Wavelet Analysis: Theory and Applications

Wavelet analysis has attracted attention for its ability to analyze rapidly changing transient signals. Any application using the Fourier transform can be formulated using wavelets to provide more accurately localized temporal and frequency information. This paper gives an overview of wavelet analysis and describes a software toolbox created by HP Laboratories Japan to aid in the development of wavelet applications.

by Daniel T.L. Lee and Akio Yamamoto

Wavelet analysis (also called wavelet theory, or just wavelets) has attracted much attention recently in signal processing. It has been successfully applied in many applications such as transient signal analysis, image analysis, communications systems, and other signal processing applications. It is not a new theory in the sense that many of the ideas and techniques involved in wavelets (subband coding, quadrature mirror filters, etc.) were developed independently in various signal processing applications and have been known for some time. What is new is the development of recent results on the mathematical foundations of wavelets that provide a unified framework for the subject.

Within this framework a common link is established between the many diversified problems that are of interest to different fields, including electrical engineering (signal processing, data compression), mathematical analysis (harmonic analysis, operator theory), and physics (fractals, quantum field theory). Wavelet theory has become an active area of research in these fields. There are opportunities for further development of both the mathematical understanding of wavelets and a wide range of applications in science and engineering.

Like Fourier analysis, wavelet analysis deals with expansion of functions in terms of a set of basis functions. Unlike Fourier analysis, wavelet analysis expands functions not in terms of trigonometric polynomials but in terms of *wavelets,* which are generated in the form of translations and dilations of a fixed function called the *mother wavelet.* The wavelets obtained in this way have special scaling properties. They are localized in time and frequency, permitting a closer connection between the function being represented and their coefficients. Greater numerical stability in reconstruction and manipulation is ensured.

The objective of wavelet analysis is to define these powerful wavelet basis functions and find efficient methods for their computation. It can be shown that every application using the fast Fourier transform (FFT) can be formulated using wavelets to provide more localized temporal (or spatial) and frequency information. Thus, instead of a frequency spectrum, for example, one gets a wavelet spectrum. In signal processing, wavelets are very useful for processing nonstationary signals.

Wavelets have created much excitement in the mathematics community (perhaps more so than in engineering) because the mathematical development has followed a very interesting path. The recent developments can be viewed as resolving some of the difficulties inherent in Fourier analysis. For example, a typical question is how to relate the Fourier coefficients to the global or local behavior of a function. The development of wavelet analysis can be considered an outgrowth of the Littlewood-Paley theory[1] (first published in 1931), which sought a new approach to answer some of these difficulties. Again, it is the unifying framework made possible by recent results in wavelet theory related to problems of harmonic analysis (also to similar problems in operator theory called the Calderón-Zygmund theory[1]) that has generated much of the excitement.

In electrical engineering, there have been independent developments in the analysis of nonstationary signals, specifically in the form of the short-term Fourier transform, a variation of which called the Gabor transform was first published in 1946.[2] A major advance in wavelet theory was the discovery of smooth mother wavelets whose set of discrete translations and dilations forms an orthonormal basis for $L^2(\mathbf{R})$, where $\mathbf{R}$ is the real numbers and L^2 is the set of all functions, f, that have bounded energy, that is, functions for which

$$\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty .$$

This is a main difference from the Gabor transform. In the Gabor case, no orthonormal basis can be generated from smooth wavelets. Thus the unifying framework brought about a better understanding and a new approach that overcomes the difficulties in the short-term Fourier transform methods.

In the next section we give an overview of the main features of wavelet analysis and then turn to a software toolbox that

HP Laboratories Japan has developed to help in the development of wavelet applications.

For an excellent tutorial introduction to the subject, see Rioul and Vetterli[2] and the references therein (it lists 106 references). Daubechies' book[1] is a standard reference on the subject.

Fundamentals of Wavelet Theory

This section gives a quick overview of the main formulas.

The Analyzing Wavelet

Consider a complex-valued function ψ satisfying the following conditions:

$$\int_{-\infty}^{\infty} |\psi(t)|^2 dt < \infty \qquad (1)$$

$$c_\psi = 2\pi \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty, \qquad (2)$$

where Ψ is the Fourier transform of ψ. The first condition implies finite energy of the function ψ, and the second condition, the admissibility condition, implies that if $\Psi(\omega)$ is smooth then $\Psi(0) = 0$.

The function ψ is the mother wavelet.

Continuous Wavelet Transform

If ψ satisfies the conditions described above, then the *wavelet transform* of a real signal s(t) with respect to the wavelet function $\psi(t)$ is defined as:

$$S(b, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \psi'\left(\frac{t-b}{a}\right) s(t) dt, \qquad (3)$$

where ψ' denotes the complex conjugate of ψ, and this is defined on the open (b,a) half-plane ($b \in \mathbf{R}$, $a > 0$). The parameter b corresponds to the time shift and the parameter a corresponds to the scale of the analyzing wavelet.

If we define $\psi_{a,b}(t)$ as

$$\psi_{a,b}(t) = a^{-1/2} \psi\left(\frac{t-b}{a}\right), \qquad (4)$$

which means rescaling by a and shifting by b, then equation 3 can be written as a scalar or inner product of the real signal s(t) with the function $\psi_{a,b}(t)$:

$$S(b, a) = \int_{-\infty}^{\infty} \psi'_{a,b}(t) s(t) dt. \qquad (5)$$

When function $\psi(t)$ satisfies the admissibility condition, equation 2, the original signal s(t) can be obtained from the wavelet transform S(b,a) by the following inverse formula:

$$s(t) = \frac{1}{c_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(b, a) \psi_{a,b}(t) \frac{da\,db}{a^2}. \qquad (6)$$

Discrete Wavelet Transform

In the discrete domain, the scale and shift parameters are discretized as $a = a_0^m$ and $b = nb_0$, and the analyzing wavelets are also discretized as follows:

$$\psi_{m,n}(t) = a_0^{-m/2} \psi\left(\frac{t - nb_0}{a_0^m}\right), \qquad (7)$$

where m and n are integer values. The discrete wavelet transform and its inverse transform are defined as follows:

$$S_{m,n} = \int_{-\infty}^{\infty} \psi'_{m,n}(t) s(t) dt, \qquad (8)$$

$$s(t) = k_\psi \sum_m \sum_n S_{m,n} \psi_{m,n}(t), \qquad (9)$$

where k_ψ is a constant value for normalization.

The function $\psi_{m,n}(t)$ provides sampling points on the scale-time plane: linear sampling in the time (b-axis) direction but logarithmic in the scale (a-axis) direction.

The most common situation is that a_0 is chosen as:

$$a_0 = 2^{1/v}, \qquad (10)$$

where v is an integer value, and that v pieces of $\psi_{m,n}(t)$ are processed as one group, which is called a *voice.* The integer v is the number of voices per octave; it defines a well-tempered scale in the sense of music. This is analogous to the use of a set of narrowband filters in conventional Fourier analysis.

Wavelet analysis is not limited to dyadic scale analysis. By using an appropriate number of voices per octave, wavelet analysis can effectively perform the 1/3-octave, 1/6-octave, or 1/12-octave analyses that are used in acoustics.

The main focus of current research is on finding optimal wavelet basis functions and efficient algorithms for computing the corresponding wavelet transforms. The wavelet basis function can be implemented as an FIR (finite impulse response) filter or an IIR (infinite impulse response) filter depending on the particular properties needed.

Graphical Representation

This section describes how to display complex-valued functions such as equations 3 and 8 so that useful information about the signal s(t) can be highlighted. There are two aspects to consider.

The open (b,a) half-plane on which the wavelet transform is defined can be mapped onto the full plane (b,–log(a)). This representation is indispensable if we want to display, in a single picture, information with a wide range of scale parameters. For example, for sound signals in the audible range, a spread of ten octaves is common. A disadvantage of this representation, on the other hand, is that straight lines on the open (b,a) half-plane become exponential curves in the logarithmic representation.

Expressions 3 and 8 depend on the choice of the analyzing wavelet ψ. To obtain full quantitative information about the signal s(t) from its transform S(b,a), we need to know the analyzing wavelet ψ. There are, however, many features of the signal that are independent of the choice of ψ. Such features involve the phase of the complex-valued functions. Therefore, it is useful to represent separately the modulus and the phase of the complex-valued function S(b,a) to be described.

Shown in Figs. 1 and 2 is an example of the wavelet transform of a localized pulse that approximates a delta function.

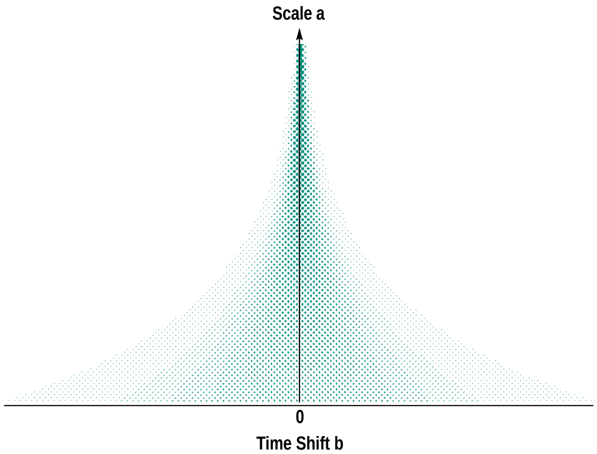

Fig. 1. Magnitude of the wavelet transform of a delta function.

The horizontal axis is time in both the magnitude picture, Fig. 1, and the phase picture, Fig. 2, and the vertical axis is scale, with small scale at the top.

In Fig. 1, the magnitude increases toward the top of the picture. The modulus or magnitude, $|S(b,a)|$, is converted to grayscale and is normalized to its maximum, that is, the plot shows x, where:

$$x = \frac{|S|}{|S_{max}|} \le 1. \tag{11}$$

The phase of S(b,a) is given by a grayscale picture in which a phase of 0 corresponds to white and a phase of 2π to black. This convention is quite useful in interpreting the resulting picture. When the phase reaches 2π, it is wrapped around to the value 0. The lines where the density drops abruptly to zero are clearly visible on the picture and play an important role in the interpretation as a visible line of constant phase. In Fig. 2, one can see the lines of constant phase pointing to the location of the delta function.

Examples of Wavelet Functions

Haar Wavelet. The Haar wavelet is the simplest kind of wavelet function. Suppose that $\phi(t)$ is a box function satisfying the following:

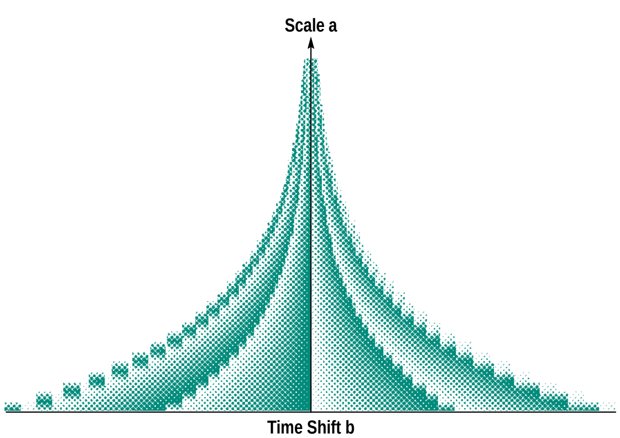

Fig. 2. Phase of the wavelet transform of a delta function.

$$\phi(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

If we define the function $\psi(t)$ as $\psi(t) = \phi(2t) - \phi(2t-1)$, we can obtain the following function:

$$\psi(t) = \begin{cases} 1 & \text{if } 0 < t \le 1/2 \\ -1 & \text{if } 1/2 < t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

The function $\phi(t)$ is the Haar scaling function, and $\psi(t)$ is the Haar wavelet. This function is orthogonal to its own translations and dilations, that is, the family

$$\psi_{m,n}(t) = 2^{-m/2}\,\psi(2^{-m}t - n), \qquad m,n \in \mathbf{Z}, \tag{14}$$

where $\mathbf{Z}$ is the real integers, constitutes an orthonormal basis for $L^2(\mathbf{R})$. Historically the Haar function was the original wavelet. This wavelet is not continuous, and its Fourier transform $\Psi(\omega)$ decays only like $|\omega|^{-1}$, corresponding to bad frequency localization.

Meyer Wavelet. Yves Meyer constructed a smooth orthonormal wavelet basis as follows. First of all, define the Fourier transform $\Phi(\omega)$ of a scaling function $\phi(t)$ as:

$$\Phi(\omega) = \begin{cases} 1 & \text{if } |\omega| \le \frac{2}{3}\pi \\ \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\omega| - 1\right)\right] & \text{if } \frac{2}{3}\pi \le |\omega| \le \frac{4}{3} \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where ν is a smooth function satisfying the following conditions:

$$\nu(t) = \begin{cases} 0 & \text{if } t \le 0 \\ 1 & \text{if } t \ge 1 \end{cases} \tag{16}$$

with the additional property

$$\nu(t) + \nu(1-t) = 1. \tag{17}$$

This function Φ is plotted in Fig. 3.

In this case, the wavelet function ψ can be found easily from Φ. First, we find the Fourier transform of ψ:

$$\Psi(\omega) = e^{i\omega/2}\sum_{l\in\mathbf{Z}}\Phi(\omega + 2\pi(2l+1))\Phi(\omega/2) \tag{18}$$

$$= e^{i\omega/2}[\Phi(\omega + 2\pi) + \Phi(\omega - 2\pi)]\Phi(\omega/2). \tag{19}$$

The function Ψ is plotted in Fig. 4.

Now since Ψ is compactly supported (its duration is finite and nonzero) and $\Psi \in C_k$ where k is arbitrary and may be ∞ (i.e., Ψ has at least k derivatives), ψ can be obtained by taking the inverse Fourier transform. Fig. 5 shows a graph of the Meyer wavelet $\psi(t) \in C^4$.

Morlet Wavelet. This particular function was most often used by R. Kronland-Martinet and J. Morlet. Its Fourier transform is a shifted Gaussian, adjusted slightly so that $\Psi(0) = 0$:

$$\Psi(\omega) = e^{-(\omega-\omega_0)^2/2} - e^{-\omega^2/2}e^{-\omega_0^2/2} \tag{20}$$

$$\psi(t) = \left(e^{-i\omega_0 t} - e^{-\omega^2/2}\right)e^{-t^2/2}. \tag{21}$$

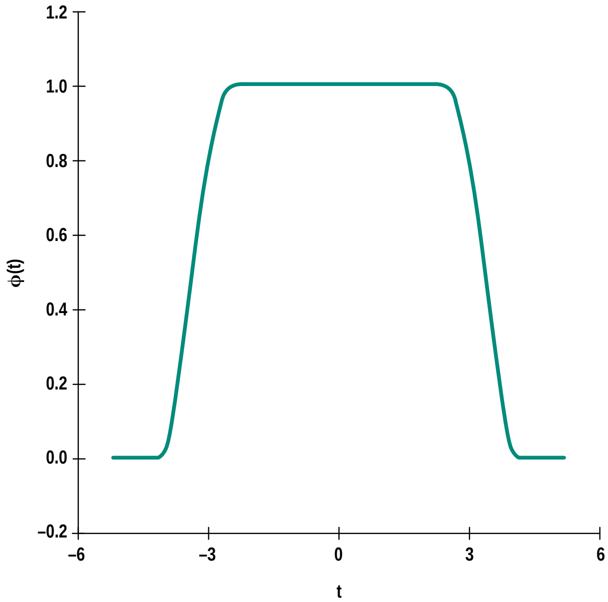

Fig. 3. Fourier transform of the scaling function for the Meyer basis.

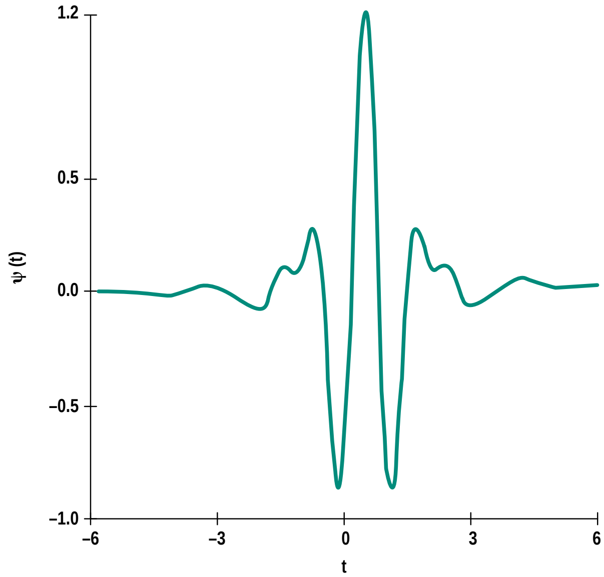

Fig. 5. The Meyer wavelet.

Often ω_0 is chosen so that the ratio of the highest maximum of ψ to the second highest maximum is approximately 1/2, that is,

$$\omega_0 = (2/\ln 2)^{1/2} \simeq 5.3364\ldots \qquad (22)$$

In practice one often takes $\omega_0 = 5$. For this value of ω_0, the second term in equation 20 is so small that it can be neglected in practice. Consequently, the Morlet wavelet can be considered as a modulated Gaussian waveform. Its real and imaginary parts for $\omega_0 = 5$ are shown in Figs. 6 and 7, respectively.

The Morlet wavelet is complex, even though most applications in which it is used involve only real signals. The wavelet transform of a real signal with this complex wavelet is plotted in modulus-phase form, that is, one plots $|\langle s, \psi_{m,n}\rangle|$ and $\tan^{-1}[\mathrm{Im}\langle s, \psi_{m,n}\rangle/\mathrm{Re}\langle s, \psi_{m,n}\rangle]$, where the brackets indicate the scalar or inner product of the signal waveform s with the basis function $\psi_{m,n}$, that is,

$$\langle s, \psi_{m,n}\rangle = \int_{-\infty}^{\infty} s(t)\psi'_{m,n}(t)dt.$$

The phase plot is particularly suited for the detection of singularities.

Daubechies Wavelet. Except for the Haar basis, all of the examples of orthonormal wavelet bases consist of infinitely supported functions. Ingrid Daubechies constructed an orthonormal wavelet in which ψ is compactly supported. The way to ensure compact support for the wavelet ψ is to choose a scaling function ϕ with compact support.

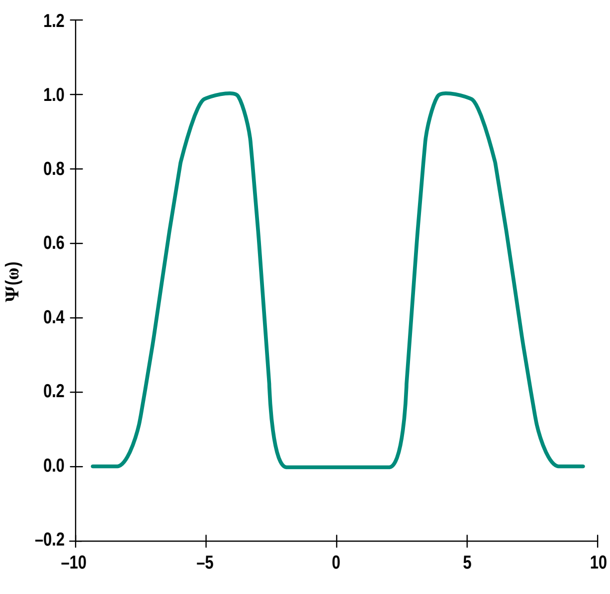

Fig. 4. Fourier transform of the Meyer wavelet.

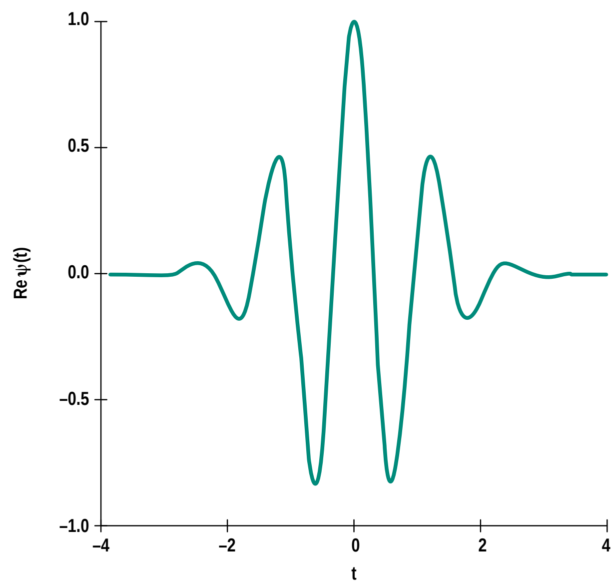

Fig. 6. Real part of the Morlet wavelet for $\omega_0 = 5$.

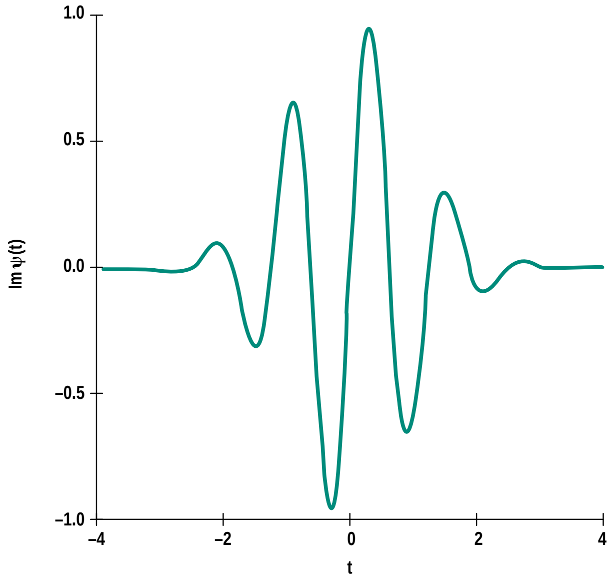

Fig. 7. Imaginary part of the Morlet wavelet for $\omega_0 = 5$.

First of all, find a progression $\{\alpha_k; k \in \mathbf{Z}\}$ satisfying the following four conditions for all integer $N \geq 2$:

$$\alpha_k = 0 \quad \text{if } k < 0 \text{ or } k > 2N \tag{23}$$

$$\sum_{k=-\infty}^{\infty} \alpha_k \alpha_{k+2m} = \delta_{0m} \quad \text{for all integer m} \tag{24}$$

$$\sum_{k=-\infty}^{\infty} \alpha_k = \sqrt{2} \tag{25}$$

$$\sum_{k=-\infty}^{\infty} \beta^k k^m = 0, \quad 0 \leq m \leq N-1, \tag{26}$$

where $\beta_k = (-1)^k \alpha_{-k+1}$.

If $N = 1$, then $\alpha_0 = \alpha_1 = 1$, corresponding to the Haar basis.

We can find a compactly supported scaling function $\phi(t)$ from the above progression $\{\alpha_k\}$. The function $\phi(t)$ is one solution of a functional equation:

$$\phi(t) = \sum_{k=-\infty}^{\infty} \alpha_k \sqrt{2}\, \phi(2t - k). \tag{27}$$

It is continuous and compactly supported and satisfies $\int \phi(t)dt = 1$ for integer N and the corresponding progression $\{\alpha_k\}$. The support of $\phi(t)$ is $[0, 2N-1]$.

Furthermore, if β_k is defined as the condition 26, the function $\psi(t)$ satisfying a functional equation

$$\psi(t) = \sum_{k=-\infty}^{\infty} \beta_k \sqrt{2}\, \phi(2t - k) \tag{28}$$

is compactly supported and fulfills the following:

- $\int \psi(t) t^m dt = 0$ for all integers $0 \leq m \leq N-1$.
- $\phi(t), \psi(t) \in C^{\lambda(N)}$ for Holder spaces $C^{\lambda(N)}$, where $\lambda(N)$ is an integer parameter and the elements of $C^{\lambda(N)}$ are functions that have $\lambda(N)$ derivatives.

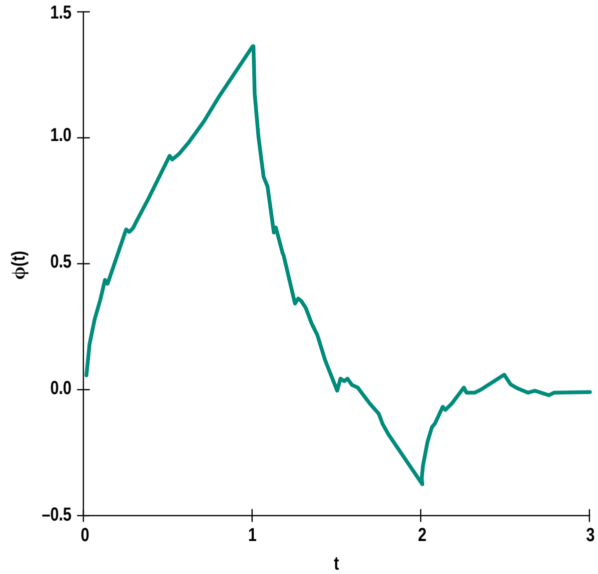

Fig. 8. The Daubechies scaling function for N = 2.

Figs. 8 and 9 show graphs of the Daubechies scaling function ϕ and the corresponding wavelet ψ for the value of $N = 2$.

Software Tools: Khoros System

The wavelet analysis software developed by HP Laboratories Japan is implemented as a toolbox in the Khoros system. The Khoros system is an integrated software development environment for information processing and visualization, based on the X Window System. It is distributed in the public domain and has been ported to the HP-UX* operating system.[3]

Khoros components include a visual programming language, code generators for extending the visual language and adding new application packages to the system, an interactive user interface editor, an interactive image display package, an

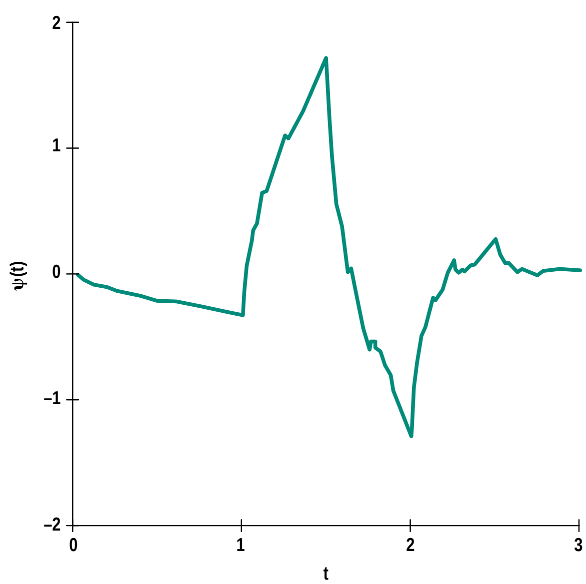

Fig. 9. The Daubechies wavelet for N = 2.

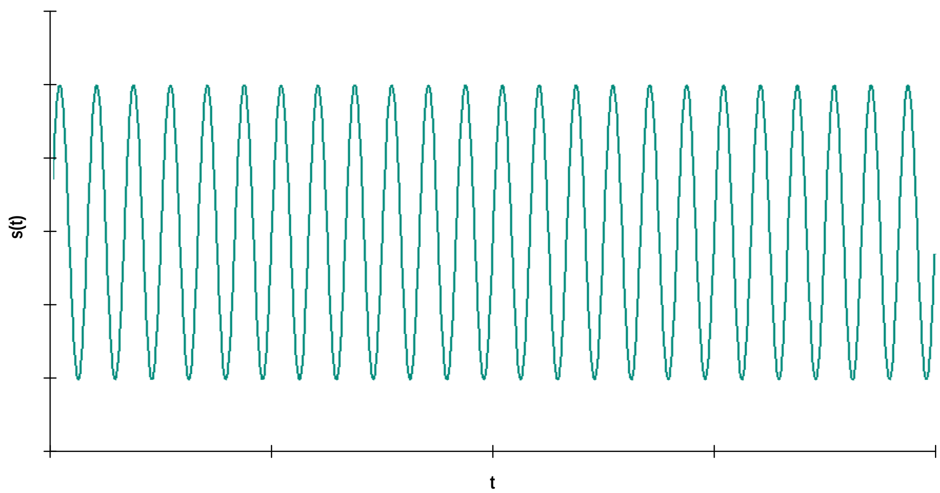

Fig. 10. Sinusoid with constant frequency.

extensive library of image processing, numerical analysis, and signal processing routines, and 2D/3D plotting packages.

The Khoros system also supports the toolbox update method for new routines created by another person or developed on another machine. A toolbox contributed by HP Laboratories Japan, the HPLJ Toolbox, contains wavelet application development tools, image data compression utilities, and other utilities.

Wavelet Analysis Examples

The following examples illustrate the advantages of the time-scale resolution properties of the wavelet transform and a related concept, the *chirplet transform,* for the analysis of various input signals, including a delta function, a step or box function, a differentially discontinuous function, a ramp function, sinusoidal functions, a chirp signal, and a sum of gliding tones.

Wavelet Analysis

This section gives several application examples of wavelet-based signal analysis, including both stationary and nonstationary signal analysis. These results were obtained with a Morlet wavelet, that is, a complex sinusoid windowed with a Gaussian envelope, expressed as follows:

$$\psi(t) = e^{ict}\exp\left(-\frac{1}{2}t^2\right), \qquad (29)$$

where c is a constant value of 5 so that the function $\psi(t)$ satisfies the admissibility condition.

As for the number of voices discussed earlier, we use v = 6 in the following three examples of synthesized data analysis.

Example 1. The first example gives the analysis of two sinusoids. Fig. 10 shows a sinusoid with a single constant frequency, and Figs. 11 and 12 represent its wavelet transform. The horizontal axis is in time in both the magnitude picture, Fig. 11, and the phase picture, Fig. 12. The vertical axis is scale, small scale at the top. Certain features of the signal are evident: horizontal strips of constant magnitude, and vertical lines in step with the phase of the signal.

Fig. 13 shows a sinusoid with linearly increasing frequency. The wavelet transform analysis results for this signal are shown in Figs. 14 and 15. Clearly visible is the upward slope corresponding to the increase of frequency.

Example 2. The second example is the analysis of the superposition of two delta functions and two sinusoids, as shown in Fig. 16. One delta function is larger than the sinusoidal signals and is visible in Fig. 16, but the other is much smaller and does not appear.

Figs. 17 and 18 show the wavelet transform representations. We can easily see the two peaks at smaller scale that correspond to the discontinuities contained in the input signal.

Example 3. This example shows the analysis of a sum of three sinusoids with different starting times. The input signal shown in Fig. 19 is not discontinuous, but its first derivative is.

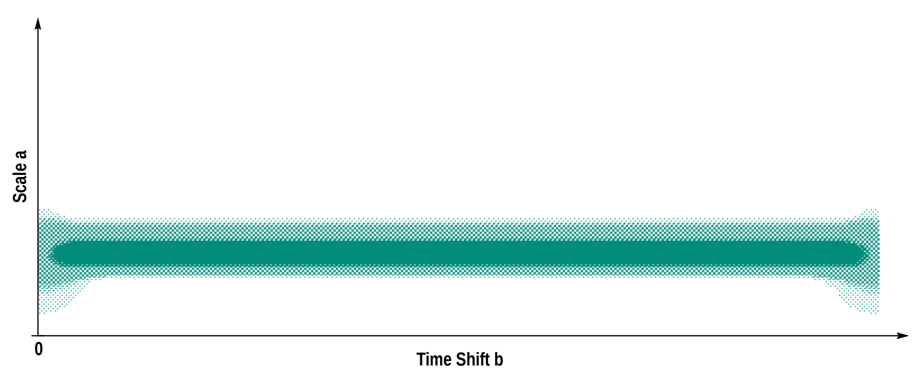

Fig. 11. Magnitude of the wavelet transform of a constant-frequency sinusoid.

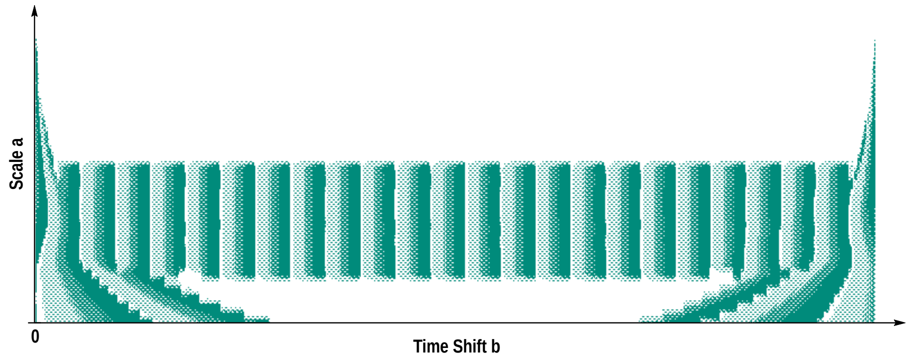

Fig. 12. Phase of the wavelet transform of a constant-frequency sinusoid.

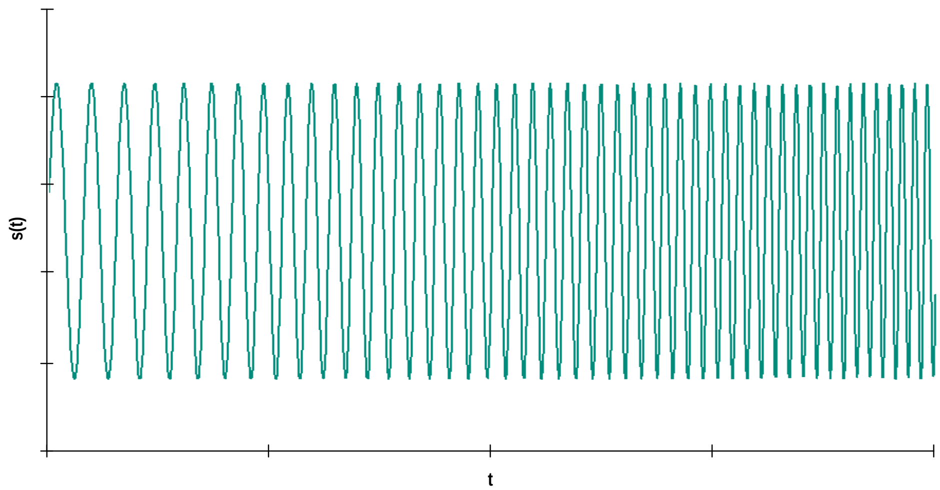

Fig. 13. Sinusoid with linearly increasing frequency.

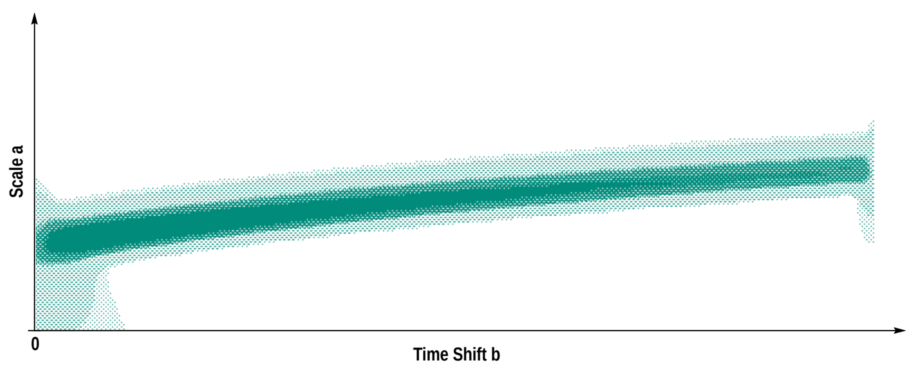

Fig. 14. Magnitude of the wavelet transform of a sinusoid with linearly increasing frequency.

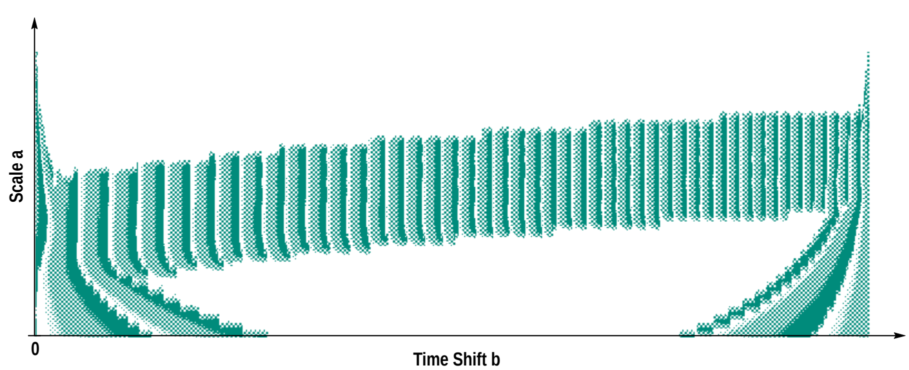

Fig. 15. Phase of the wavelet transform of a sinusoid with linearly increasing frequency.

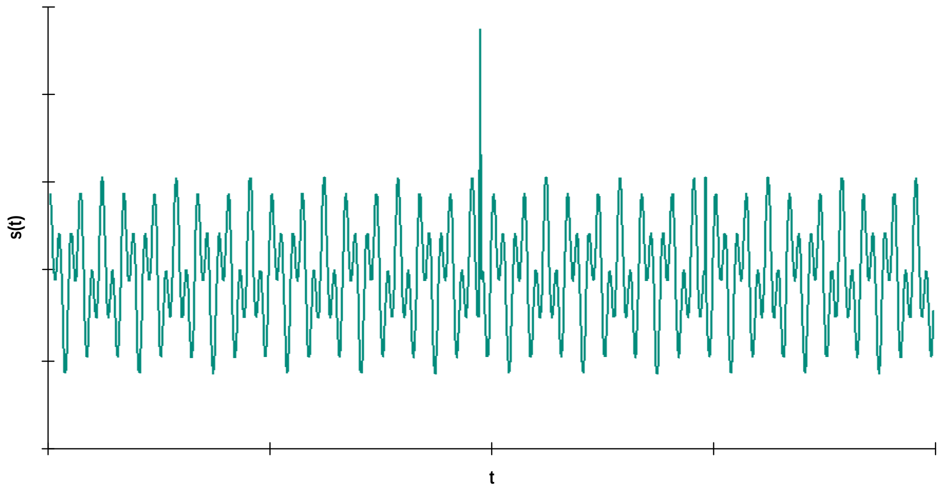

Fig. 16. Two delta functions and two sinusoids.

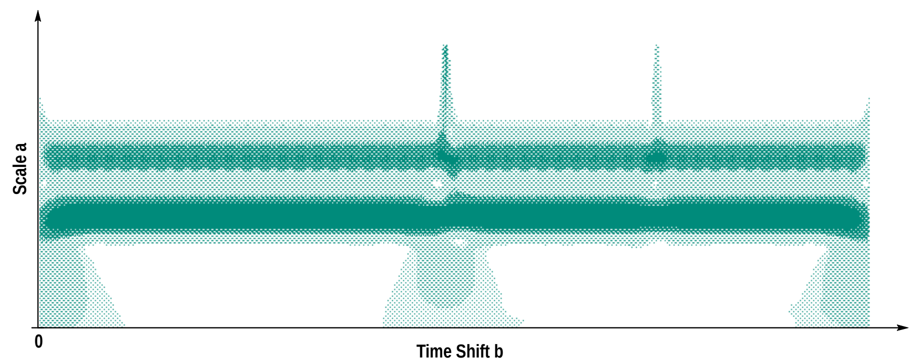

Fig. 17. Magnitude of the wavelet transform of the sum of two delta functions and two sinusoids.

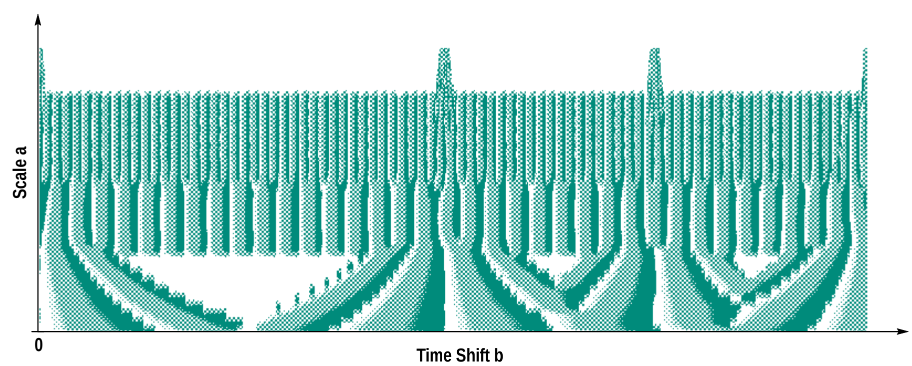

Fig. 18. Phase of the wavelet transform of the sum of two delta functions and two sinusoids.

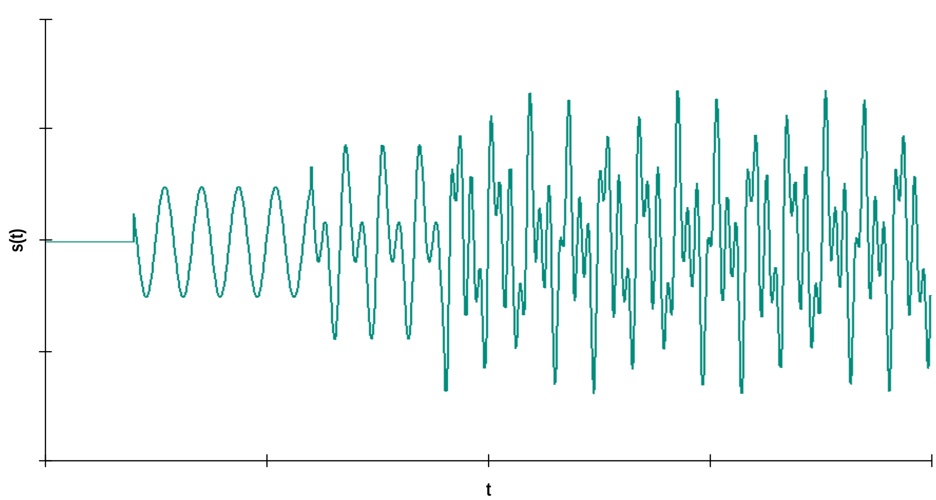

Fig. 19. Three sinusoids with different starting times.

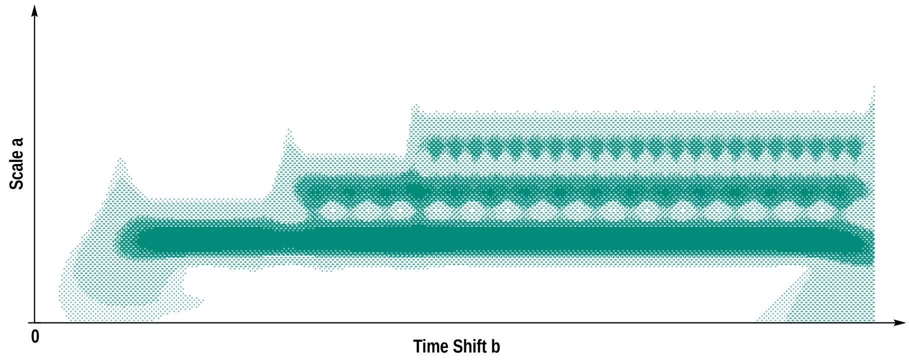

Fig. 20. Magnitude of the wavelet transform of three sinusoids with different starting times.

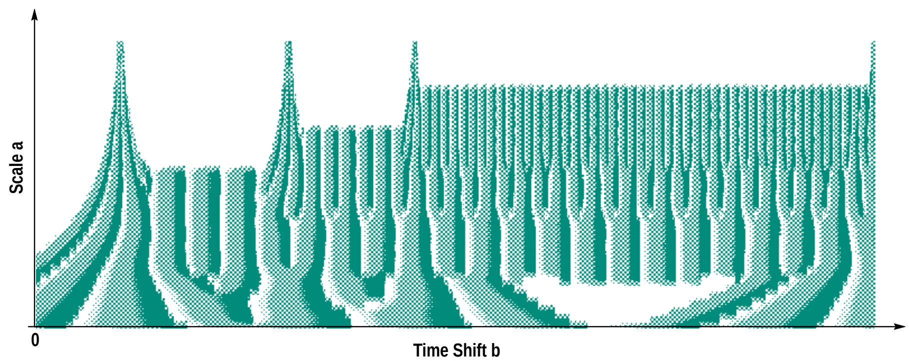

Fig. 21. Phase of the wavelet transform of three sinusoids with different starting times.

Both the frequencies and the beginnings of the components are very clearly visible in the wavelet transform pictures,

Figs. 20 and 21. A low-frequency sinusoid starts first, followed by a medium-frequency and a high-frequency sinusoid.

Real Data Analysis. This section shows the results of real data analysis. This data, provided by the Lake Stevens Instrument Division, is the transmitter turn-on data of a dual-band transceiver that was taken at a center frequency of 146.52 MHz with the measurement span set to 39.0625 kHz. In other words, the data is filtered to approximately a 40-kHz bandwidth. The time interval between points is 20 μs.

The input signal is plotted in Fig. 22. In this case, the transform was performed for the value of v = 12, and the magnitude and phase of the wavelet transform are shown in Figs. 23 and 24, respectively.

Chirplet Analysis

The wavelet transform has the effect of dissecting the time-scale plane into time-invariant cells with an aspect ratio dependent on the scale parameter. This property is important in spectral processing of signals but does not affect dynamic spectrum displays where time t is advanced by small increments relative to the cell width.

The representation of signals may benefit if the cell shape is not held time-invariant throughout. This time-dependent adjustment can be performed adaptively. Such a technique, called the *chirplet transform,* has been proposed. It uses oblique cells adapted to the local structure, permitting separation of the signal components.

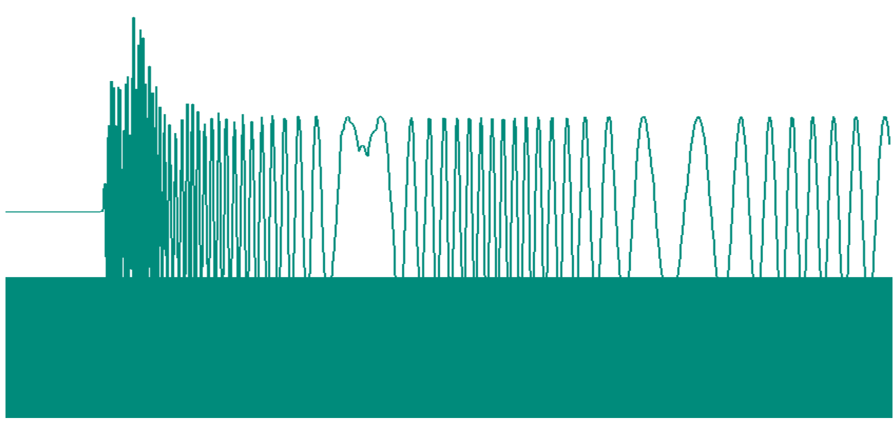

Fig. 22. Transmitter turn-on data.