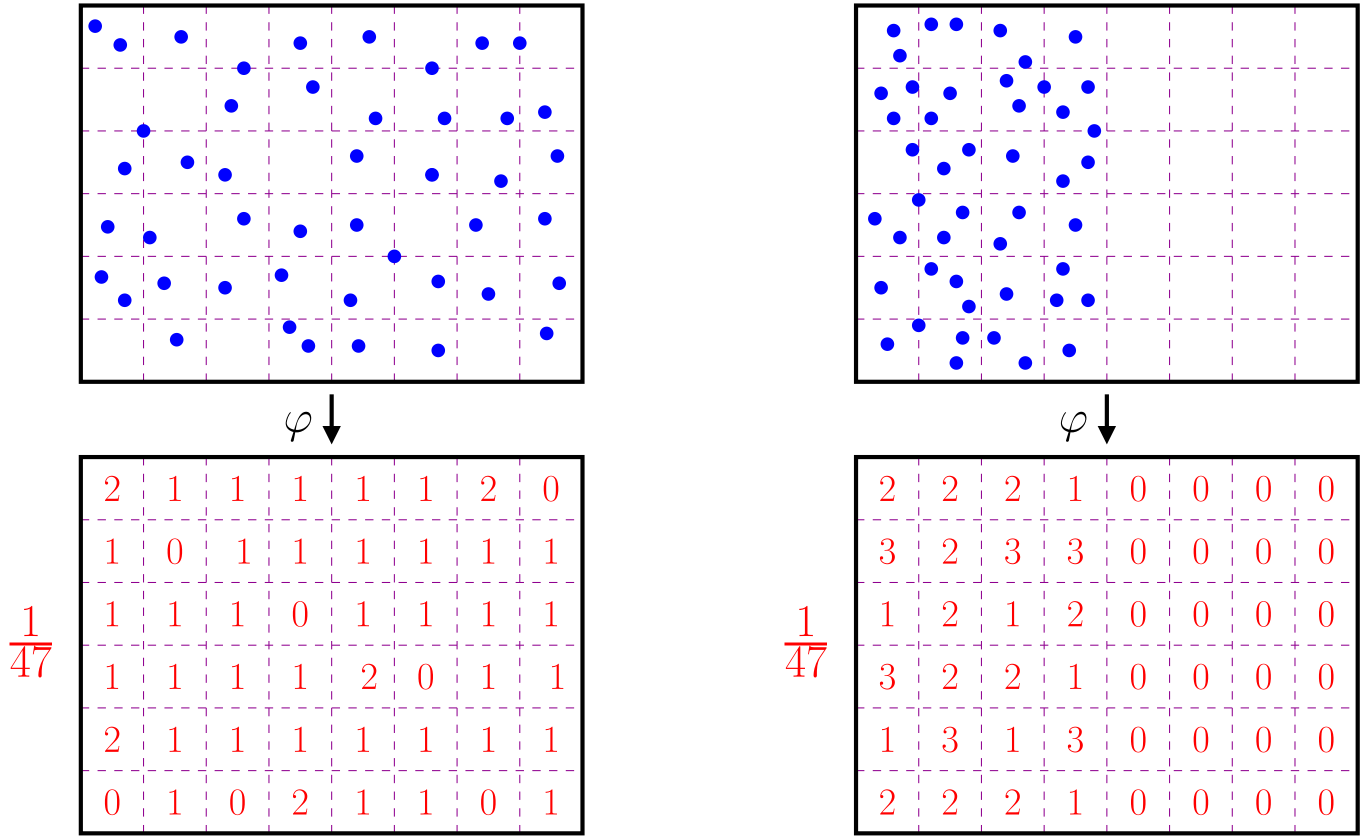
2	1	1	1	1	1	2	0
1	0	1	1	1	1	1	1
1	1	1	0	1	1	1	1
1	1	1	1	2	0	1	1
2	1	1	1	1	1	1	1
0	1	0	2	1	1	0	1

2	2	2	1	0	0	0	0
3	2	3	3	0	0	0	0
1	2	1	2	0	0	0	0
3	2	2	1	0	0	0	0
1	3	1	3	0	0	0	0
2	2	2	1	0	0	0	0

If $P(\text{microstate}) =$ const. for all microstates then

$$\begin{aligned} P(\text{macrostate}) &\propto \#\{\text{microstate} : \ \varphi(\text{microstate}) = \text{macrostate}\} \\ &= \#\varphi^{-1}(\text{macrostate}) \end{aligned}$$

Particles in a Box



microstate

$\varphi \downarrow$

macrostate

$\frac{1}{47}$

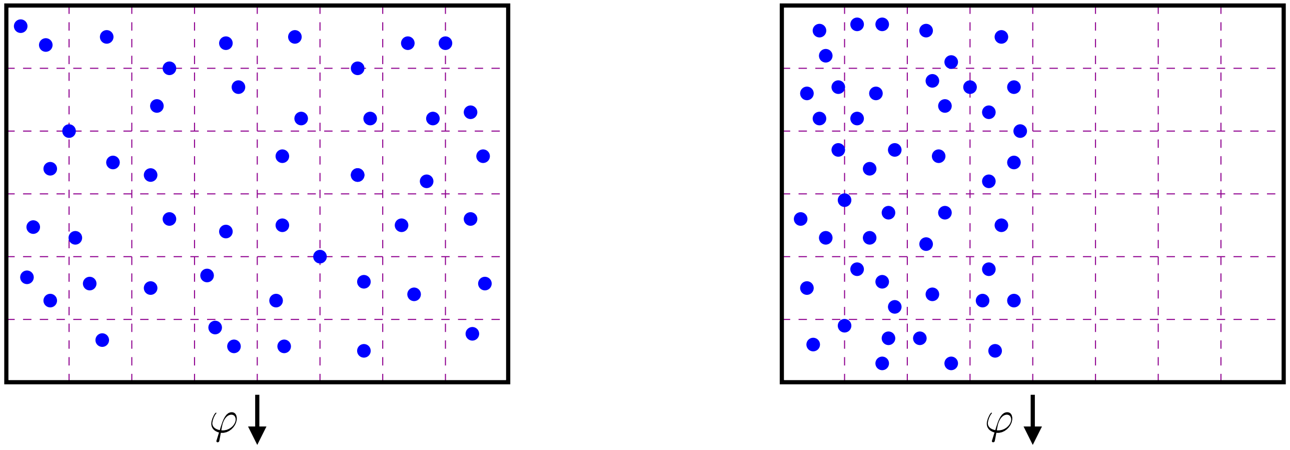
2	1	1	1	1	1	2	0
1	0	1	1	1	1	1	1
1	1	1	0	1	1	1	1
1	1	1	1	2	0	1	1
2	1	1	1	1	1	1	1
0	1	0	2	1	1	0	1

$\frac{1}{47}$

2	2	2	1	0	0	0	0
3	2	3	3	0	0	0	0
1	2	1	2	0	0	0	0
3	2	2	1	0	0	0	0
1	3	1	3	0	0	0	0
2	2	2	1	0	0	0	0

Here: $\#\varphi^{-1}(\text{macrostate } 1) \gg \#\varphi^{-1}(\text{macrostate } 2)$

Particles in a Box



microstate

$\varphi\downarrow$

macrostate

$\frac{1}{47}$

2	1	1	1	1	1	2	0
1	0	1	1	1	1	1	1
1	1	1	0	1	1	1	1
1	1	1	1	2	0	1	1
2	1	1	1	1	1	1	1
0	1	0	2	1	1	0	1

$\frac{1}{47}$

2	2	2	1	0	0	0	0
3	2	3	3	0	0	0	0
1	2	1	2	0	0	0	0
3	2	2	1	0	0	0	0
1	3	1	3	0	0	0	0
2	2	2	1	0	0	0	0

Here: $\#\varphi^{-1}(\text{macrostate }1) \gg \#\varphi^{-1}(\text{macrostate }2)$

$$\Rightarrow P(\text{macrostate }1) \gg P(\text{macrostate }2)$$

Particles in a Box with Gradient Field

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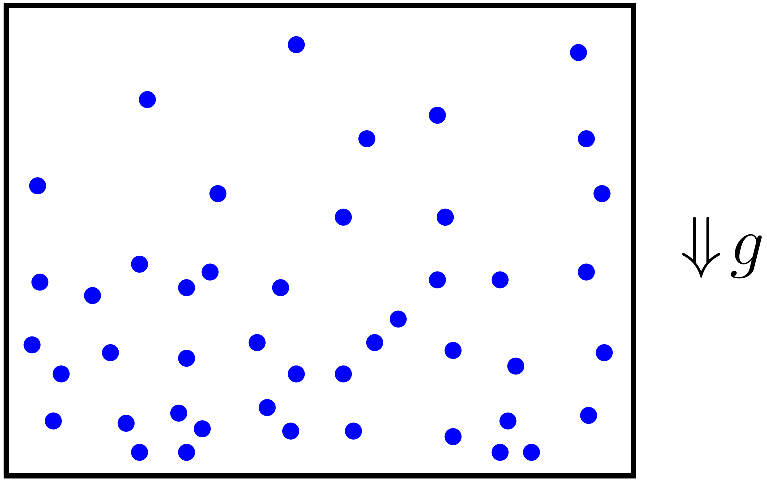

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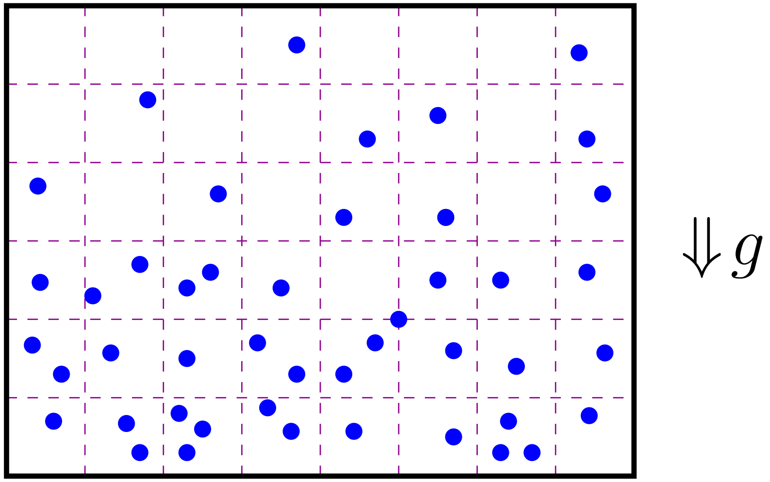

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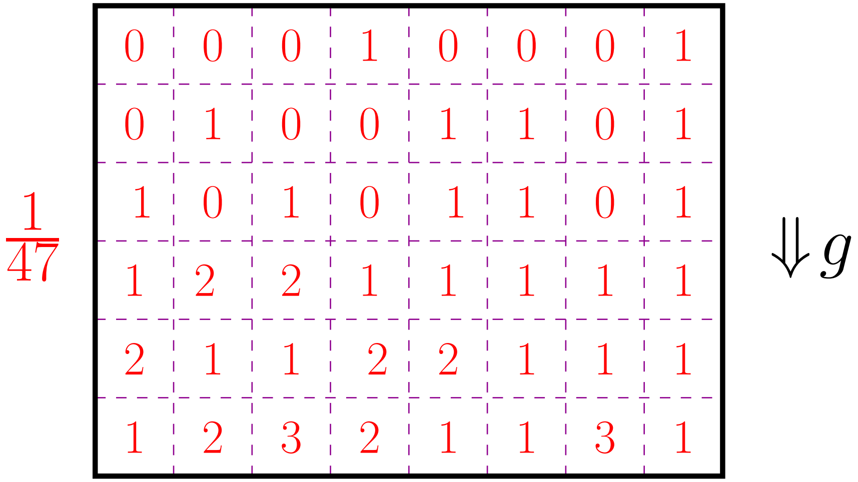
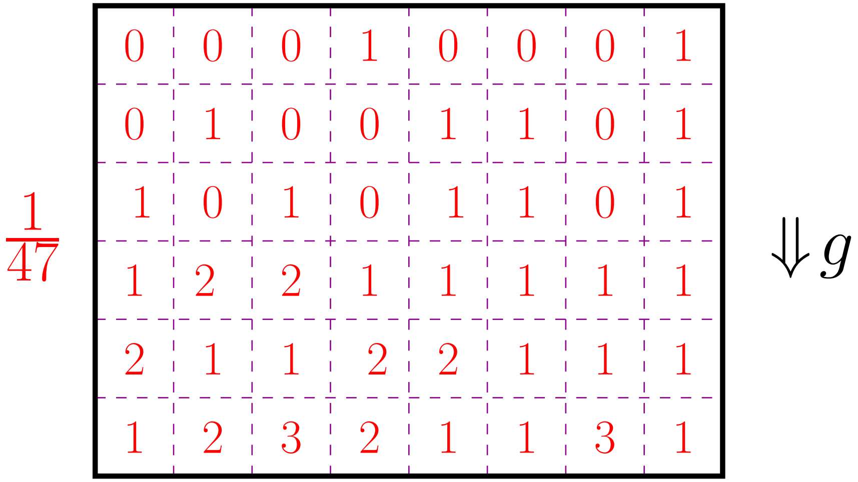


$\frac{1}{47}$

0	0	0	1	0	0	0	1
0	1	0	0	1	1	0	1
1	0	1	0	1	1	0	1
1	2	2	1	1	1	1	1
2	1	1	2	2	1	1	1
1	2	3	2	1	1	3	1

$\Downarrow g$

Particles in a Box with Gradient Field



$\frac{1}{47}$

0	0	0	1	0	0	0	1
0	1	0	0	1	1	0	1
1	0	1	0	1	1	0	1
1	2	2	1	1	1	1	1
2	1	1	2	2	1	1	1
1	2	3	2	1	1	3	1

$\Downarrow g$

If $P(\text{microstate}) \;\propto\; \exp\Big(-M \cdot E(\varphi(\text{microstate}))\Big)$

Particles in a Box with Gradient Field

$\frac{1}{47}$

0	0	0	1	0	0	0	1
0	1	0	0	1	1	0	1
1	0	1	0	1	1	0	1
1	2	2	1	1	1	1	1
2	1	1	2	2	1	1	1
1	2	3	2	1	1	3	1

$\Downarrow g$

If $P(\text{microstate}) \;\propto\; \exp\Big(-M \cdot E\big(\varphi(\text{microstate})\big)\Big)$ then

$$P(\text{macrostate}) \;\propto\; \exp\big(-M \cdot E(\text{macrostate})\big)$$

$$\cdot\; \#\big\{\text{microstate} : \;\varphi(\text{microstate}) = \text{macrostate}\big\}$$

Particles in a Box with Gradient Field

$\frac{1}{47}$

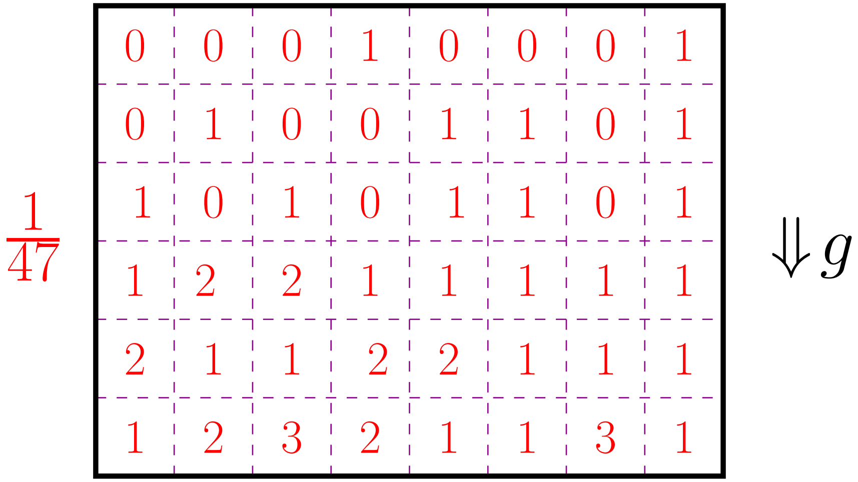
0	0	0	1	0	0	0	1
0	1	0	0	1	1	0	1
1	0	1	0	1	1	0	1
1	2	2	1	1	1	1	1
2	1	1	2	2	1	1	1
1	2	3	2	1	1	3	1

$\Downarrow g$

If $P(\text{microstate}) \;\propto\; \exp\Big(-M \cdot E\big(\varphi(\text{microstate})\big)\Big)$ then

$$\begin{aligned} P(\text{macrostate}) \;&\propto\; \exp\big(-M \cdot E(\text{macrostate})\big) \\ &\quad \cdot \#\big\{\text{microstate} :\; \varphi(\text{microstate}) = \text{macrostate}\big\} \\ &=\; \exp\big(-M \cdot E(\text{macrostate})\big) \;\cdot\; \#\varphi^{-1}(\text{macrostate}) \end{aligned}$$

Particles in a Box with Gradient Field

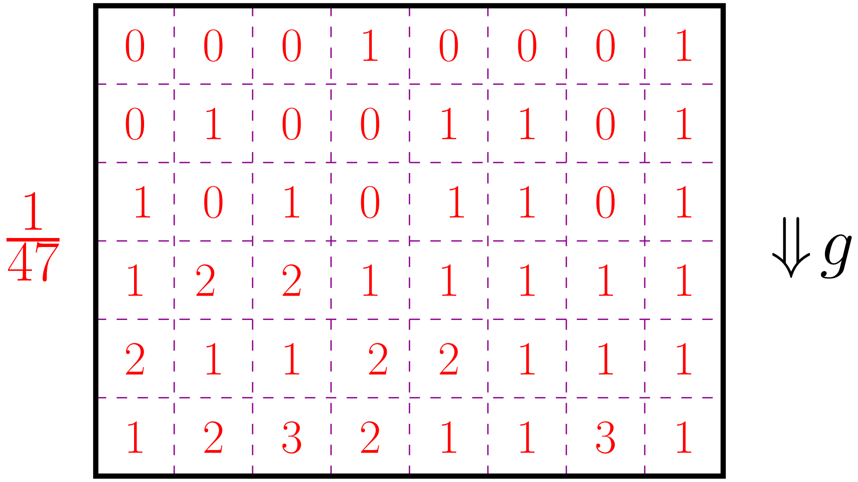


$\frac{1}{47}$

0	0	0	1	0	0	0	1
0	1	0	0	1	1	0	1
1	0	1	0	1	1	0	1
1	2	2	1	1	1	1	1
2	1	1	2	2	1	1	1
1	2	3	2	1	1	3	1

$\Downarrow g$

Particles in a Box with Gradient Field

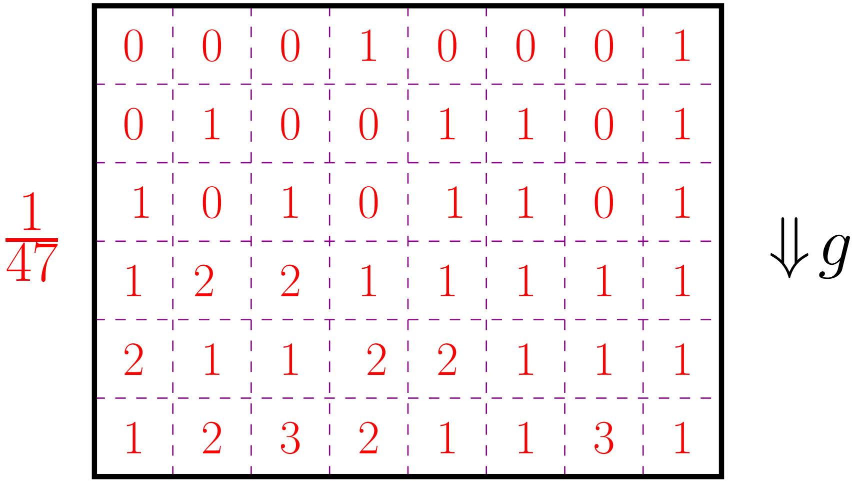


$$\frac{1}{47}\begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 & 1 & 1 & 3 & 1 \\ \hline \end{array} \quad \Downarrow g$$

Let

$$H_M(\text{macrostate}) \triangleq \frac{1}{M}\log\left(\#\varphi^{-1}(\text{macrostate})\right).$$

Particles in a Box with Gradient Field

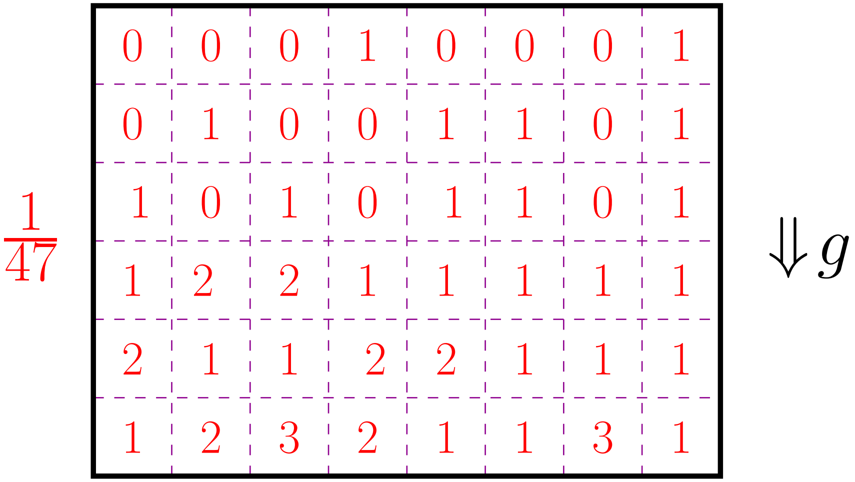


0	0	0	1	0	0	0	1
0	1	0	0	1	1	0	1
1	0	1	0	1	1	0	1
1	2	2	1	1	1	1	1
2	1	1	2	2	1	1	1
1	2	3	2	1	1	3	1

Let
$$H_M(\text{macrostate}) \triangleq \frac{1}{M} \log\left(\#\varphi^{-1}(\text{macrostate})\right) .$$

$$\Rightarrow \; P(\text{macrostate}) \propto \exp\left(- M \cdot E(\text{macrostate})\right) \cdot \#\varphi^{-1}(\text{macrostate})$$

Particles in a Box with Gradient Field



0	0	0	1	0	0	0	1
0	1	0	0	1	1	0	1
1	0	1	0	1	1	0	1
1	2	2	1	1	1	1	1
2	1	1	2	2	1	1	1
1	2	3	2	1	1	3	1

Let $$H_M(\text{macrostate}) \triangleq \frac{1}{M} \log\left(\#\varphi^{-1}(\text{macrostate})\right) .$$

$$\begin{aligned} \Rightarrow \quad P(\text{macrostate}) &\propto \exp\left(- M \cdot E(\text{macrostate})\right) \cdot \#\varphi^{-1}(\text{macrostate}) \\ &= \exp\left(M \cdot \left(-E(\text{macrostate}) + H_M(\text{macrostate})\right)\right) \end{aligned}$$

Static vs. Dynamic Setup

Static setup:

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$$P(\mathrm{M}) \;\propto\; \exp\big(-M \cdot E(\mathrm{M})\big) \;\cdot\; \#\varphi^{-1}(\mathrm{M})$$

Static vs. Dynamic Setup

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$$P(\mathrm{M}) \;\propto\; \exp\big(-M\cdot E(\mathrm{M})\big) \;\cdot\; \#\varphi^{-1}(\mathrm{M})$$

Dynamic setup:

$$P\big(\mathrm{M}(t+\Delta t)\bigm|\mathrm{m}(t)\big)$$

$$\propto\; \exp\Big(-M\cdot E\big(\mathrm{M}(t+\Delta t)\bigm|\mathrm{m}(t)\big)\Big) \;\cdot\; \#\varphi^{-1}\big(\mathrm{M}(t+\Delta t)\bigm|\mathrm{m}(t)\big)$$

Static vs. Dynamic Setup

Static setup:

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Dynamic setup:

$$P\Big(\mathrm{M}(t+\Delta t)\;\big|\;\mathrm{m}(t)\Big)$$
$$\propto\; \exp\bigg(-\,M \cdot E\Big(\mathrm{M}(t+\Delta t)\;\big|\;\mathrm{m}(t)\Big)\bigg) \;\cdot\; \#\varphi^{-1}\Big(\mathrm{M}(t+\Delta t)\;\big|\;\mathrm{m}(t)\Big)$$

"Better" dynamic setup:

$$P\Big(\mathrm{M}(t+\Delta t)\;\big|\;\mathrm{M}(t)\Big)$$
$$\propto\; \exp\bigg(-\,M \cdot E\Big(\mathrm{M}(t+\Delta t)\;\big|\;\mathrm{M}(t)\Big)\bigg) \;\cdot\; \#\varphi^{-1}\Big(\mathrm{M}(t+\Delta t)\;\big|\;\mathrm{M}(t)\Big)$$

Static vs. Dynamic Setup

Static setup: will model fix points of the SPA

$$P(\mathrm{M}) \;\propto\; \exp\big(-M \cdot E(\mathrm{M})\big) \;\cdot\; \#\varphi^{-1}(\mathrm{M})$$

Dynamic setup:

$$P\big(\mathrm{M}(t+\Delta t) \mid \mathrm{m}(t)\big)$$
$$\propto\; \exp\Big(-M \cdot E\big(\mathrm{M}(t+\Delta t) \mid \mathrm{m}(t)\big)\Big) \;\cdot\; \#\varphi^{-1}\big(\mathrm{M}(t+\Delta t) \mid \mathrm{m}(t)\big)$$

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"Better" dynamic setup: will model the transient part of the SPA

$$P\big(\mathrm{M}(t+\Delta t) \mid \mathrm{M}(t)\big)$$

$$\propto\; \exp\Big(-M \cdot E\big(\mathrm{M}(t+\Delta t) \mid \mathrm{M}(t)\big)\Big) \;\cdot\; \#\varphi^{-1}\big(\mathrm{M}(t+\Delta t) \mid \mathrm{M}(t)\big)$$

Forney-style Factor Graphs (FFGs)

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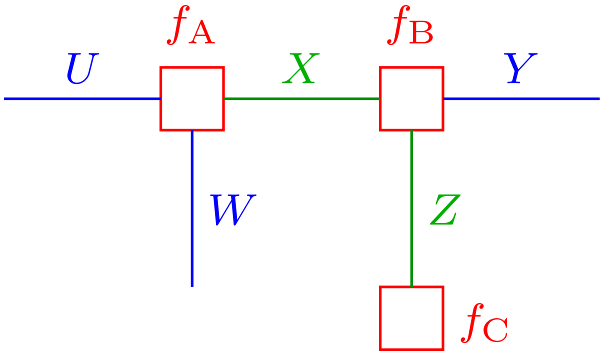

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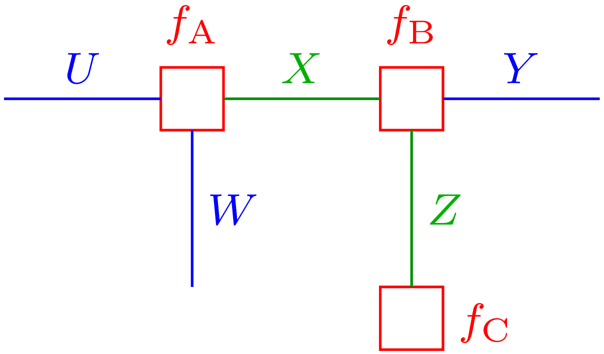

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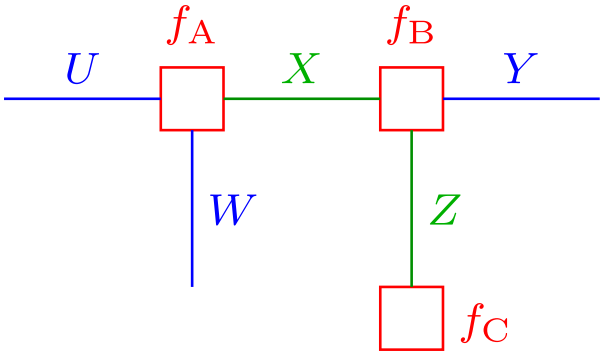

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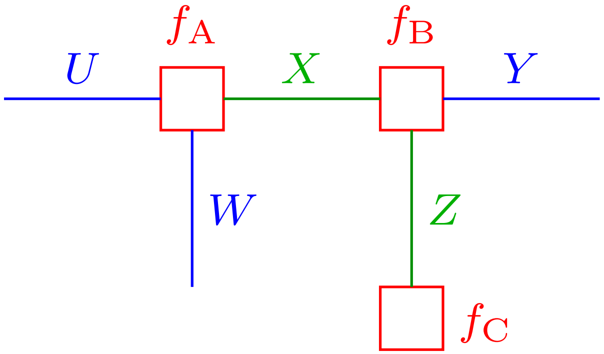

- Factor graphs were defined in [Kschischang:Frey:Loeliger:01].
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$\Rightarrow$ We will call them Forney-style Factor graphs (FFGs).

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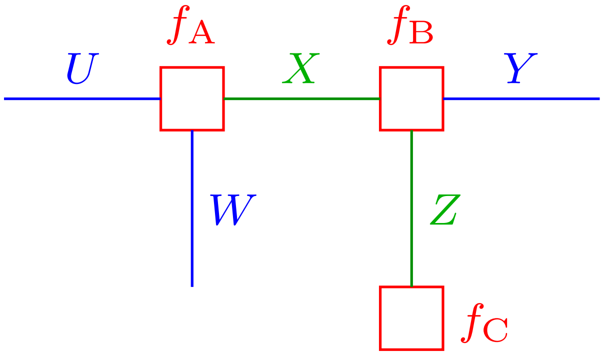

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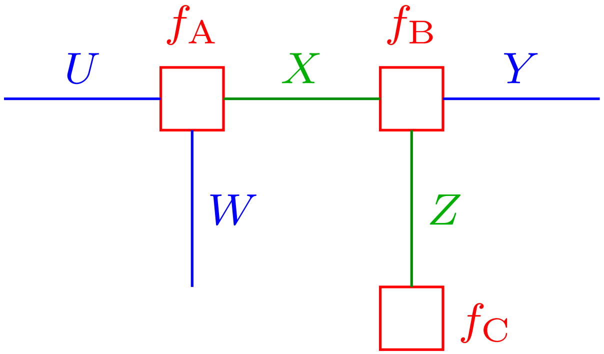

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- the local functions f_{A}, f_{B}, and f_{C},

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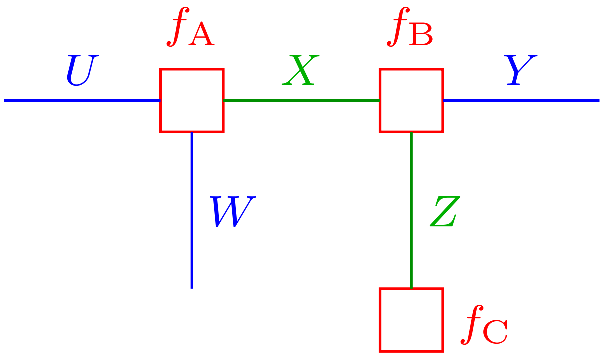

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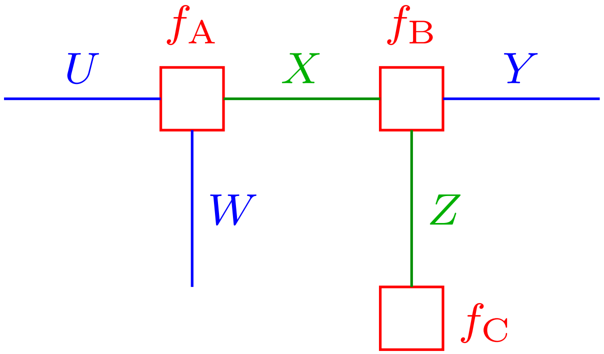

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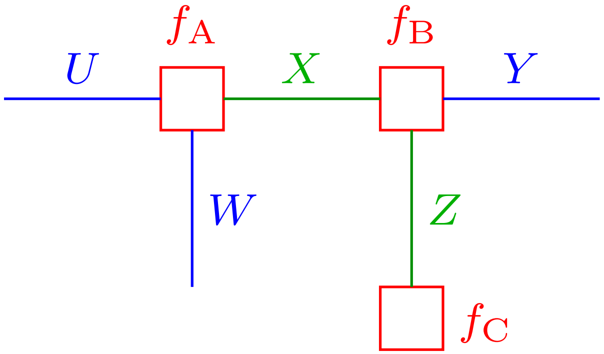

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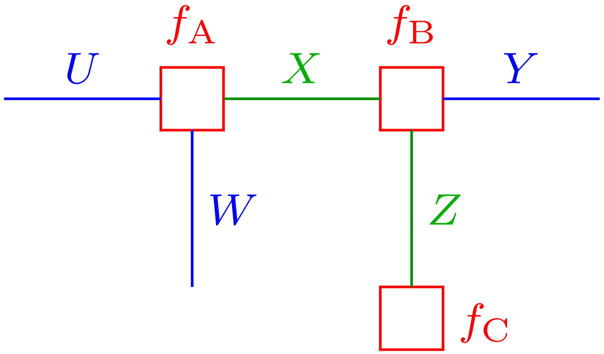
- the half edges corresponding to the variables U, W, and Y,
- and finally the global function $f_{\mathrm{A}}(u, w, x) \cdot f_{\mathrm{B}}(x, y, z) \cdot f_{\mathrm{C}}(z)$.

Forney-style Factor Graphs (FFGs)



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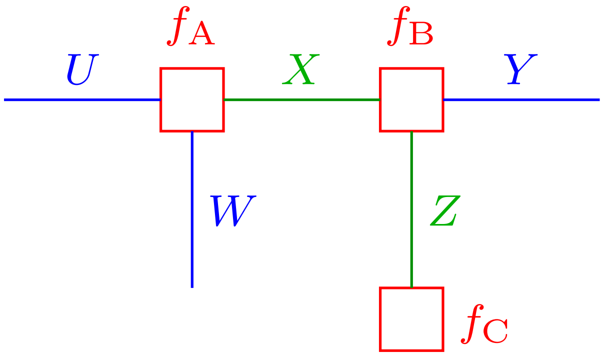

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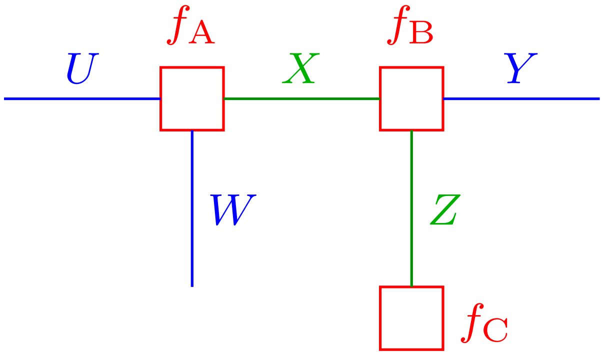

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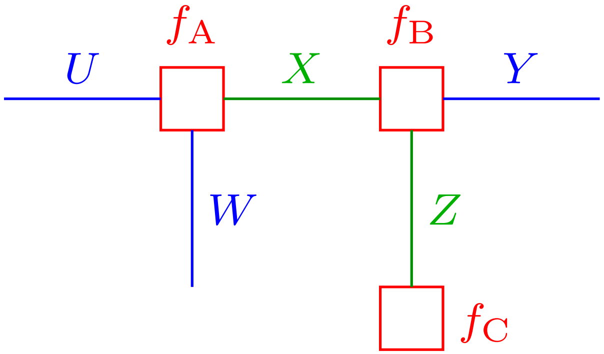

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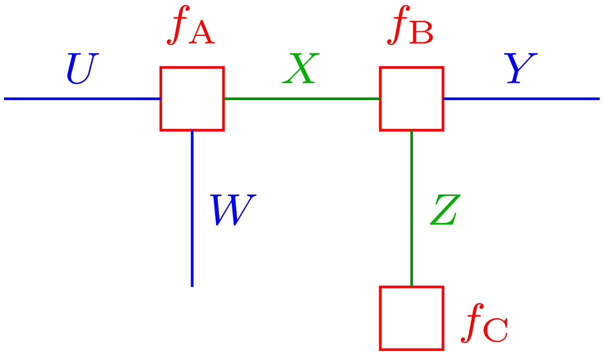

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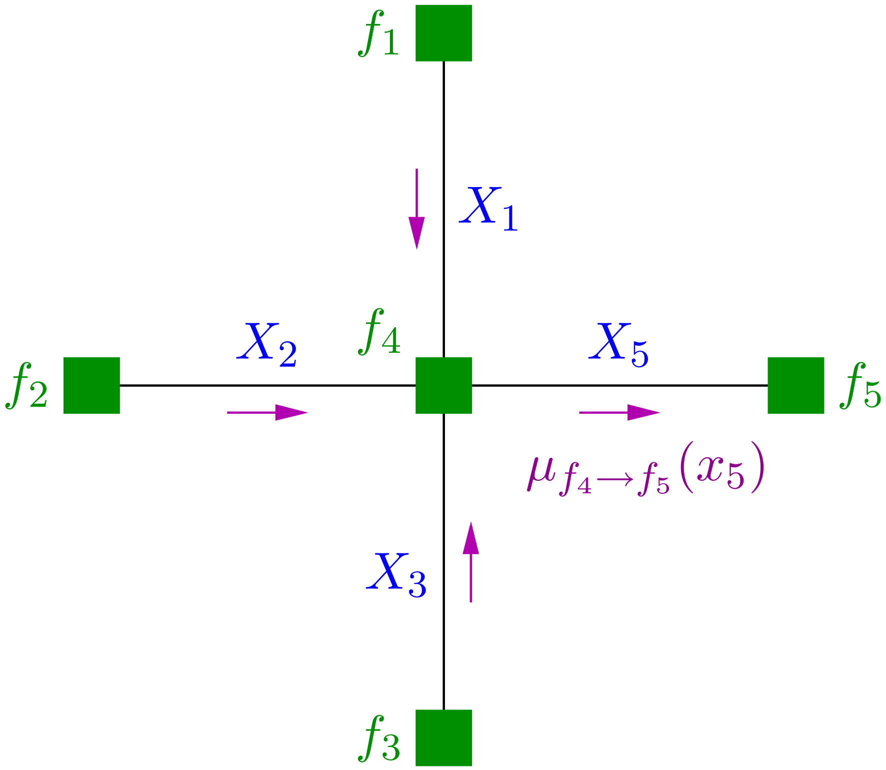
- System variable: $X : \Omega \to A_X : \boldsymbol{\omega} \mapsto x = X(\boldsymbol{\omega})$.

The Sum-Product Algorithm (SPA)

SPA: Update Rule



$$\mu_{f_4 \to f_5}(x_5)$$

$$= \frac{1}{Z_{f_4 \to f_5}} \sum_{x_1} \sum_{x_2} \sum_{x_3} f_4(x_1, x_2, x_3, x_5) \cdot \mu_{f_1 \to f_4}(x_1) \cdot \mu_{f_2 \to f_4}(x_2) \cdot \mu_{f_3 \to f_4}(x_3)$$

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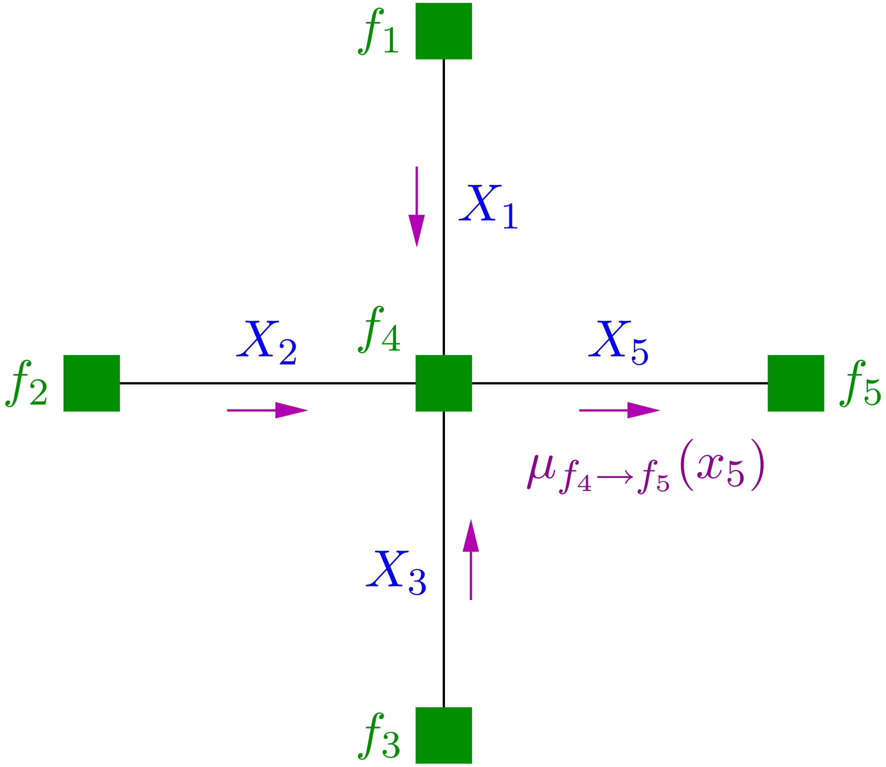

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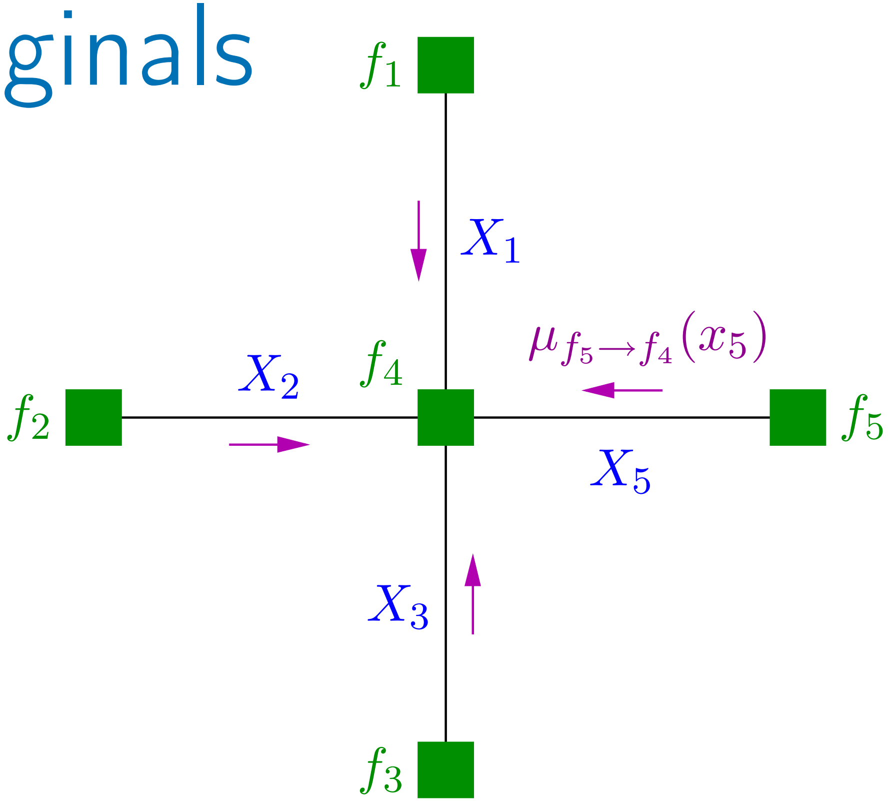
Note: $\frac{1}{Z_{f_4\to f_5}}$ is suitably chosen depending on the setup.

SPA: Computing Marginals



$$\eta_{f_4}(x_1, x_2, x_3, x_5)$$

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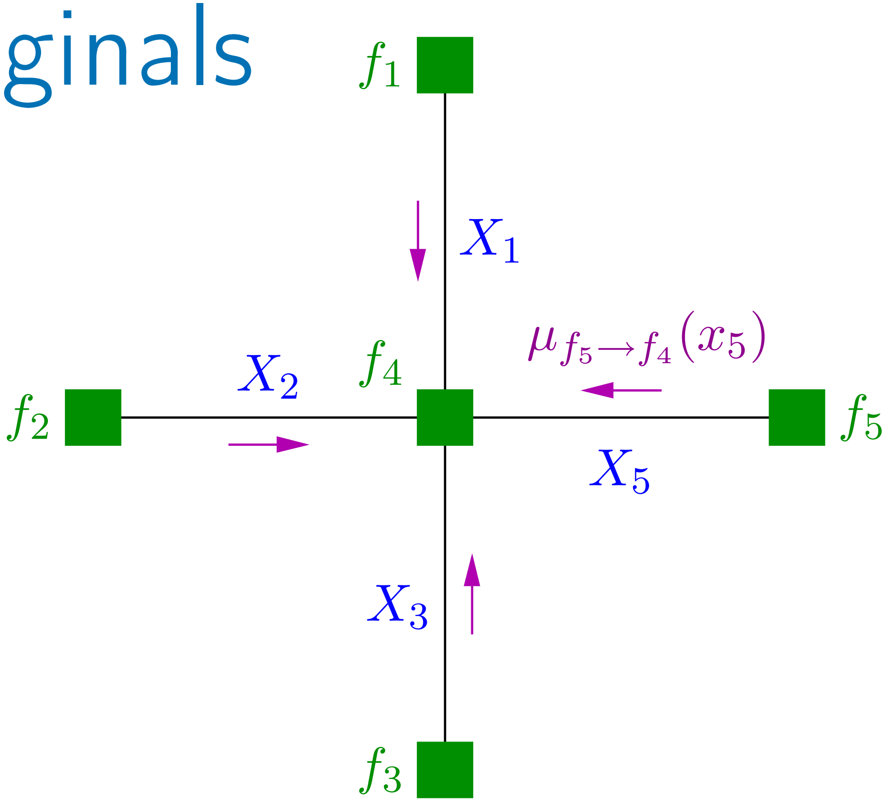

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Fixed points of the SPA correspond to stationary points of the Variational Bethe free energy (VBFE).

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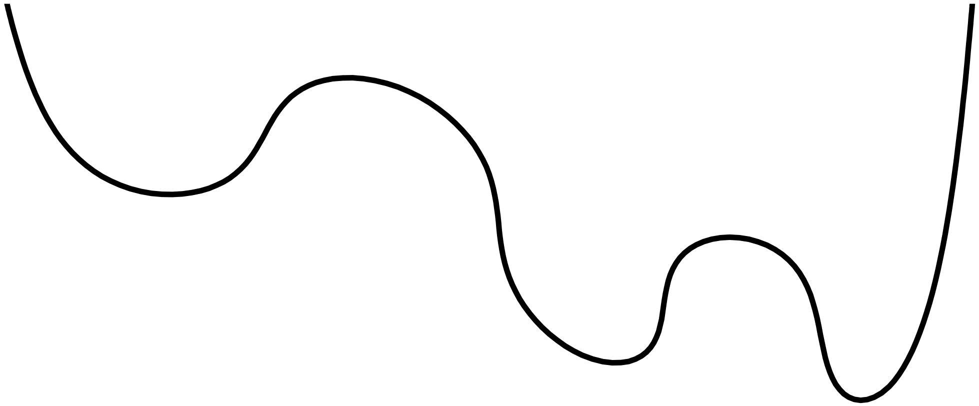

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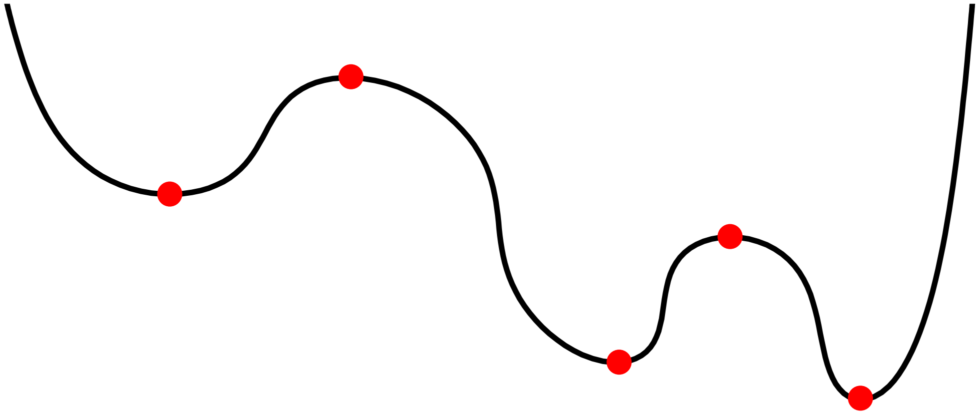

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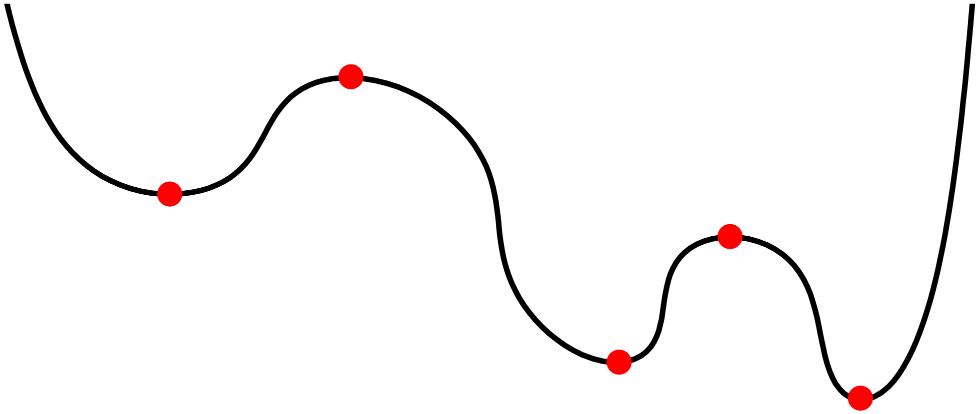

Note that the VBFE is an approximation of the Var. Gibbs free energy:

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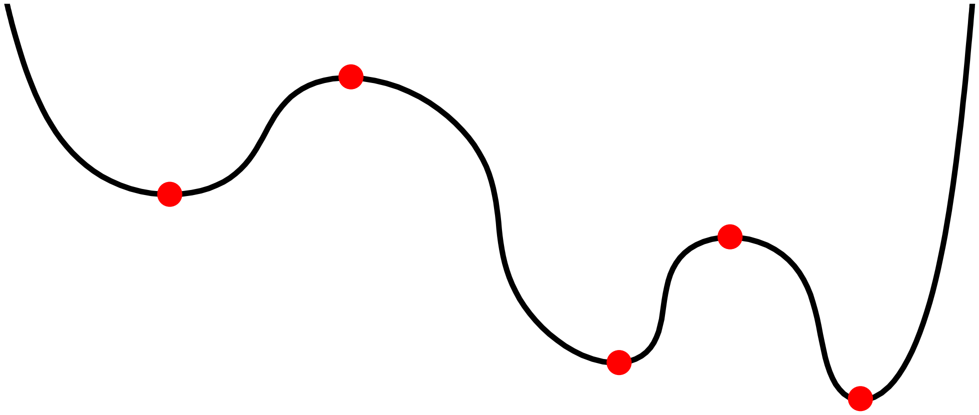

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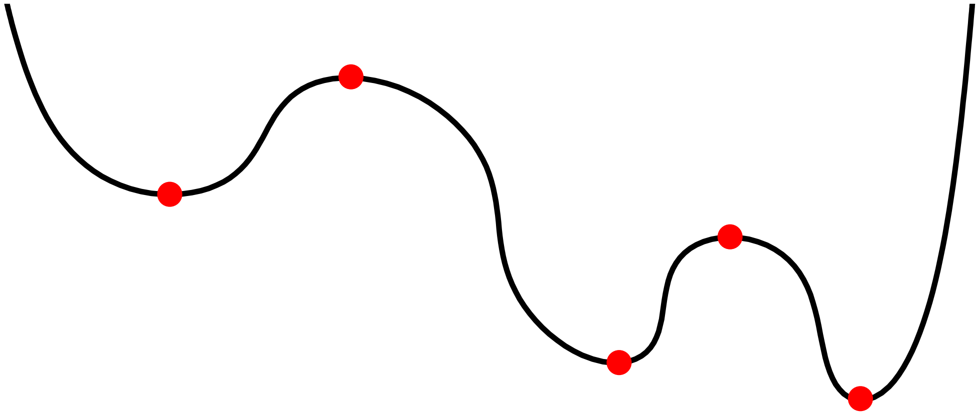


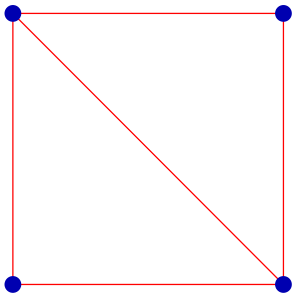
Note that the VBFE is an approximation of the Var. Gibbs free energy:

- If it is an approximation, how is it possible that we obtain an exact result like in the above theorem?
- What is the meaning of the VBFE?

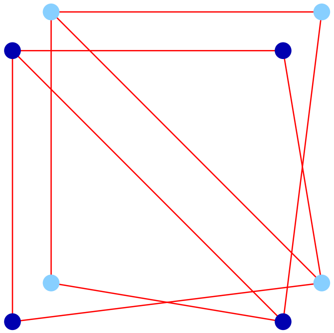
Graph Covers and

Counting Valid Configurations Therein

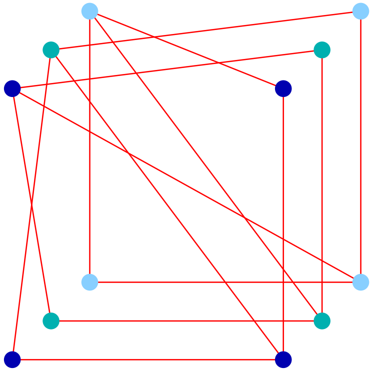
Graph Covers



original graph



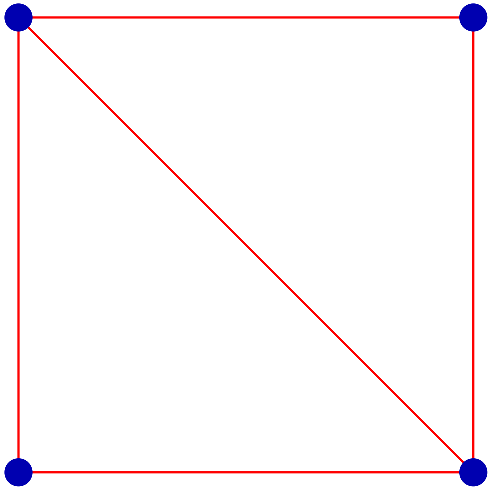
(a possible)
double cover of
the original graph



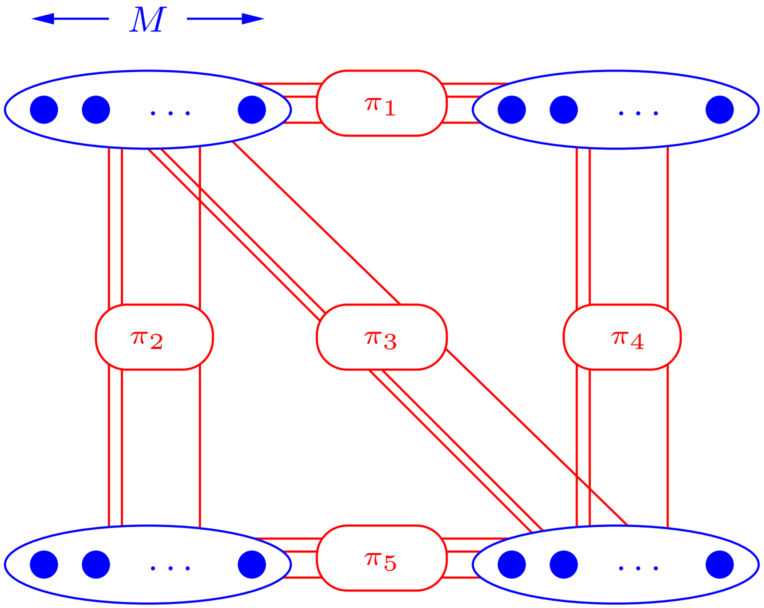
(a possible)
triple cover of
the original graph

. . .

Graph Covers



original graph

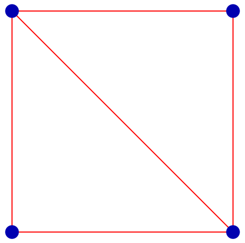


(possible)
M-fold cover of
original graph

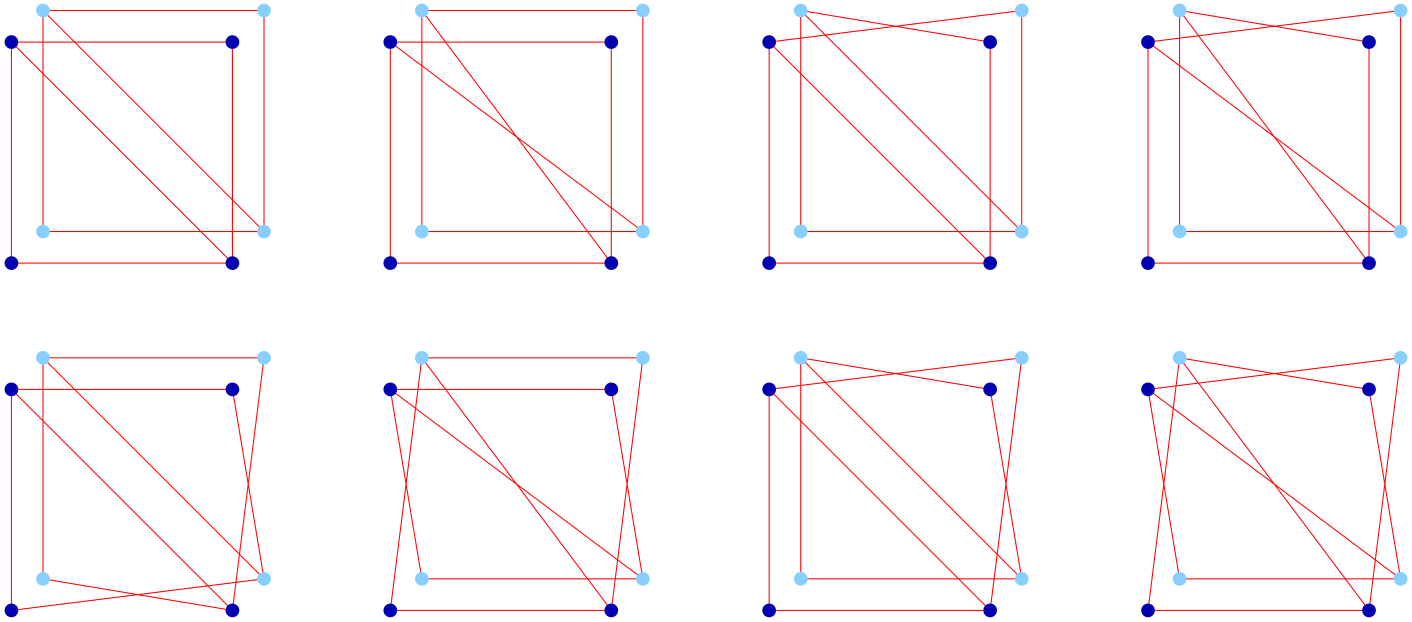
An M-fold cover is also called a cover of degree M. Do not confuse this degree with the degree of a vertex!

Note: there are many possible M-fold covers of a graph.

Graph Covers

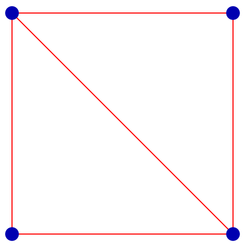


original graph

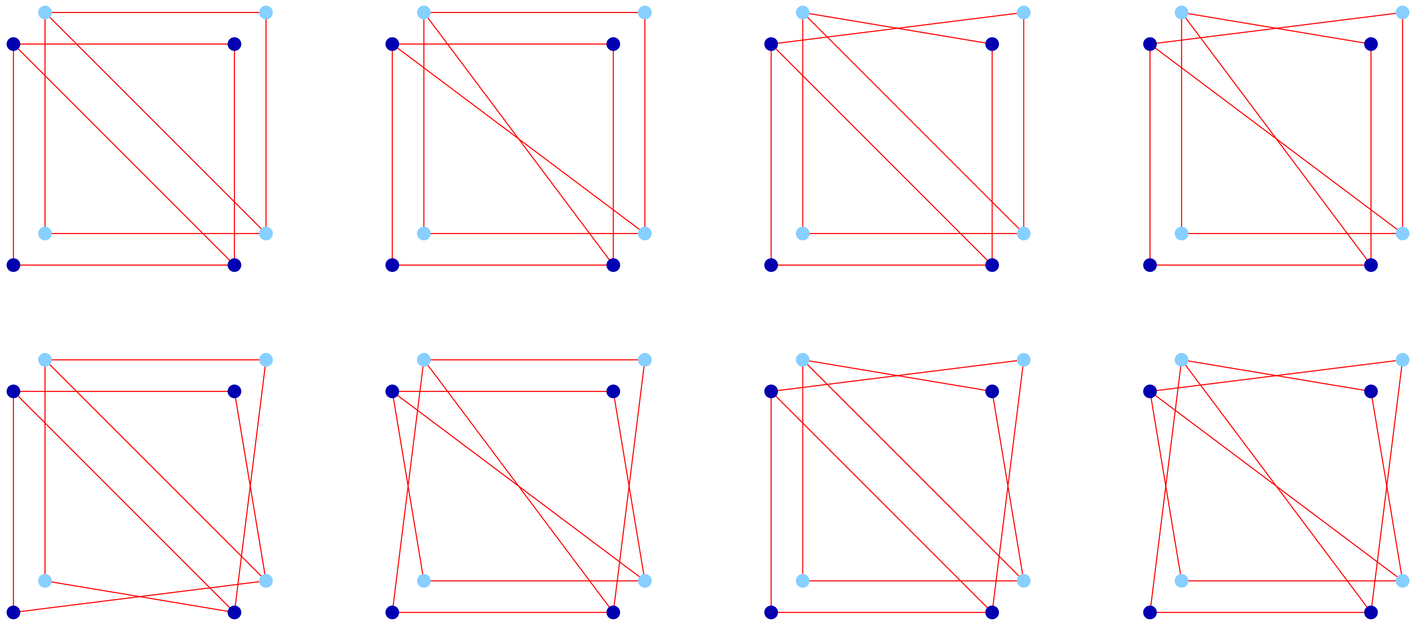


sample of possible
double covers of
the original graph

Graph Covers



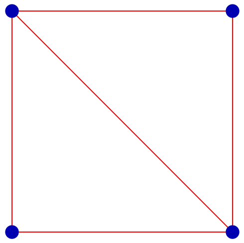
original graph



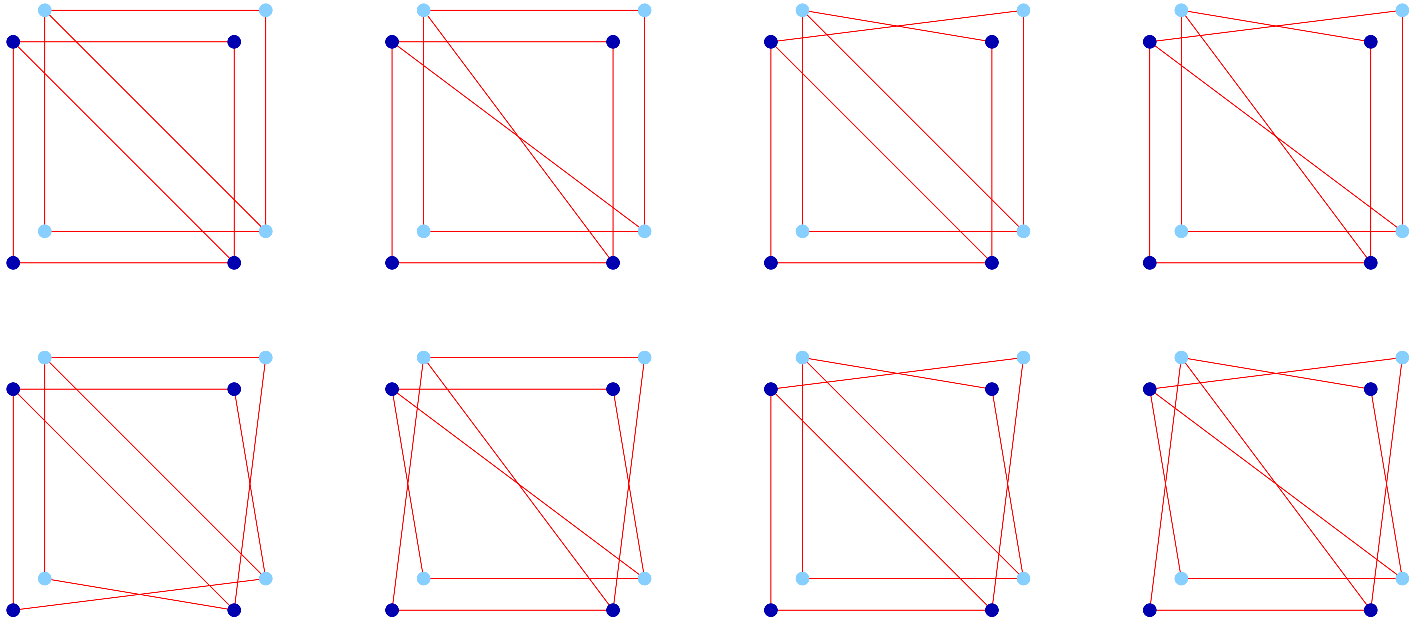
sample of possible
double covers of
the original graph

Note: the above graph has $2! \cdot 2! \cdot 2! \cdot 2! \cdot 2! = 32$ double covers.

Graph Covers



original graph



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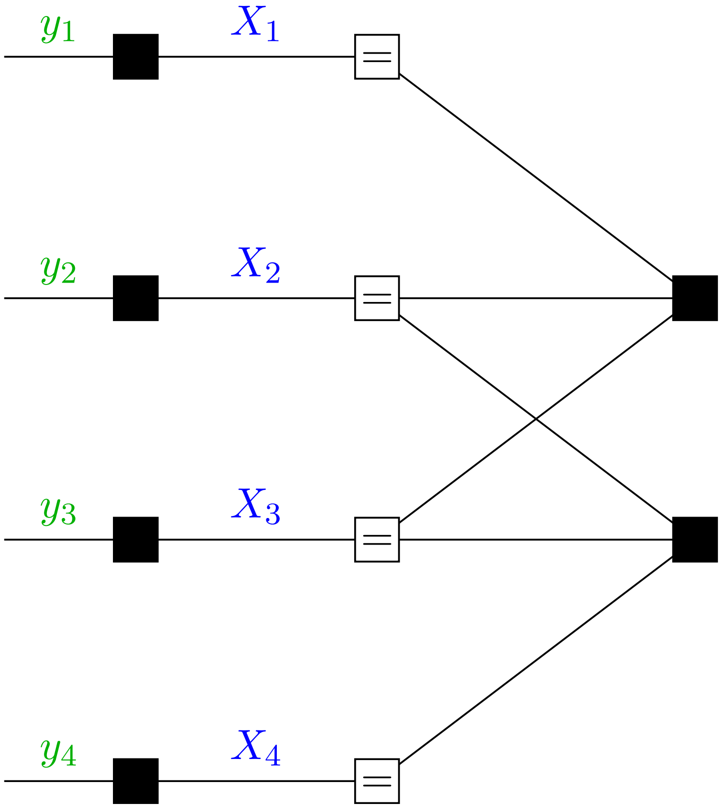
In general: $\#\widetilde{\mathcal{G}}_M \; = \; \#(M\text{-covers of G}) \; = \; (M!)^{\#\text{Edges}(\text{G})}$.

Graph Covers and Valid Configurations Therein



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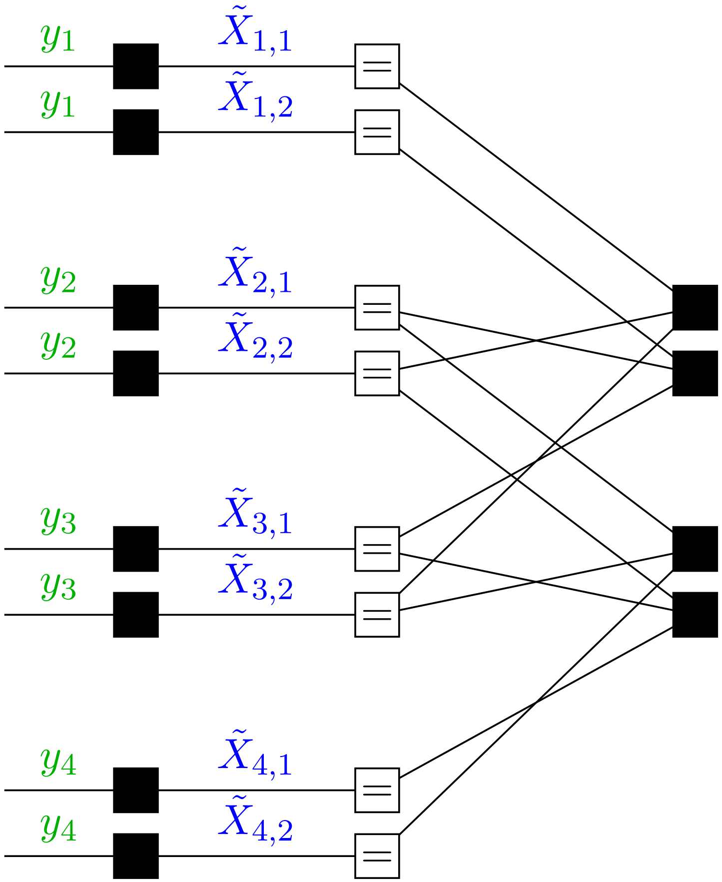

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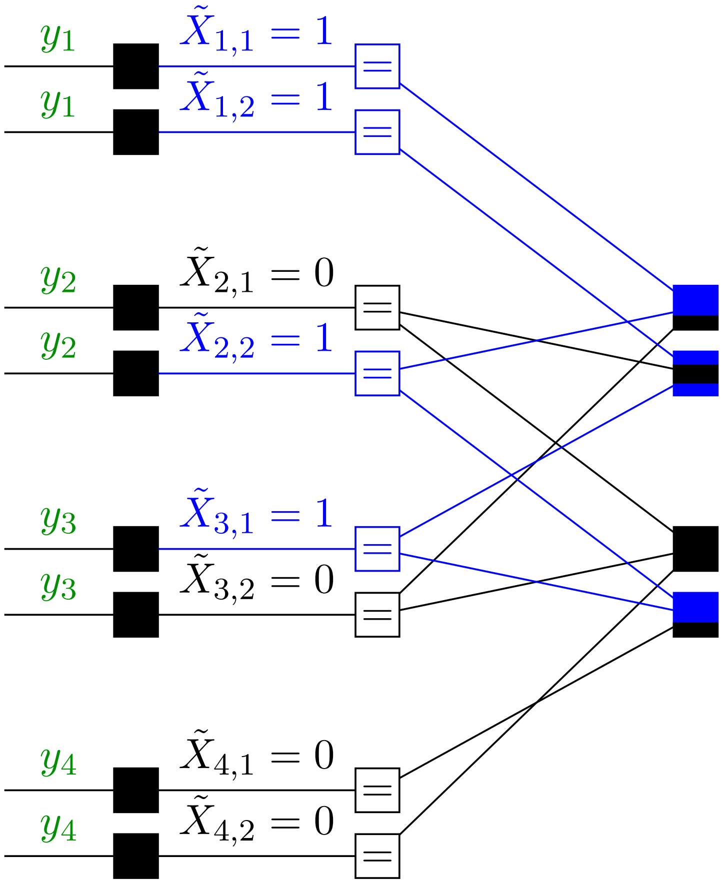

Graph Covers and Valid Configurations Therein

The components of the pseudo-codeword

$$\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$$

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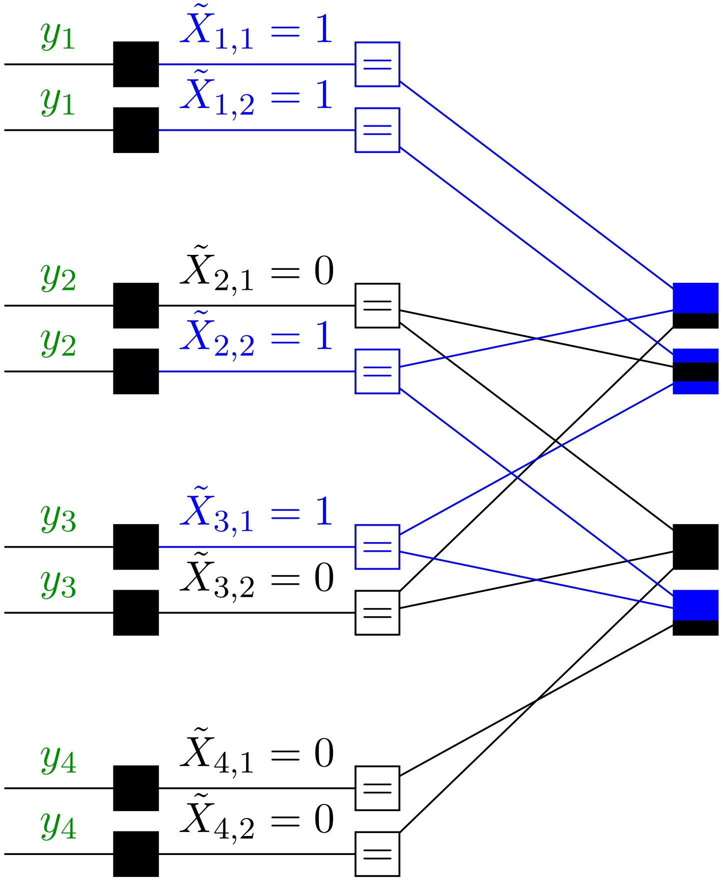
$$\omega_i \triangleq \frac{1}{M} \sum_{m \in [M]} \tilde{x}_{i,m} \ .$$



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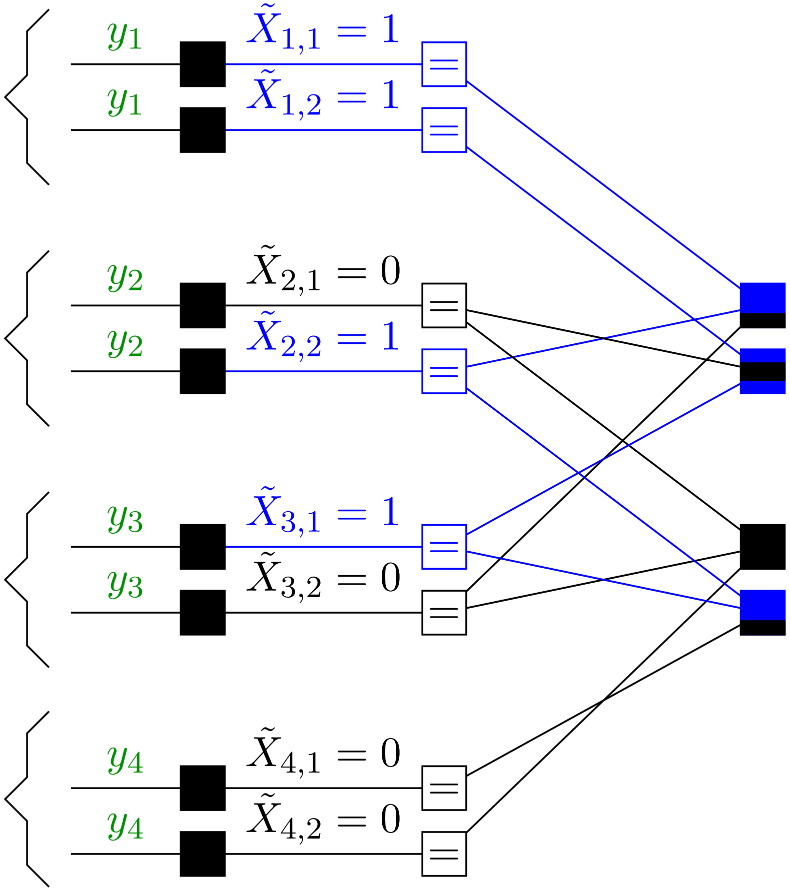

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- Set of $\mathrm{microstates} \triangleq$ set of $\mathrm{microstates}_M$

$$\triangleq \left((\widetilde{\mathsf{G}}, \widetilde{\mathbf{x}}) \;\middle|\; \widetilde{\mathsf{G}} \in \widetilde{\mathcal{G}}_M,\ \widetilde{\mathbf{x}} \text{ is a valid configuration in } \widetilde{\mathcal{G}}_M \right)$$

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- Set of $\mathrm{macrostates} \triangleq$ set of $\mathrm{macrostates}_M$

$$\triangleq \boldsymbol{\varphi}_M(\text{set of } \mathrm{microstates})$$

Graph Covers and Valid Configurations Therein

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- Note:

$$\#(\text{set of } \mathrm{macrostates}) = \mathrm{poly}(M)$$

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$$\mathrm{closure}\left(\lim_{M\to\infty}(\text{set of } \mathrm{macrostates})\right) = \text{fundamental polytope}$$

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Then

$$P(\text{macrostate}) = \exp\Big(-M \cdot \langle \text{macrostate}, \boldsymbol{\lambda}\rangle\Big) \cdot \#\varphi^{-1}(\text{macrostate})$$

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A fixed point of the SPA corresponds to a macrostate $\boldsymbol{\omega}$, i.e., a pseudo-codeword $\boldsymbol{\omega}$, that is a *stationary point* of

$$P(\boldsymbol{\omega}) \ \propto \ \exp\Big(-M \cdot \langle \boldsymbol{\omega}, \boldsymbol{\lambda} \rangle\Big) \cdot \#\varphi^{-1}(\boldsymbol{\omega})$$

when M goes to infinity.

Fixed Points of the SPA

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Fixed points of the SPA correspond to *local minima* of the Variational Bethe free energy (VBFE).

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Justifying this Result (Ingredient 1)

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- Note: The ratio

$$\frac{\#\boldsymbol{\varphi}_M^{-1}(\boldsymbol{\omega})}{\#\widetilde{\mathcal{G}}_M}$$

represents the average number of valid configurations $\tilde{\mathbf{x}}$ per M-fold cover with associated pseudo-codeword $\boldsymbol{\omega}$. Therefore, $H_{\mathrm{Bethe}}(\boldsymbol{\omega})$ gives the asymptotic growth rate of that quantity.

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  - [Fogal/McEliece/Thorpe, 2005],
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- To the best of our knowledge, the above interpretation of the Bethe entropy cannot be found in the literature (besides the talk that we gave at the 2008 Allerton Conference.)

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with

$$\begin{aligned}
-U_{\mathrm{Bethe}}(\boldsymbol{\omega}) &= \langle \boldsymbol{\omega}, \boldsymbol{\lambda} \rangle \\
+H_{\mathrm{Bethe}}(\boldsymbol{\omega}) &= \lim_{M \to \infty} \left(\frac{1}{M} \log \#\boldsymbol{\varphi}_M^{-1}(\boldsymbol{\omega}) - \frac{1}{M} \log \#\widetilde{\mathcal{G}}_M \right)
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The Transient Part of the SPA

Graph-Dynamical Systems

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- Let Γ be a set of graphs.
- Let Ψ be some (possibly random) mapping from Γ to Γ.
- Because the domain and the range of Ψ are equal, it makes sense to study the repeated application of the mapping Ψ:

$$\Gamma \quad \xrightarrow{\Psi} \quad \Gamma \quad \xrightarrow{\Psi} \quad \cdots \quad \xrightarrow{\Psi} \quad \Gamma$$

Review

(of the setup used in the re-interpretation of f.p.s of the SPA)

- Set of microstates

$$\triangleq \left((\widetilde{\mathsf{G}}, \widetilde{\mathbf{x}}) \;\middle|\; \widetilde{\mathsf{G}} \in \widetilde{\mathcal{G}}_M,\ \widetilde{\mathbf{x}} \text{ is a valid configuration in } \widetilde{\mathcal{G}}_M \right)$$

- Mapping $\boldsymbol{\varphi}_M$

$$\text{maps} \quad (\widetilde{\mathsf{G}}, \widetilde{\mathbf{x}}) \quad \text{to} \quad \boldsymbol{\omega}(\widetilde{\mathbf{x}})$$

- Set of macrostates

$$\triangleq \boldsymbol{\varphi}_M(\text{set of microstates})$$

Corresponding Setup for the Transient Part of the SPA

- Set of microstates

$$???$$

- Mapping φ_M

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Note: $\Gamma =$ set of M-covers of G and valid configurations therein is obviously not sufficient.

Corresponding Setup for the Transient Part of the SPA

- Set of microstates

  $\Rightarrow$ $\Gamma =$ set of what we call colored hypergraph M-cover

  or colored twisted M-cover

- Mapping $\boldsymbol{\varphi}_M$

  ???

- Set of macrostates

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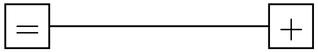
- Set of macrostates

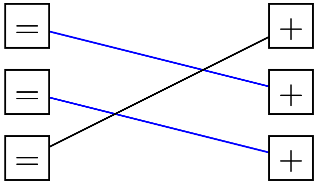
set of all possible marginals on the LHS function nodes
× set of all possible marginals on the RHS function nodes

Comment on Microstates



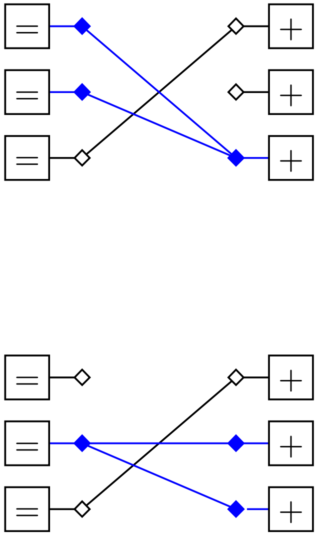
edge

in FFG



corrsponding edges

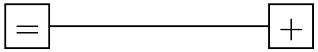
in some colored 3-cover



corresponding edges

in colored hypergraph 3-cover

Comment on Microstates



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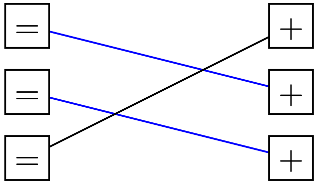
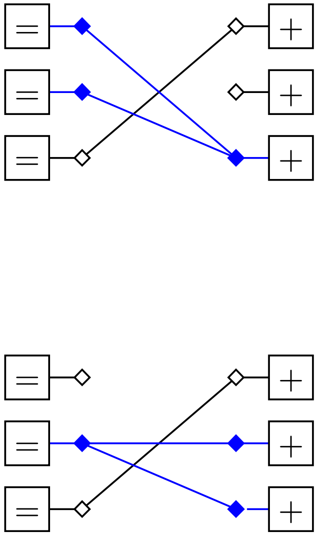


corrsponding edges

in some colored 3-cover

LHS and RHS marginals must match



corresponding edges

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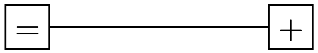

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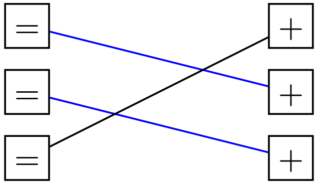

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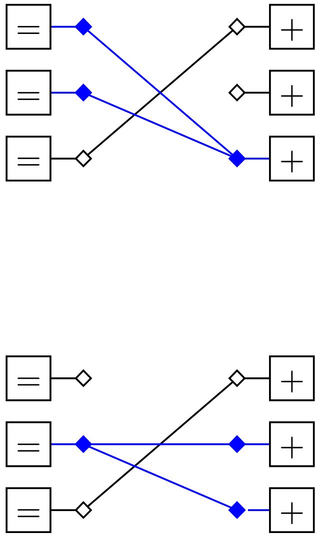

corresponding edges

in colored hypergraph 3-cover

LHS and RHS marginals
do not have to match

Comment on Macrostates

$$\begin{tikzcd}
\overleftarrow{\boldsymbol{\mu}}(0) \arrow[r] \arrow[d] & \overrightarrow{\boldsymbol{\mu}}(0.5) \arrow[r] \arrow[d] & \overleftarrow{\boldsymbol{\mu}}(1) \arrow[r] \arrow[d] & \overrightarrow{\boldsymbol{\mu}}(1.5) \arrow[r] \arrow[d] & \overleftarrow{\boldsymbol{\mu}}(2) \arrow[r] \arrow[d] & {} \\
\boldsymbol{\eta}_{\text{left}}(0) & \boldsymbol{\eta}_{\text{right}}(0.5) & \boldsymbol{\eta}_{\text{left}}(1) & \boldsymbol{\eta}_{\text{right}}(1.5) & \boldsymbol{\eta}_{\text{left}}(2) & .
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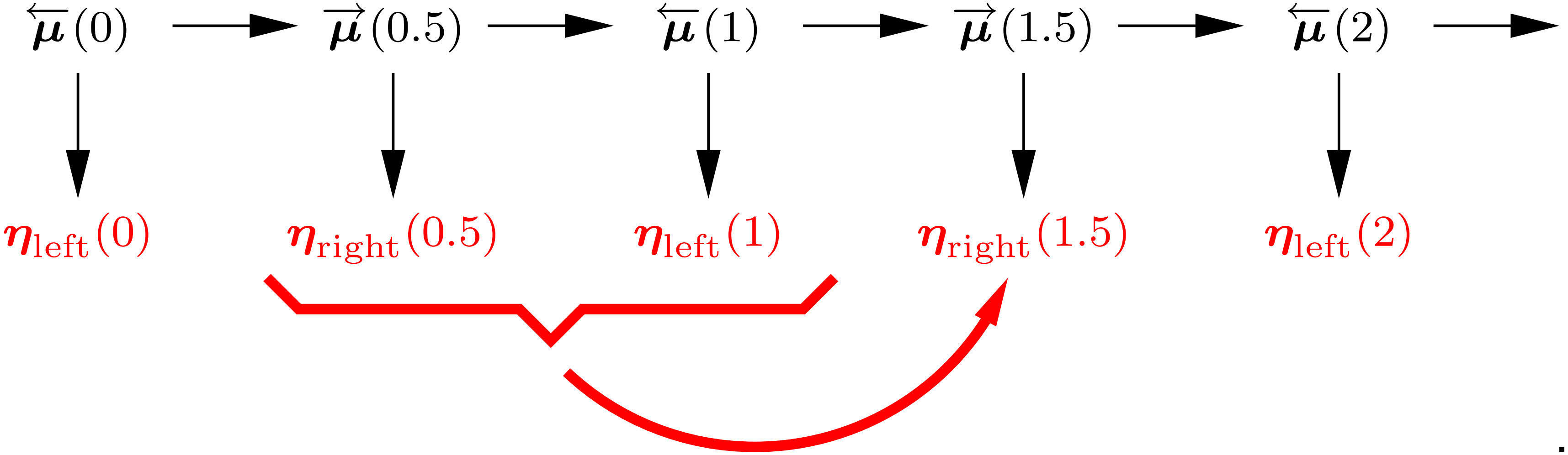

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\boxed{\begin{array}{c}\boldsymbol{\eta}_{\text{left}}(0)\\ \boldsymbol{\eta}_{\text{right}}(-.5)\end{array}} \arrow[r] & \boxed{\begin{array}{c}\boldsymbol{\eta}_{\text{left}}(0)\\ \boldsymbol{\eta}_{\text{right}}(0.5)\end{array}} \arrow[r] & \boxed{\begin{array}{c}\boldsymbol{\eta}_{\text{left}}(1)\\ \boldsymbol{\eta}_{\text{right}}(0.5)\end{array}} \arrow[r] & \boxed{\begin{array}{c}\boldsymbol{\eta}_{\text{left}}(1)\\ \boldsymbol{\eta}_{\text{right}}(1.5)\end{array}} \arrow[r] & \boxed{\begin{array}{c}\boldsymbol{\eta}_{\text{left}}(2)\\ \boldsymbol{\eta}_{\text{right}}(1.5)\end{array}} \arrow[r] & {} .
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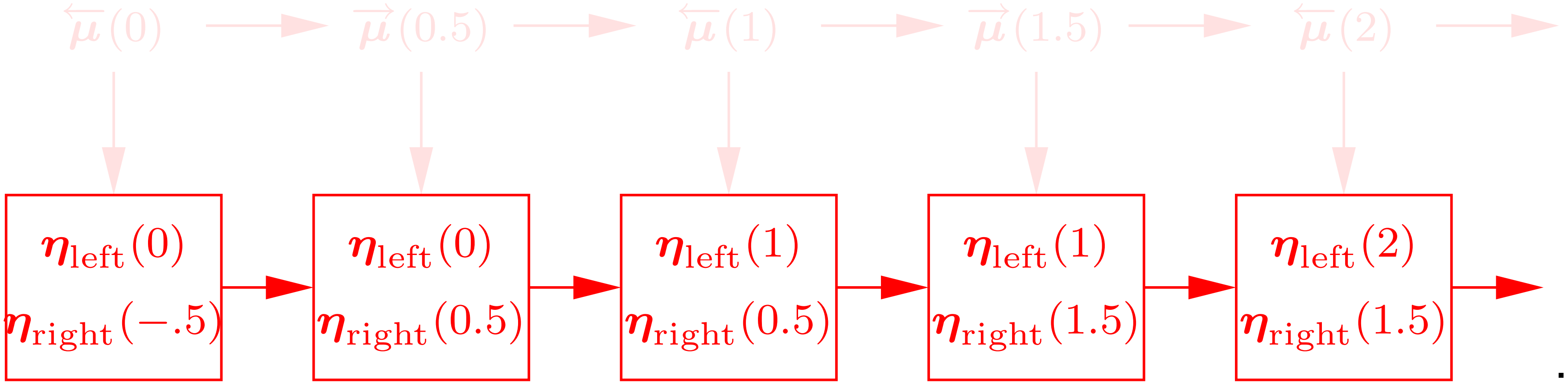

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$\boldsymbol{\eta}_{\text{left}}(0)$ $\boldsymbol{\eta}_{\text{right}}(-.5)$

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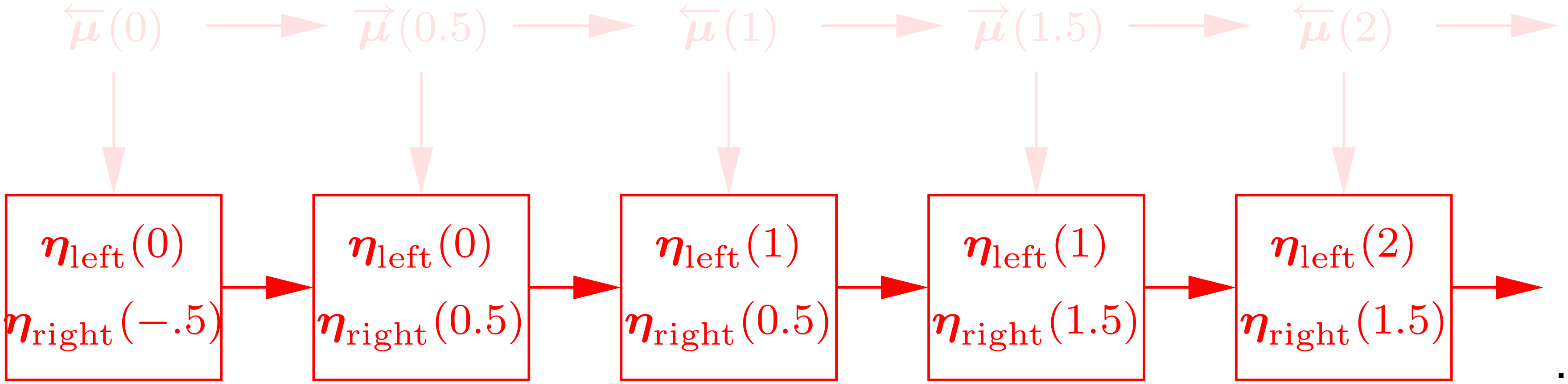

⇒ This can be considered as a "message-free version of the SPA".

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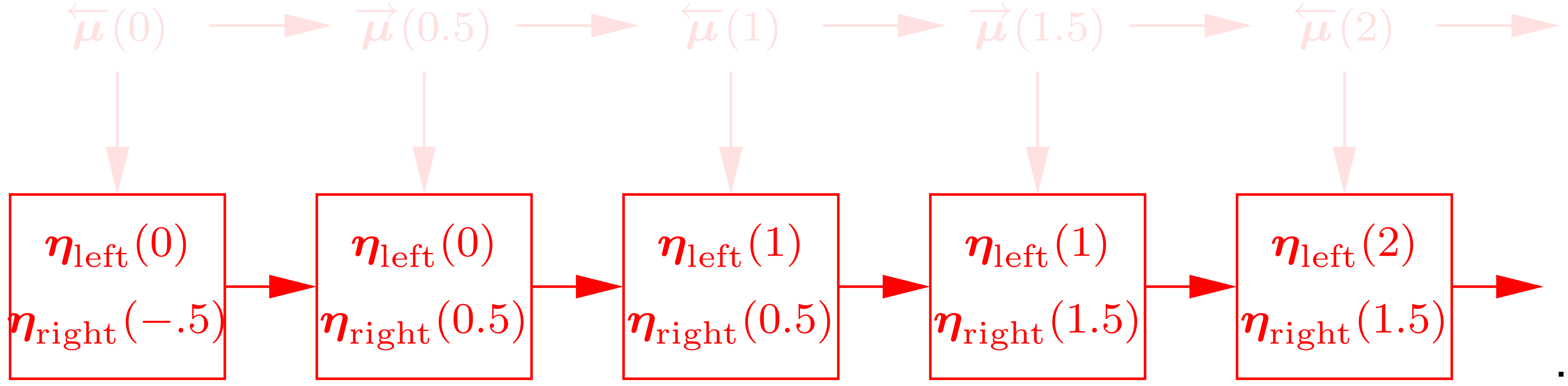

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Cf. “Message-free version of belief-propagation”
in [Wainwright/Jaakkola/Willsky, 2003].

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- Talked about microstates, macrostates, and their uses.
- Given a re-interpretation of fixed points of the SPA
  in terms of graph covers and valid configurations therein.
- Touched upon a re-interpretation of the transient part of the SPA
  in terms of a graph-dynamical system.

Thank you!

LABS hp